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SUPERSYMMETRY, NONTRIVIAL FERMIONIC SHELL AND NILPOTENT MECHANICS

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A need of the introduction of nilpotent even variables to the consistent classical supersymmetric theory is justyfied on the example of simple supersymmetric system. It is claimed that any classical supersymmetric theory on fermionic shell is equivalent to the corresponding nilpotent mechanics.

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Natural formulation of supersymmetric models is based on the formalism involving so called Grassmannian coordinates [1, 2], i.e. functions taking values in an appropriate Grassmann algebra [3]. Usually nilpotence is associated to the fermions due to the anticommutativity of objects describing them. However even in simple supersymmetric mechanical models (for review see e.g. [4]) it turns out that we must consider nilpotent even coordinates to describe consistently the model [5, 6] and to have nontrivial description on the fermionic shell [7]. These nilpotent even elements appear as an extension of the field of numbers used in the formalism. In our approach such an algebra of the so called Study numbers [8] or dual numbers [9, 10] replaces the basic field of numbers (complex or real) and resulting configuration and phase spaces have a geometric structure which we shall call a *nilfold*.

In the present paper we want to show that the even nilpotent sector of the supersymmetric models is very important and should not be neglected. Moreover, it is connected in a natural way to the two-time physics approach proposed by Bars at al. [11, 12, 13]. The notion of even nilpotent “directions” as such is not new. It appears in supermanifold theory [14, 15, 16, 17] and classical SUSY models [18, 19, 20, 21, 22, 23] (see also [24, 25, 26, 27] and [28]). Some interesting properties of nilpotent directions in mechanics were discussed in [29, 6, 7]. A generalization of the notion of supermanifold including even nilpotent coordinates has also been considered in [30, 31].

CLASSICAL SUPERSYMMETRY. TOY MODEL

Let us consider $(1|2)$ dimensional superspace and in the chiral basis (t, θ^+, θ^-) and a scalar superfield $\Phi(t, \theta^+, \theta^-)$ which has the expansion $\Phi(t, \theta^+, \theta^-) = q(t) + i\theta^+\psi_-(t) + i\theta^-\psi_+(t) + \theta^+\theta^- F(t)$, where $q(t)$ is a bosonic coordinate, $F(t)$ is an auxiliary bosonic field, $\psi_\pm(t)$ are fermionic coordinates. The superfield Lagrangian of the model with a superpotential $V(\Phi)$ can be written as

$$\mathcal{L} = \frac{1}{2} D^+\Phi \cdot D^-\Phi + V(\Phi), \tag{1}$$

where $D^\pm = \partial/\partial\theta^\mp + i\theta^\pm\partial/\partial t$ and the equations of motion are

$$\frac{1}{2}\left[D^+, D^-\right]\Phi - V'(\Phi) = 0.$$

The action has standard form $S = \int dt\mathcal{L}$, where $L = \int d\theta^- d\theta^+ \mathcal{L}$ is the component Lagrangian. After expansion in θ and using the nondynamical equation for auxiliary field $F(t) = -V'(q)$ it takes the following form

$$L = \frac{\dot{q}^2}{2} - \frac{V'^2(q)}{2} + i\psi_+\dot{\psi}_- + V''(q)\psi_+\psi_-. \tag{2}$$

From (2) the equations of motion are

$$\ddot{q} + V'(q)V''(q) - V'''(q)\psi_+\psi_- = 0, \tag{3}$$
$$\dot{\psi}_\pm \pm iV''(q)\psi_\pm = 0. \tag{4}$$

The bosonic equation (3) cannot be satisfied by real $q(t)$ outside trivial fermionic shell $\psi_\pm(t) = 0$. Taking nontrivial solutions of the fermionic equations of motion (4), what means that fermionic coordinates are not zero, we obtain

$$\psi_\pm(t) = \lambda_\pm e^{\mp ip(q_0, E)}, \tag{5}$$

where $\sin p\left(q_0,E\right)=\frac{V_0'}{\sqrt{2E}}$, $V_0'=V'\left(q_0\left(t\right)\right)$, E is the energy of the system, and $q_0\left(t\right)$ is a standard solution for one-dimensional nonsupersymmetric system (see below (19)). In this case we obviously have

$$\psi_+\left(t\right)\psi_-\left(t\right)=\lambda_+\lambda_-=\mathbf{e}=const, \tag{6}$$

and therefore $\mathbf{e}$ is a nilpotent even Grassmann number $\mathbf{e}^2=0$.

