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CODERIVATIONS AND CODIFFERENTIAL CALCULI ON FINITE GROUPS

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Various aspects of noncommutative geometry which became recently an important tool of building theoretical models of elementary particle physics on noncommutative space-time are studied. The way of consistent construction of codifferential calculi, a dual analog of differential calculi, which is the ground of any particle model, is shown. The dual analog of derivation, a coderivation, and covariant maps are introduced and corresponding consistency equations are presented for coalgebras and comoduli. Concrete examples for some finite groups, which are important for particle physics models on discrete space-time, are developed manifestly. Codifferential calculi on finite groups are constructed, where covariant conditions lead to some special system of functional equations which are solved in various cases. It is outlined that further investigation can be connected with properly dualizing of corresponding complexes and bicomplexes.
KEY WORDS: noncommutative geometry, codifferential calculus, finite group, complex, bicomplex, coalgebra, comoduli

Recently field theories on noncommutative spaces gained a lot of interest because of the appearance of such theories as certain limits of string, D-brane and M-theory (see [1, 2] and the refs therein). In view of application of noncommutative geometry [3, 4, 5, 6, 7] and quantum groups [8, 9, 10] in modern theoretical models of particle physics [11, 12, 13, 14, 15] the extension of the notion of derivation is very important and promising. In the noncommutative geometry one replaces the algebra of smooth functions on a manifold by some associative and noncommutative algebra $\mathcal{A}$, and in search of analogs of vector fields one defines them as derivations of $\mathcal{A}$. Such a point of view has been pioneered by Robert Hermann [16]. In order to be able to formulate dynamics and field theories on or with such 'generalized spaces', a convenient tool appears to be a 'differential calculus' [17, 18, 19, 20] on it which is an algebraic analogue of the calculus of differential forms on a manifold (see also [21, 22]). From another side discrete spaces play a fundamental role in the building of elementary particle models in the framework of noncommutative geometry [23, 24, 25, 26]. However the algebra of functions on a finite set do not admit any derivation besides the trivial one which maps everything to zero. In some special cases there is a distinguished set of maps which satisfy a modified derivation rules, but a suitable general concept is missing.

In this paper we consider another approach which starts from observation [27] that a derivation of an algebra induces a coderivation on a subspace of its dual [28, 29]. We study generalization of coderivation to comodules of coalgebras, coderivations between bicomodules, and then generalized bicovariant maps from which coderivations emerge as special cases. Then we apply these results to maps between spaces associated with differential calculi over finite groups and prove some useful statements.

COALGEBRAS AND HOPF ALGEBRAS

Here we review some properties of coalgebras and Hopf algebras [30, 31] starting with definitions to make the paper self-contained. An (associative) *algebra* is a triple $(\mathcal{A}, m, \eta)$ (usually denoted by $\mathcal{A}$), where $\mathcal{A}$ is a linear space over a field $\mathbb{K}$ together with the tensor product $\otimes_{\mathbb{K}}$ and two maps $m : \mathcal{A}\otimes_{\mathbb{K}}\mathcal{A} \to \mathcal{A}$ and $\eta : \mathbb{K} \to \mathcal{A}$ such that 1) m and η are linear; 2) $m(m\otimes_{\mathbb{K}}1) = m(1\otimes_{\mathbb{K}}m)$ (associativity); 3) $m(1\otimes_{\mathbb{K}}\eta) = m(\eta\otimes_{\mathbb{K}}1) = \mathrm{id}$ (unit). The products $a\otimes_{\mathbb{K}}b$ and $m(a\otimes_{\mathbb{K}}b)$ are usually denoted by $a\otimes b$ and $a \cdot b$ (or ab for obvious cases), and so the associativity is $a\,(bc) = (ab)\,c$, $a, b, c \in \mathcal{A}$. Property 3) is an unusual way of saying that $\mathcal{A}$ has a unit element, for let $\alpha\otimes a\epsilon\mathcal{A}\otimes\mathbb{K} \simeq \mathcal{A}$, then $m(\eta\otimes1)(\alpha\otimes a) = \eta(\alpha)a$ which by property 3) is equal to αa. This means that $\eta(\alpha) = \alpha \cdot 1$ where 1 is the unit element of $\mathcal{A}$. Let $(\mathcal{A}, m_{\mathcal{A}}, \eta_{\mathcal{A}})$ and $(\mathcal{B}, m_{\mathcal{B}}, \eta_{\mathcal{B}})$ be two algebras, then the tensor product space $\mathcal{A}\otimes\mathcal{B}$ is naturally endowed with the structure of an algebra. The multiplication $m_{\mathcal{A}\otimes\mathcal{B}}$ on $\mathcal{A}\otimes\mathcal{B}$ is defined by $m_{\mathcal{A}\otimes\mathcal{B}} = (m_{\mathcal{A}}\otimes m_{\mathcal{B}})(1\otimes\tau\otimes1)$ where τ is the so called flip map $\tau(a\otimes b) = b\otimes a$, or more explicitly multiplication on $\mathcal{A}\otimes\mathcal{B}$ reads $(a_1\otimes b_1)\cdot(a_2\otimes b_2) = (a_1 \cdot a_2)\otimes(b_1 \cdot b_2)$. It follows that the set of algebras is closed under taking tensor products. The algebra is called *commutative*, if $m = m \circ \tau$ (or $ab = ba$).

Despite these abstract algebras are of considerable importance themselves, in physics one is primarily interested in their representations, the homomorphisms from the algebra to an algebra of linear operators on some vector space. Let $(\mathcal{A}, m, \eta)$ be an algebra, V a linear space and ρ a map from $\mathcal{A}$ to the space of linear operators in V, then (V, ρ) is called a *representation* of $\mathcal{A}$ if 1) ρ is linear; 2) $\rho(xy) = \rho(x)\rho(y)$. In physics it often happens that one has to compose two representations (for example when adding angular momenta). This happens when two physical systems each within a certain representation interact (for example two spin $1/2$ particles). Mathematically this means that one must consider tensor product representations of the underlying abstract algebra. Suppose (ψ_1, V_1) and(ψ_2, V_2) are two representations of an algebra $\mathcal{A}$. How do we define an action of $\mathcal{A}$ an $V_1\otimes V_2$ using ψ_1 and ψ_2 ? There are only two reasonable possibilities:

1) the action of $a \in \mathcal{A}$ on $v_1 \otimes v_2 \in V_1 \otimes V_2$ is: $a \cdot (v_1 \otimes v_2) = (\psi_1(a)v_1) \otimes (\psi_2(a)v_2)$; 2) the action of a on $v_1 \otimes v_2$ is $a \cdot (v_1 \otimes v_2) = \psi_1(a)v_1 \otimes v_2 + v_1 \otimes \psi_2(a)v_2$. Definition 1) certainly does not satisfy the required properties since such a tensor product representation would not be linear. Definition 2) has the problem that the homomorphism property does not hold unless the multiplication on $\mathcal{A}$ is antisymmetric (as is the case in for example a Lie algebra). There seems no way out, we have to endow the algebra with some *extra structure*, that is comultiplication Δ which makes it possible to define tensor product representations. Consider a map $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and define the tensor product representation by $\Psi = (\psi_1 \otimes \psi_2)\Delta$, where Ψ is linear, satisfy the homomorphism property and the representations $(V_1 \otimes V_2) \otimes V_3$ and $V_1 \otimes (V_2 \otimes V_3)$ are equal, therefore the set of representations becomes a ring. These requirements lead to the following conditions on Δ: 1) Δ is linear; 2) $\Delta(ab) = \Delta(a)\Delta(b)$ (algebra homomorphism); 3) $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$ (coassociativity). A map $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ with these properties is called a *comultiplication* (or *coproduct*). Associated with the multiplication on $\mathcal{A}$ there is a map η whose properties signaled the existence of a unit in $\mathcal{A}$. Analogously, a *counit* is a map $\varepsilon : \mathcal{A} \to \mathbb{K}$ such that $(1 \otimes \varepsilon)\Delta = (\varepsilon \otimes 1)\Delta = \mathrm{id}$. The set $(\mathcal{A}, \Delta, \varepsilon)$ is called a *coalgebra*, and the set $(\mathcal{A}, m, \Delta, \eta, \varepsilon)$ is called a *bialgebra*, if all the above properties are satisfied. A coalgebra is *cocommutative* if $\Delta = \tau \circ \Delta$. In the Heyneman-Sweedler notation [31] $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$ (usually dropping summation symbol). If $\Delta(c) = c \otimes c$ and $\varepsilon(c) = 1$, then the element $c \in \mathcal{A}$ is called *grouplike*. Let $(\mathcal{A}, m_{\mathcal{A}}, \eta_{\mathcal{A}})$ and $(\mathcal{B}, m_{\mathcal{B}}, \eta_{\mathcal{B}})$ be algebras, than the linear map $f : \mathcal{A} \to \mathcal{B}$ is an *algebra map*, if $f \circ m_{\mathcal{A}} = m_{\mathcal{B}} \circ f$ (or $f(a \cdot b) = f(a) \cdot f(b)$) and $f \circ \eta_{\mathcal{A}} = \eta_{\mathcal{B}}$ (or $f(1_{\mathcal{A}}) = 1_{\mathcal{B}}$), $a \in \mathcal{A}$, $b \in \mathcal{B}$. Let $(C, \Delta_C, \varepsilon_C)$ and $(\mathcal{D}, \Delta_{\mathcal{D}}, \varepsilon_{\mathcal{D}})$ be coalgebras, than the linear map $f : C \to \mathcal{D}$ is a *coalgebra map*, if $\Delta_D(f(c)) = f(c_{(1)}) \otimes f(c_{(2)})$ and $\varepsilon_D(f(c)) = \varepsilon_C(c)$, $c \in C$. A coalgebra map is isomorphism, if it is one-one and onto. Also $C \otimes \mathcal{D}$ is a coalgebra with $\Delta(c \otimes d) = (c_{(1)} \otimes d_{(1)}) \otimes (c_{(2)} \otimes d_{(2)})$ and $\varepsilon(c \otimes d) = \varepsilon_C(c)\varepsilon_{\mathcal{D}}(d)$. A *star multiplication* $\star$ is defined by $(f \star g)(c) = f(c_{(1)})\, g(c_{(2)}) = m(f \otimes g)\Delta$.