On the fermionic shell the original SUSY lagrangian (2) takes the form

$$L_{FermiShell}\left(q,V|\mathbf{e}\right)=\frac{\dot{q}^2}{2}-\frac{V'^2\left(q\right)}{2}+V''\left(q\right)\mathbf{e}. \tag{7}$$

Let us represent it as the one-dimensional lagrangian

$$L_{NilMech}\left(q,\mathbf{U}\right)=\frac{\dot{q}^2}{2}-\frac{\mathbf{U}'^2}{2}, \tag{8}$$

where

$$\mathbf{U}\equiv\mathbf{U}\left(q|\mathbf{e}\right)=V\left(q\right)+\mathbf{e}\ln\frac{1}{V'\left(q\right)}. \tag{9}$$

Thus

$$L_{FermiShell}\left(q,V|\mathbf{e}\right)=L_{NilMech}\left(q,\mathbf{U}\right) \tag{10}$$

and we see that the lagrangian of supersymmetric mechanics on fermionic shell is equal to a lagrangian with potential of the special form (9) in which some even nilpotent part is added. We expect that such property may take place in larger class of theories, and this equivalence can be the fundamental property of any classical supersymmetric theory on a fermionic shell.

The lagrangian (2) is invariant with respect to the following supersymmetry transformations

$$\delta q=i\left(\varepsilon_+\psi_-+\varepsilon_-\psi_+\right), \tag{11}$$

$$\delta\psi_+=\varepsilon_+\left(-\dot{q}+iV'\right),\quad\delta\psi_-=\varepsilon_-\left(-\dot{q}-iV'\right). \tag{12}$$

On the fermionic shell the parameters $\varepsilon_\pm$ take the form

$$\varepsilon_\pm=k_\pm\lambda_\pm, \tag{13}$$

where $k_\pm$ are some even parameters. Now the form of the supersymmetry parameters $\varepsilon_\pm$ is fixed: they are expressed a product of arbitrary even parameter $k_\pm$ defining the transformation and fixed odd constants of fermionic motion $\lambda_\pm$. Then the bosonic part of the supersymmetry transformation (11) becomes

$$\delta q=i\mathbf{e}\left(k_+\sin p\left(q_0,E\right)+k_-\cos p\left(q_0,E\right)\right)=i\mathbf{e}\frac{k_+V_0'+k_-\sqrt{2E-V_0'^{\,2}}}{\sqrt{2E}}, \tag{14}$$

which means that this transformation is pure nilpotent and numerical part of q does not change. On the fermionic shell the bosonic equation (3) takes the form

$$\ddot{q}+V'\left(q\right)V''\left(q\right)-V'''\left(q\right)\mathbf{e}=0. \tag{15}$$

Therefore the bosonic solution on the fermionic shell can be written as follows

$$\mathbf{q}\left(t\right)=q_0\left(t\right)+\mathbf{e}q_N\left(t\right). \tag{16}$$

Then for $q_0\left(t\right)$ and $q_N\left(t\right)$ we have the equations

$$\ddot{q}_0+V_0'V_0''=0, \tag{17}$$

$$\ddot{q}_N+\left(V_0'V_0''\right)'q_N-V_0'''=0. \tag{18}$$

After first integration we formally obtain

$$\dot{q}_0^2+V_0'^{\,2}=2E_0, \tag{19}$$

$$\dot{q}_0\dot{q}_N+V_0''q_N-V_0''=E_N, \tag{20}$$

where E_0 and E_N are even integration constants and $V_0 \equiv V(q_0)$. Then the trajectory of such classical system is given by

$$t = \int \frac{dq_0}{\sqrt{2E_0 - V_0'^{\,2}}}, \tag{21}$$

$$q_N = 2\sqrt{2E_0 - V_0'^{\,2}} \int dq_0 \frac{E_N + V_0''}{\left(2E_0 - V_0'^{\,2}\right)^{3/2}} + C\sqrt{2E_0 - V_0'^{\,2}}. \tag{22}$$