Let C^* be the space of linear maps $C \to \mathbb{K}$ and $\langle\ ,\ \rangle$ denotes the pairing. If we define a linear map $m^* : C^* \otimes C^* \to \mathbb{K}$ by $m^*(a^* \otimes b^*) = a^* b^*$, where $\langle a^* b^*, c \rangle = \langle a^* \otimes b^*, \Delta(c) \rangle = \langle a^*, c_{(1)} \rangle \langle b^*, c_{(2)} \rangle$, $a^*, b^* \in C$, $\eta^*(1) = \varepsilon$ and $\Delta(c) = c_{(1)} \otimes c_{(2)}$, then the triple (C^*, m^*, η^*) is the dual algebra of the coalgebra (C, Δ, ε).

An analog of inverse for groups, a map I from the group to itself such that $I(g)g = gI(g) = e$, where e is the unit element of the group, there is an extra structure called an antipode or coinverse. A *Hopf algebra* is a bialgebra $(\mathcal{A}, m, \eta, \Delta, \varepsilon)$ together with a map $S : \mathcal{A} \to \mathcal{A}$ called an *antipode*, with the property $m(S \otimes \mathrm{id})\Delta = m(\mathrm{id} \otimes S)\Delta = \eta \circ \varepsilon S$. As an example one can consider G as a compact topological group with the space of continuous functions on G denoted by $C(G)$, which is a Hopf algebra together with the maps $(f.h)(g) = f(g)h(g)$, $\Delta(f)(g_1 \otimes g_2) = f(g_1 g_2)$, $\eta(x) = x1$ where $1(g) = 1$ for all $g \epsilon G$, $\varepsilon(f) = f(e)$, where e is the unit element of G, $S(f)(g) = f(g^{-1})$, where $g_1, g_2, g \in G$, $x \in \mathbf{C}$ and $f, h \in C(G)$.

Suppose that $(\mathcal{H}, m, \Delta, \eta, \varepsilon, S)$ is a Hopf algebra and that $\mathcal{H}^*$ is its dual space, then using the structure maps of $\mathcal{H}$ one defines the structure maps $(m^*, \Delta^*, \eta^*, \varepsilon^* S^*)$ on $\mathcal{H}^*$ as follows: $\langle m^*(a^* \otimes b^*), a \rangle = \langle a^* \otimes b^*, \Delta(a) \rangle$, $\langle \Delta^*(a^*), a \otimes b \rangle = \langle a^*, ab \rangle$, $\langle \eta^*(\alpha), a \rangle = \alpha.\varepsilon(a)$, $\varepsilon^*(a^*) = \langle a^*, 1 \rangle$, $\langle S^*(a^*), a \rangle = \langle a^*, S(a) \rangle$, where $a^*, b^* \in \mathcal{H}^*$ and $a, b \in \mathcal{H}$. The dual contraction between $\mathcal{H}^*$ and $\mathcal{H}$ satisfies $\langle a^* \otimes b^*, a \otimes b \rangle = \langle a^*, a \rangle \langle b^*, b \rangle$. It is easy, using the fact that $\mathcal{H}$ is a Hopf algebra, to verify that $\mathcal{H}^*$ is also a Hopf algebra. Note that if $\mathcal{H}$ is commutative then $\mathcal{H}^*$ is cocommutative, and if $\mathcal{H}$ is cocommutative then $\mathcal{H}^*$ is commutative. This is so because the multiplication on $\mathcal{H}$ induces the comultiplication on $\mathcal{H}^*$, and the comultiplication on $\mathcal{H}$ induces the multiplication on $\mathcal{H}^*$.

CODERIVATIONS AND COALGEBRAS

A linear endomorphism $\delta : \mathcal{A} \to \mathcal{A}$ of an algebra $(\mathcal{A}, m, \eta)$ is a *derivation* of $\mathcal{A}$, if $\delta(a \cdot b) = a \cdot \delta(b) + \delta(a) \cdot b$ or

$$\delta \circ m = m \circ (\mathrm{id} \otimes \delta) + m \circ (\delta \otimes \mathrm{id}). \tag{1}$$

Dualizing this definition we obtain the notion of coderivation. A linear endomorphism $\delta_\Delta : C \to C$ of a coalgebra (C, Δ, ε) is a *coderivation* of C, if

$$\Delta \circ \delta_\Delta = (\mathrm{id} \otimes \delta_\Delta) \circ \Delta + (\delta_\Delta \otimes \mathrm{id}) \circ \Delta, \tag{2}$$

which can be called a *co-Leibniz rule*. The dual $\delta_\Delta^* : C^* \to C^*$ of δ_Δ is determined by the pairing

$$\left\langle \delta_\Delta^* c^*, c \right\rangle = \langle c^*, \delta_\Delta c \rangle. \tag{3}$$

If δ_Δ^* is a derivation of C^*, then δ_Δ is a coderivation of C, which is seen from the following identities

$$\left\langle \delta_\Delta^*(a^* \cdot b^*), c \right\rangle = \langle a^* \cdot b^*, \delta_\Delta c \rangle = \left\langle a^*, (\delta_\Delta c)_{(1)} \right\rangle \left\langle b^*, (\delta_\Delta c)_{(2)} \right\rangle = \langle a^* \otimes b^*, \Delta(\delta_\Delta c) \rangle \tag{4}$$

and

$$\begin{aligned} \langle a^* \otimes b^*, (\mathrm{id} \otimes \delta_\Delta) \circ \Delta(c) + (\delta_\Delta \otimes \mathrm{id}) \circ \Delta(c) \rangle &= \langle a^*, c_{(1)} \rangle \left\langle b^*, (\delta_\Delta c)_{(2)} \right\rangle + \left\langle a^*, (\delta_\Delta c)_{(1)} \right\rangle \langle b^*, c_{(2)} \rangle = \\ \langle a^*, c_{(1)} \rangle \left\langle \delta_\Delta^* b^*, c_{(2)} \right\rangle + \left\langle \delta_\Delta^* a^*, c_{(1)} \right\rangle \langle b^*, c_{(2)} \rangle &= \left\langle \left(\delta_\Delta^* a^*\right) \cdot b^* + a^* \left(\delta_\Delta^* b^*\right), c \right\rangle, \end{aligned} \tag{5}$$

where $a^*, b^* \in C^*$, $c \in C$. Thus $\delta_\Delta \longmapsto \delta_\Delta^*$ presents a one-one correspondence between the coalgebra derivation of C and the *continuous* derivations of C^*. From coassociativity $(\Delta\otimes\mathrm{id})\Delta = (\mathrm{id}\otimes\Delta)\Delta$ and the identity

$$\begin{aligned}\langle (a^*\cdot b^*)\cdot c^*, c\rangle &= \langle (a^*\cdot b^*)\otimes c^*, \Delta c\rangle = \langle (a^*\otimes b^*)\otimes c^*, (\Delta\otimes\mathrm{id})\circ\Delta c\rangle = \\ \langle a^*\left(\otimes b^*\otimes c^*\right), (\mathrm{id}\otimes\Delta)\circ\Delta c\rangle &= \langle a^*\otimes (b^*\cdot c^*), \mathrm{id}\otimes c\rangle = \langle a^*\cdot (b^*\cdot c^*), c\rangle\end{aligned}$$

it follows the associativity of the product in C^*, and conversely, associativity of the dual algebra (C^*, m^*, η^*) implies the coassociativity of the coalgebra (C, Δ, ε).

A natural generalization of the derivation (1) is *$\rho\sigma$-derivation* [28] defined by $\delta^{\rho\sigma}(a\cdot b) = \rho(a)\cdot\delta(b) + \delta(a)\cdot\sigma(b)$ or

$$\delta^{\rho\sigma}\circ m = m\circ(\rho\otimes\delta^{\rho\sigma}) + m\circ(\delta^{\rho\sigma}\otimes\sigma), \tag{6}$$

where ρ, σ are algebra homomorphisms $\mathcal{A} \to \mathbb{K}$. By analogy dualizing (6) we define a *$\mu\nu$-coderivation* as a linear endomorphism $\delta_\Delta^{\mu\nu} : C \to C$ of the coalgebra (C, Δ, ε) by

$$\Delta\circ\delta_\Delta^{\mu\nu} = \left(\mu\otimes\delta_\Delta^{\mu\nu}\right)\circ\Delta + \left(\delta_\Delta^{\mu\nu}\otimes\nu\right)\circ\Delta, \tag{7}$$

where μ, ν are coalgebra homomorphisms $C \to \mathbb{K}$. Similarly (4)–(5) we derive $\rho = \mu^*$ and $\eta = \nu^*$.