Let us assume that $q_0(t), q_N(t) \in \mathbb{R}$ (for simplicity, but more complicated cases are possible). From the above example one can see that classical "bosonic" trajectory of supersymmetric one dimensional system takes values not in the usual real or complex numbers but *in the larger structure*. This structure is known as the Study numbers [8] which form the so called dual algebra $\mathbf{D}_1(\boldsymbol{\iota};\mathbb{R})$ with one nilpotent generator, where $\boldsymbol{\iota} = \mathbf{e}$. Dual numbers were introduced in 1873 by Clifford [32]. The dual algebra $\mathbf{D}_n(\boldsymbol{\iota};\mathbb{R})$ is defined as an associative algebra with unit and *nilpotent* generators $\boldsymbol{\iota}_1, \ldots, \boldsymbol{\iota}_n,\ \boldsymbol{\iota}_k^2 = 0,\ k = 1, \ldots, n$ with *commutative* multiplication $\boldsymbol{\iota}_k \boldsymbol{\iota}_m = \boldsymbol{\iota}_m \boldsymbol{\iota}_k,\ k \neq m$. The general element of $\mathbf{D}_n(\boldsymbol{\iota};\mathbb{C})$ has the form $\mathbf{a} = a_0 + \sum_{p=1}^{2^n-1} \sum_{k_1<\ldots<k_p} a_{k_1\ldots k_p} \boldsymbol{\iota}_{k_1} \ldots \boldsymbol{\iota}_{k_p}, \quad a_0, a_{k_1\ldots k_p} \in \mathbb{C}$.For $n = 1$ we have $\mathbf{D}_1(\boldsymbol{\iota}_1;\mathbb{C}) \ni$ $\mathbf{a}= a_0 + a_1\boldsymbol{\iota}_1$, i.e. dual (or Study) numbers, when $a_0, a_1 \in \mathbb{C}$. For $n = 2$ the general element of $\mathbf{D}_2(\boldsymbol{\iota}_1, \boldsymbol{\iota}_2;\mathbb{C})$ is written as follows: $\mathbf{a} = a_0 + a_1\boldsymbol{\iota}_1 + a_2\boldsymbol{\iota}_2 + a_{12}\boldsymbol{\iota}_1\boldsymbol{\iota}_2$. A function of a dual argument is defined by its Taylor expansion $f(\mathbf{x}) = f(x_0 + \boldsymbol{\iota}_k x_N) = f(x_0) + \boldsymbol{\iota}_k x_N f'(x_0)$. The dual numbers have many applications, e.g. in such fields as 3D measurement problem [33], Lie and Hopf algebra contractions [10, 34].

NILPOTENT MECHANICS. LAGRANGIAN FORMULATION

Let us consider the one-dimensional trajectory as a *dual algebra* valued function[1] $\mathbf{x}(\mathbf{T})$, $\mathbf{x}, \mathbf{T} \in \mathbf{D}_1(\boldsymbol{\iota};\mathbb{R})$ (i.e. with values in a Banach superalgebra Λ_0 with two generators $1,\ \boldsymbol{\iota}$, where $\boldsymbol{\iota}^2 = 0$ [35, 36]). Then

$$\mathbf{T} = t + \boldsymbol{\iota} t_N, \quad \mathbf{x}(\mathbf{T}) = x_0(\mathbf{T}) + \boldsymbol{\iota} x_N(\mathbf{T}). \tag{23}$$

The derivative is of the following form [35, 36]

$$\frac{\partial}{\partial \mathbf{T}} = \frac{\partial}{\partial t} + \boldsymbol{\iota} \frac{\partial}{\partial t_N}, \tag{24}$$

the conjugation is defined as

$$\frac{\partial}{\partial \bar{\mathbf{T}}} = \frac{\partial}{\partial t} - \boldsymbol{\iota} \frac{\partial}{\partial t_N}, \quad \bar{\mathbf{T}} = t_N, \tag{25}$$

and the Cauchy-Riemann conditions are

$$\frac{\partial x_0(\mathbf{T})}{\partial t_N} = 0, \quad \frac{\partial x_0(\mathbf{T})}{\partial t} = \frac{\partial x_N(\mathbf{T})}{\partial t_N}. \tag{26}$$

Now the full derivative is

$$\frac{d\mathbf{x}(\mathbf{T})}{d\mathbf{T}} = \dot{x}_0(t) + \boldsymbol{\iota}\left[\dot{x}_N(t) + \ddot{x}_0(t)\, t_N\right], \tag{27}$$

where dot denotes differentiation by t.