If $(C, \Delta_C, \varepsilon_C)$ and $(\mathcal{D}, \Delta_\mathcal{D}, \varepsilon_\mathcal{D})$ are two coalgebras, then we define a covariant map $\delta_\Delta^{C\mathcal{D}} : C \to \mathcal{D}$ by

$$\Delta_\mathcal{D}\circ\delta_\Delta^{C\mathcal{D}} = \left(\mathrm{id}\otimes\delta_\Delta^{C\mathcal{D}}\right)\circ\Delta_C + \left(\delta_\Delta^{C\mathcal{D}}\otimes\mathrm{id}\right)\circ\Delta_C, \tag{8}$$

which can be used to study various properties of coalgebras. Note that the structure of the right-hand side of (8) requires $\Delta_\mathcal{D} : \mathcal{D} \to (C\otimes\mathcal{D})\oplus(\mathcal{D}\otimes C) \subset (\mathcal{D}\otimes\mathcal{D})$.

CODERIVATIONS AND COMODULES

Let (C, Δ, ε) be a coalgebra over a field $\mathbb{K}$ of characteristic zero. A *left C-comodule* $(\mathcal{M}_L, \Delta_L)$ is a linear space over $\mathbb{K}$ with a linear map $\Delta_L : \mathcal{M}_L \to C\otimes\mathcal{M}_L$ such that

$$(\mathrm{id}\otimes\Delta_L)\circ\Delta_L = (\Delta\otimes\mathrm{id})\circ\Delta_L. \tag{9}$$

A *right C-comodule* $(\mathcal{M}_R, \Delta_R)$ is a linear space over $\mathbb{K}$ with a linear map $\Delta_R : \mathcal{M}_R \to \mathcal{M}_R\otimes C$ such that

$$(\Delta_R\otimes\mathrm{id})\circ\Delta_R = (\mathrm{id}\otimes\Delta)\circ\Delta_R. \tag{10}$$

A *C-bicomodule* $(\mathcal{M}, \Delta_L, \Delta_R)$ is a left and right C-comodule. Notice that C itself is a left C-comodule with $\Delta_L = \Delta$ and a right C-comodule with $\Delta_R = \Delta$, or a C-bicomodule with $\Delta_L = \Delta$, $\Delta_R = \Delta$.

Let us consider covariant maps between comodules. In case $(\mathcal{M}_L, \Delta_L)$ and $\left(\mathcal{M}'_L, \Delta'_L\right)$ are two left C-comodules, a comodule map $\delta_\Delta^L : \mathcal{M}_L \to \mathcal{M}'_L$ is called *left-covariant*, if

$$\Delta'_L\circ\delta_\Delta^L = \left(\mathrm{id}\otimes\delta_\Delta^L\right)\circ\Delta_L. \tag{11}$$

For $(\mathcal{M}_R, \Delta_R)$ and $\left(\mathcal{M}'_R, \Delta'_R\right)$ are two right C-comodules, a comodule map $\delta_\Delta^R : \mathcal{M}_R \to \mathcal{M}'_R$ is called *right-covariant*, if

$$\Delta'_R\circ\delta_\Delta^R = \left(\delta_\Delta^R\otimes\mathrm{id}\right)\circ\Delta_R. \tag{12}$$

In case $(\mathcal{M}, \Delta_L, \Delta_R)$ and $\left(\mathcal{M}', \Delta'_L, \Delta'_R\right)$ are two C-bicomodules, a map $\delta_\Delta^{LR} : \mathcal{M} \to \mathcal{M}'$ is called *bicovariant*, if

$$(\Delta'_L + \Delta'_R)\circ\delta_\Delta^{LR} = \left(\mathrm{id}\otimes\delta_\Delta^{LR}\right)\circ\Delta_L + \left(\delta_\Delta^{LR}\otimes\mathrm{id}\right)\circ\Delta_R. \tag{13}$$

Some generalization of (13) can be made if consider a map to the coalgebra, indeed, if $(C', \Delta', \varepsilon')$ is a coalgebra, a linear map $\delta_\Delta^{bicov} : \mathcal{M} \to C'$ is called *bicovariant*, if

$$\Delta'\circ\delta_\Delta^{bicov} = \left(\mathrm{id}\otimes\delta_\Delta^{bicov}\right)\circ\Delta_L + \left(\delta_\Delta^{bicov}\otimes\mathrm{id}\right)\circ\Delta_R. \tag{14}$$

Setting $\Delta' = \Delta'_L + \Delta'_R$ the conditions of left- (11) and right-covariance (12) for δ_Δ^{bicov} implies (14). Additionally, the coassociativity of Δ' leads to the *compatibility condition* of Δ_L and Δ_R

$$(\mathrm{id}\otimes\Delta_R)\circ\Delta_L = (\Delta_L\otimes\mathrm{id})\circ\Delta_R. \tag{15}$$

If $(\mathcal{M}, \Delta_L, \Delta_R)$ is a C-bicomodule with the compatibility condition, a linear map $\delta_\Delta^{bicovder} : \mathcal{M} \to C$ satisfying (14) with $\Delta' = \Delta$

$$\Delta \circ \delta_\Delta^{bicovder} = \left(\mathrm{id} \otimes \delta_\Delta^{bicovder}\right) \circ \Delta_L + \left(\delta_\Delta^{bicovder} \otimes \mathrm{id}\right) \circ \Delta_R. \tag{16}$$

is called a *bicomodule coderivation*. Thus, a *first order codifferential calculus* on a coalgebra C consists of a C-bicomodule $(\mathcal{M}, \Delta_L, \Delta_R)$ and a bicomodule coderivation $\delta_\Delta^{bicovder}$ satisfying (16). Note that covariant maps subject to one of the above conditions form a linear space over $\mathbb{K}$.

In a coordinate language [32, 33] a left comodule with a basis $\{e_k\}$ is defined by

$$\triangle_L(e_k) = L_k^i \otimes e_i, \tag{17}$$

where C-valued matrix elements L_k^i have to satisfy the following relations (see e.g. [30])

$$\triangle(L_k^i) = L_k^m \otimes L_m^i, \quad \varepsilon(L_k^i) = \delta_k^i.$$

A right comodule structure $\triangle_R$ can be encoded by

$$\triangle_R(e_k) = e_i \otimes R_k^i, \tag{18}$$

with

$$\triangle(R_k^i) = R_m^i \otimes R_k^m, \quad \varepsilon(R_k^i) = \delta_k^i.$$

The compatibility condition (15) reads as

$$L_k^i \otimes R_i^m = L_j^m \otimes R_k^j. \tag{19}$$

Writing a bicomodule coderivation as $\delta_\Delta^{bicovder} = e^k \otimes \delta_k$, the relation (16) becomes a restriction on the coefficients δ_k as

$$\triangle(\delta_k) = L_k^i \otimes \delta_i + \delta_i \otimes R_k^i. \tag{20}$$

In analogy with the algebra case, the elements of the bicomodule can be called 1-*form cofields* and the (left) dual can be called (left) *vector cofields* [32]. With any vector cofield $X \doteq X^i \otimes e_i$ one can associate an endomorphism $X^\delta \in \mathrm{End}\, C$

$$X^\delta = \delta \circ X = X^i \otimes \delta_i. \tag{21}$$

which is the dual counterpart of the Cartan formula (see e.g.[34]).

A given a group G with product $*$ determines a coproduct $\Delta : \mathcal{A} \mapsto \mathcal{A} \otimes \mathcal{A}$ on the algebra $\mathcal{A}$ of functions on the group

$$\Delta(f)(g, g') = f(g * g'), \tag{22}$$

where $f \in \mathcal{A}$, $g, g' \in G$ which is an algebra homomorphism. This coproduct encodes all the information about the underlying group structure on the level of the algebra $\mathcal{A}$. Moreover, this 'reformulation' of the group product allows deformations which, in particular, is the origin of 'quantum symmetries'.

Let (C, Δ_L) be a left $\mathcal{A}$-comodule. This means that C is a left $\mathcal{A}$-module and $\Delta_L : C \to \mathcal{A} \otimes C$ is a linear map such that

$$\Delta_L(f \cdot \alpha) = \Delta(f) \circ \Delta_L(\alpha)$$

or all $f \in \mathcal{A}$ and $\alpha \in C$. Let (C, Δ_R) be a right $\mathcal{A}$-comodule. This means that C is a right $\mathcal{A}$-module and $\Delta_R : C \to C \otimes \mathcal{A}$ is a linear map such that

$$\Delta_R(\alpha \cdot f) = \Delta_R(\alpha) \circ \Delta(f)$$

for all $f \in \mathcal{A}$ and $\alpha \in C$. Let (C, Δ_L, Δ_R) be an $\mathcal{A}$-bicomodule. This is a left and right $\mathcal{A}$-comodule such that

$$\begin{aligned} \Delta_L(f \cdot \alpha \cdot f') &= \Delta(f) \circ \Delta_L(\alpha) \circ \Delta(f') \\ \Delta_R(f \cdot \alpha \cdot f') &= \Delta(f) \circ \Delta_R(\alpha) \circ \Delta(f') \end{aligned}$$

for all $f, f' \in \mathcal{A}$ and $\alpha \in C$. Special examples of (bi)comodules are determined by first order bicovariant differential calculi on Hopf algebras [20] and in particular on finite groups [35]. Indeed, the derivation $\mathrm{d} : \Omega^r \to \Omega^{r+1}$ of a left- (or right-) covariant differential calculus is a left- (right-) covariant map [20]. For a bicovariant differential calculus, d then satisfies (11) and (12), which imply (14) with $\Delta' = \Delta_L + \Delta_R$ on the space Ω^1 of 1-forms (see also [36]). In the following we explore the meaning of other covariant maps associated with spaces determined by differential calculi on Hopf algebras and in particular on finite groups [35].