Let $\mathbf{U}(\mathbf{x}) = U_0(\mathbf{x}) + \boldsymbol{\iota} U_N(\mathbf{x})$, then the *dual algebra* valued lagrangian $\mathbf{L}$ has the form

$$\mathbf{L} = \frac{1}{2}\left(\frac{d\mathbf{x}(\mathbf{T})}{d\mathbf{T}}\right)^2 - \mathbf{U}(\mathbf{x}(\mathbf{T})). \tag{28}$$

Expanding in series with respect to $\boldsymbol{\iota}$ we obtain decomposition $\mathbf{L} = L_0 + \boldsymbol{\iota} L_N$ (cf. (7) with $\mathbf{e} = \boldsymbol{\iota}$), where

$$L_0 = \frac{1}{2}\dot{x}_0^2(t) - U_0(x_0(t)), \tag{29}$$

$$L_N = \dot{x}_0(t)\,\dot{x}_N(t) - x_N(t)\,U_0'(x_0(t)) - U_N(x_0(t)) + t_N \dot{x}_0(t)\left[\ddot{x}_0(t) - U_0'(x_0(t))\right], \tag{30}$$

The prime here denotes differentiation with respect to the variable.

The second time parameter t_N emerges naturally within this approach and suggests relation to the two-time theory developed by Bars at al. [37, 38, 11, 12, 39, 13] and allows various holographic pictures of the supersymmetric model. We will discuss this in the forthcoming paper. Here we consider only the simplest picture where one takes into account single time t with $t_N = 0$.

[1] Dual valued variables will be written in bold.

Let us consider a model of the nilpotent mechanics defined by lagrangians (30) on hyperplane $t_N = 0$. In this case nilpotent mechanics is described by two lagrangians

$$L_0 = \frac{1}{2}\dot{x}_0^2 - U_0(x_0), \tag{31}$$

$$L_N = \dot{x}_0\dot{x}_N - x_N U_0'(x_0) - U_N(x_0), \tag{32}$$

where $L_0 = L_0(x_0, \dot{x}_0)$ is a usual lagrangian of point particle, $L_N = L_N(x_0, x_N, \dot{x}_0, \dot{x}_N)$ is nil lagrangian which depends on two degrees of freedom [2].

At the first glance we have three generalized momenta

$$p_0 = \frac{\partial L_0}{\partial \dot{x}_0} = \dot{x}_0, \tag{33}$$

$$p_N = \frac{\partial L_N}{\partial \dot{x}_0} = \dot{x}_N, \tag{34}$$

$$p_1 = \frac{\partial L_N}{\partial \dot{x}_N} = \dot{x}_0. \tag{35}$$

Let us notice that

$$p_1 = p_0, \tag{36}$$

what is a consequence of nilpotence and $\boldsymbol{\iota}^2 = 0$.

Equations of motion are of the form

$$\ddot{x}_0 + U_0'(x_0) = 0, \tag{37}$$

$$\ddot{x}_N + x_N U_0''(x_0) + U_N'(x_0) = 0. \tag{38}$$

So we formally have an *additional* integral of motion E_N

$$\dot{x}_0\frac{\partial L_0}{\partial \dot{x}_0} - L_0 = \frac{\dot{x}_0^2}{2} + U_0(x_0) = E_0, \tag{39}$$

$$\dot{x}_0\frac{\partial L_N}{\partial \dot{x}_0} + \dot{x}_N\frac{\partial L_N}{\partial \dot{x}_N} - L_N = \dot{x}_0\dot{x}_N + x_N U_0'(x_0) + U_N(x_0) = E_N. \tag{40}$$

Using (40) and $\partial/\partial t = \dot{x}_0\partial/\partial x_0$ we obtain

$$2\left(E_0 - U_0(x_0)\right)x_N'(x_0) + U_0'(x_0)\,x_N(x_0) = E_N - U_N(x_0). \tag{41}$$

One can see that the system has two return points

$$E_0 = U_0(x_0) \Longleftrightarrow x_N(x_0) = \frac{E_N - U_N(x_0)}{U_0'(x_0)}, \tag{42}$$

$$E_N = U_N(x_0) \Longleftrightarrow x_N(x_0) = C\sqrt{E_0 - U_0(x_0)}, \tag{43}$$

where C is integration constant. If simultaneously $E_0 = U_0(x_0)$ and $E_N = U_N(x_0)$, then $x_N(x_0) = 0$.

The solution for $x_N(x_0)$ can be obtained in the explicit form

$$x_N(x_0) = \sqrt{E_0 - U_0(x_0)}\left(\frac{1}{2}\int dx_0 \frac{E_N - U_N(x_0)}{\left(E_0 - U_0(x_0)\right)^{3/2}} + C\right). \tag{44}$$

The term with C can be removed by shifts in parameter t_N and therefore we choose $C = 0$. Then the on-shell lagrangians take the form

$$L_0 = E_0 - 2U_0(x_0), \tag{45}$$

$$L_N = E_N - 2U_N(x_0) - U_0'(x_0)\sqrt{E_0 - U_0(x_0)}\int dx_0 \frac{E_N - U_N(x_0)}{\left(E_0 - U_0(x_0)\right)^{3/2}}. \tag{46}$$

Note that we cannot put the term L_N to zero using parameters of the theory, and therefore the full lagrangian of the nilpotent mechanics always takes value in Study numbers ($L_N \neq 0$).

Standard passage to the phase space is not possible for the lagrangian $\mathbf{L}$, since its Hessian vanishes

$$\left|\frac{\partial \mathbf{L}}{\partial \dot{q}^i \partial \dot{q}^j}\right| = \boldsymbol{\iota}^2 = 0, \quad \dot{q}^i = \dot{x}_0,\, \dot{x}_N. \tag{47}$$

Nevertheless, we will not use here the Dirac constraint approach (see e.g.[40]), but we will try to provide some modification of the Legendre transformation specially for the dual space $\mathbf{D}_1(\boldsymbol{\iota};\mathbb{R})$ [29].

[2] We note that $x_0(t)$ and $x_N(t)$ are considered as 2 independent functions.

GENERALIZED LEGENDRE TRANSFORMATION

Let us consider a general function on dual space $\mathbf{D}_1(\boldsymbol{\iota};\mathbb{R}) \to \mathbf{D}_1(\boldsymbol{\iota};\mathbb{R})$ $\mathbf{y} = f(\mathbf{x}) + \boldsymbol{\iota} h(\mathbf{x})$, where $\mathbf{x} = x_0 + \boldsymbol{\iota} x_N$. Then

$$\mathbf{y} = y_0 + \boldsymbol{\iota} y_N, \tag{48}$$
$$y_0(x_0) = f(x_0), \tag{49}$$
$$y_N(x_0, x_N) = x_N f'(x_0) + h(x_0). \tag{50}$$

We construct the Legendre transformation for $y_0(x_0) \to g_0(p_0, x_0)$ and $y_N(x_0, x_N)$ separately taking into account that $y_N(x_0, x_N) \to g_N(p_N, p_1, x_0, x_N)$ is a function of two variables [41] as follows

$$g_0(p_0, x_0) \quad = \quad p_0 x_0 - y_0(x_0), \tag{51}$$
$$g_N(p_N, p_1, x_0, x_N) \quad = \quad p_N x_0 + p_1 x_N - y_N(x_0, x_N). \tag{52}$$

As usually [41] we determine parameters p_0, p_N, p_1 from condition of maximum of $g_0(p_0, x_0)$ and $g_N(p_N, p_1, x_0, x_N)$ as the functions of x_0, x_N, and obtain

$$p_0 = \frac{\partial y_0(x_0)}{\partial x_0} = f'(x_0), \tag{53}$$
$$p_N = \frac{\partial y_N(x_0, x_N)}{\partial x_0} = x_N f''(x_0) + h'(x_0), \tag{54}$$
$$p_1 = \frac{\partial y_N(x_0, x_N)}{\partial x_N} = f'(x_0). \tag{55}$$

Equality of p_0 and p_1, i.e.

$$p_0 = p_1 \tag{56}$$

is a consequence of generalized Cauchy-Riemann conditions for the analyticity of dual number valued functions [35].

$$\frac{\partial y_0}{\partial x_N} = 0, \tag{57}$$
$$\frac{\partial y_0}{\partial x_0} = \frac{\partial y_N}{\partial x_N}. \tag{58}$$

We use (52), (56) and (50) to obtain

$$g_N = p_N x_0 + p_0 x_N - y_N = p_N x_0 + f'(x_0) x_N - x_N f'(x_0) - h(x_0) = p_N x_0 - h(x_0). \tag{59}$$

Then the final form of the Legendre transformation (taking into account (56)) is

$$\mathbf{g}(p_N, p_1, x_0, x_N) = g_0(p_0, x_0) + \boldsymbol{\iota} g_N(p_N, p_1, x_0, x_N), \tag{60}$$

where

$$g_0(p_0) = p_0 x_0(p_0) - f(x_0(p_0)), \tag{61}$$
$$g_N(p_N, p_0) = p_N x_0(p_0) - h(x_0(p_0)). \tag{62}$$

The function $x_0(p_0)$ is determined from eq.(53). Notice that the difference between (61) and (62) is only in $f(x_0(p_0))$ and $h(x_0(p_0))$.