COMODULES AND COVARIANT MAPS ASSOCIATED WITH FINITE GROUPS

Let $\mathcal{A}$ be the set of $\mathbb{C}$-valued functions on a finite group G. With each element $g \in G$ we associate a function $e^g \in \mathcal{A}$ via $e^g(g') = \delta^g_{g'}$. Then

$$e^g\, e^{g'} = \delta^{g,g'}\, e^g\,, \qquad \sum_{g\in G} e^g = \mathbf{1}$$

where $\mathbf{1}$ is the unit in $\mathcal{A}$. Every function f on G can be written as

$$f = \sum_{g\in G} f_g\, e^g$$

with $f_g \in \mathbb{C}$. The group structure on G induces a coproduct $\Delta : \mathcal{A} \to \mathcal{A}\otimes\mathcal{A}$ via (22). In particular, we obtain

$$\Delta(e^g) = \sum_{h\in G} e^h \otimes e^{h^{-1}g}\,.$$

Let us consider a self-map $\delta_\Delta : \mathcal{A} \to \mathcal{A}$ with the action on functions e^g determined by

$$\delta_\Delta(e^g) = \sum_{h\in G} F^g_h\, e^h. \tag{23}$$

This map is covariant in the sense of (16). Inserting (23) gives

$$F^g_{hh'} = F^{h^{-1}g}_{h'} + F^{gh'^{-1}}_{h}, \qquad \forall h, h' \in G\,. \tag{24}$$

The general solution has the form

$$F^g_h = \Phi(h^{-1}g) - \Phi(gh^{-1}) \tag{25}$$

with an arbitrary function $\Phi : G \to \mathbb{C}$. Moreover, if G is a finite group (that is our case), then every solution of (24) has the form (25). To prove this fact we change variables and rewrite (24) in the form

$$F^g_h = F^g_{hh'} - F^{h^{-1}gh'}_{h'}.$$

Notice that left-hand side does not depend from h' and then take finite sum over all $h' \in G$, which gives

$$F^g_h = \left|G^{-1}\right| \sum_{h'\in G} \left(F^g_{hh'} - F^{h^{-1}gh'}_{h'}\right).$$

After summing the right-hand side and introducing a new function of one variable

$$\Phi(g) = -\,|G|^{-1} \sum_{h\in G} F^{gh}_h$$

we obtain (25). So that the solution for the coderivation map δ_Δ is

$$\delta_\Delta(e^g) = \sum_{h\in G} [\Phi(h^{-1}g) - \Phi(gh^{-1})]\, e^h\,. \tag{26}$$

Since $\Delta(\mathbf{1}) = \mathbf{1}\otimes\mathbf{1}$, we have in particular

$$\Delta\circ\delta_\Delta(\mathbf{1}) = \mathbf{1}\otimes\delta_\Delta(\mathbf{1}) + \delta_\Delta(\mathbf{1})\otimes\mathbf{1}$$

and

$$\delta_\Delta(\mathbf{1}) = \delta_\Delta(\sum_{g\in G} e^g) = \sum_{g\in G} \delta_\Delta(e^g)\,.$$

Using (26), we obtain

$$\delta_\Delta(\mathbf{1}) = 0.$$

Moreover, directly from (26) it follows that for commutative finite groups, where $\Phi(h^{-1}g) = \Phi(gh^{-1})$, the only solution of (24) is trivial one $F^g_h = 0$ and $\delta_\Delta = 0$. Thus, the map δ_Δ "measure" noncommutativity of a finite group G.

DIFFERENTIAL CALCULI ON FINITE GROUPS

A *differential calculus* on $\mathcal{A}$ is an extension of $\mathcal{A}$ to a differential algebra (Ω, d). Here $\Omega = \bigoplus_{r=0}^{\infty} \Omega^r$ is a graded associative algebra where $\Omega^0 = \mathcal{A}$. Ω^{r+1} is generated as an $\mathcal{A}$-bimodule via the action of a linear operator $\mathrm{d} : \Omega^r \to \Omega^{r+1}$ satisfying $\mathrm{d}^2 = 0$, $\mathrm{d}\mathbf{1} = 0$, and the graded Leibniz rule $\mathrm{d}(\varphi\varphi') = (\mathrm{d}\varphi)\,\varphi' + (-1)^r \varphi\,\mathrm{d}\varphi'$ where $\varphi \in \Omega^r$. We introduce the special 1-forms (see e.g. [35])

$$e^{g,g'} = e^g\,\mathrm{d}e^{g'}, \qquad (g \neq g'), \qquad e^{g,g} = 0 \tag{27}$$

and the $(r-1)$-forms $e^{g_1,\ldots,g_r} = e^{g_1,g_2} \otimes_{\mathcal{A}} e^{g_2,g_3} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} e^{g_{r-1},g_r}$. They satisfy $e^{g_1,\ldots,g_r} \otimes_{\mathcal{A}} e^{h_1,\ldots,h_s} = \delta^{g_r,h_1}\, e^{g_1,\ldots,g_r,h_2,\ldots,h_s}$. The differential operator d acts on them as follows,

$$\mathrm{d}e^{g_1,\ldots,g_r} = \sum_{h\in G} [e^{h,g_1,\ldots,g_r} - e^{g_1,h,g_2,\ldots,g_r} + e^{g_1,g_2,h,g_3,\ldots,g_r} - \ldots + (-1)^r\, e^{g_1,\ldots,g_r,h}]. \tag{28}$$