NILPOTENT MECHANICS. HAMILTONIAN FORMULATION

Let us apply previously generalized Legendre transformation to the nilpotent mechanics lagrangian $\mathbf{L} = L_0 + \boldsymbol{\iota} L_N$, where L_0 and L_N are given by eqs. (31) and (32) respectively. In this way we obtain the following hamiltonian

$$\mathbf{H} = H_0 + \boldsymbol{\iota} H_N, \tag{63}$$

where

$$H_0(x_0, p_0) \quad = \quad p_0 \dot{x}_0 - L_0 = \frac{p_0^2}{2} + U_0(x_0), \tag{64}$$
$$H_N(x_0, x_N, p_0, p_N) \quad = \quad p_N \dot{x}_0 + p_0 \dot{x}_N - L_N = p_0 p_N + x_N U_0'(x_0) + U_N(x_0), \tag{65}$$

and we have used condition $p_1 = p_0$ (56). The resulting hamiltonian equations of motion are of the form

$$\begin{aligned} \dot{x}_0 &= \frac{\partial H_0}{\partial p_0}, & \dot{x}_0 &= \frac{\partial H_N}{\partial p_N}, & \dot{x}_N &= \frac{\partial H_N}{\partial p_0}, \\ \dot{p}_0 &= -\frac{\partial H_0}{\partial x_0}, & \dot{p}_N &= -\frac{\partial H_N}{\partial x_0}, & \dot{p}_0 &= -\frac{\partial H_N}{\partial x_N}. \end{aligned} \tag{66}$$

Analogously as for L_N now H_N contains all the information about motion of the system. Component energies are defined by

$$H_0 (x_0, p_0) = E_0, \tag{67}$$

$$H_N (x_0, x_N, p_0, p_N) = E_N \tag{68}$$

and the full energy is a dual valued number as well $\mathbf{E} = E_0 + \boldsymbol{\iota} E_N$. The component phase space has two sectors with Poisson brackets specific for each sector. Namely

$$\{A, B\}_0 = \left(\frac{\partial A}{\partial x_0} \frac{\partial B}{\partial p_0} - \frac{\partial A}{\partial p_0} \frac{\partial B}{\partial x_0} \right) \tag{69}$$

and

$$\{A, B\}_N = \left(\frac{\partial A}{\partial x_0} \frac{\partial B}{\partial p_N} - \frac{\partial A}{\partial p_N} \frac{\partial B}{\partial x_0} \right) + \left(\frac{\partial A}{\partial x_N} \frac{\partial B}{\partial p_0} - \frac{\partial A}{\partial p_0} \frac{\partial B}{\partial x_N} \right). \tag{70}$$

We note that in "nilpotent" sector of the phase space the Poisson bracket is related to unusual conjugation of canonical variables

$$x_0 \leftrightarrow p_N \text{ and } x_N \leftrightarrow p_0. \tag{71}$$

The second Poisson bracket here ω_N^{ij} can lead to additional quantization rule with additional (to Planck $\hbar$) constant $\hbar_N$ using nil Hamiltonian H_N (65). Analogous effect is also present in the quantization over the oddons of anti-bracket (supersymmetric) systems (cf. [42, 43] and references therein).

CONCLUSIONS

We emphasize that the dual numbers are necessary to formulate in a consistent way supersymmetric models. The supersymmetry transformations as well as the dynamics of the model have to be written in terms of a new nilfold language. General picture of new nilpotent models on the dual space $\mathbf{D}_1(\boldsymbol{\iota}; \mathbb{R})$ corresponding to $N = 1$ supersymmetric models requires the presence of two time parameters what is related closely to the two-time approach developed by Bars at al.[37, 38]. In case of N supersymmetries [44, 45] one should consider many-time approach. This aspect of the nilpotent mechanics as well as its relation to the $D = 11$ supersymmetric mechanics and string/brane/M theory will be presented elsewhere.

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СУПЕРСИММЕТРИЯ, НЕТРИВИАЛЬНАЯ ФЕРМИОННАЯ ПОВЕРХНОСТЬ И НИЛЬПОТЕНТНАЯ МЕХАНИКА

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Необходимость рассматривать четные нильпотентные направления в последовательной классической суперсимметричной теории продемонстрирована на примере простейшей суперсимметричной модели. Утверждается, что любая суперсимметричная теория на фермионной массовой поверхности эквивалентна соответствующей нильпотентной механике.

КЛЮЧЕВЫЕ СЛОВА: супермеханика, дуальное число, нильпотентность, фермионная массовая поверхность, преобразование Лежандра, скобки Пуассона