If no further relations are imposed, one is dealing with the 'universal differential calculus' $(\tilde{\Omega}, \tilde{\mathrm{d}})$. The $e^{g_1,\ldots,g_r}$ with $g_i \neq g_{i+1}$ $(i = 1, \ldots, r-1)$ then constitute a basis over $\mathbb{C}$ of $\tilde{\Omega}^{r-1}$ for $r > 1$. Every other differential calculus on G is obtained from $\tilde{\Omega}$ as the quotient with respect to some two-sided differential ideal. Up to first order, i.e., the level of 1-forms, every differential calculus on G is obtained by setting some of the $e^{g,g'}$ to zero, which induces relations for forms of higher grade. In addition, or alternatively, one may also factor out ideals generated by forms of higher grade. Every first order differential calculus on G can be described by a (di)graph the vertices of which are the elements of G and there is an arrow pointing from a vertex g to a vertex g' iff $e^{g,g'} \neq 0$.

A differential calculus on G (or, more generally, any Hopf algebra $\mathcal{A}$) is called *left-covariant* [20] if there is a linear map $\Delta_L : \Omega^1 \to \mathcal{A} \otimes \Omega^1$ such that

$$\Delta_L(f\,\varphi\,f') = \Delta(f)\,\Delta_L(\varphi)\,\Delta(f') \qquad \forall f, f' \in \mathcal{A},\ \varphi \in \Omega^1$$

and

$$\Delta_L \circ \mathrm{d} = (\mathrm{id} \otimes \mathrm{d}) \circ \Delta\ .$$

As a consequence of (27) and (28), we obtain

$$\Delta_L(e^{g,g'}) = \sum_{h\in G} e^{h^{-1}} \otimes e^{hg,hg'}\ . \tag{29}$$

Hence, in order to find the left-covariant differential calculi on G, we have to determine the orbits of all elements of $(G \times G)'$ where the prime indicates omission of the diagonal (i.e., $(G \times G)' = (G \times G) \setminus \{(g,g) \mid g \in G\}$) with respect to the left action $(g, g') \mapsto (hg, hg')$. In the graph picture, left-covariant first order differential calculi are obtained from the universal one (which is left-covariant) by deleting corresponding orbits of arrows.

A differential calculus on G is called *right-covariant* [20] if there is a linear map $\Delta_R : \Omega^1 \to \Omega^1 \otimes \mathcal{A}$ such that

$$\Delta_R(f\,\varphi\,f') = \Delta(f)\,\Delta_R(\varphi)\,\Delta(f'), \quad \Delta_R \circ \mathrm{d} = (\mathrm{d} \otimes \mathrm{id}) \circ \Delta\ .$$

This implies

$$\Delta_R(e^{g,g'}) = \sum_{h\in G} e^{gh,g'h} \otimes e^{h^{-1}}\ . \tag{30}$$

A differential calculus is called *bicovariant* if it is left- and right-covariant.

CODIFFERENTIAL CALCULI ON FINITE GROUPS

A *first order codifferential calculus* on a finite group G consists of a space of various forms having structure of bicomodule and a bicomodule coderivation δ_Δ satisfying

$$\Delta \circ \delta_\Delta = (\mathrm{id} \otimes \delta_\Delta) \circ \Delta_L + (\delta_\Delta \otimes \mathrm{id}) \circ \Delta_R. \tag{31}$$

Let us consider concrete finite group examples of solution of (31).

Bicovariant maps $\Omega^1 \to \mathcal{A}$. In order to determine the corresponding bicovariant maps $\delta_\Delta^{1|0} : \Omega^1 \to \mathcal{A}$, we have to evaluate

$$\Delta \circ \delta_\Delta^{1|0}(e^{g,g'}) = (\mathrm{id} \otimes \delta_\Delta^{1|0}) \circ \Delta_L(e^{g,g'}) + (\delta_\Delta^{1|0} \otimes \mathrm{id}) \circ \Delta_R(e^{g,g'}). \tag{32}$$

Writing

$$\delta_\Delta^{1|0}(e^{g,g'}) = \sum_{h\in G} F_h^{g,g'}\, e^h$$

with $F_h^{g,g'} \in \mathbb{C}$, the bicovariance condition results in the linear system

$$F_{hh'}^{g,g'} = F_{h'}^{h^{-1}g,h^{-1}g'} + F_h^{gh'^{-1}.g'h'^{-1}} \qquad \forall h,h' \in G$$

where $g \neq g'$ for the universal differential calculus. The pairs (g,g') have to be further restricted in case of smaller bicovariant differential calculi. Using the ansatz (26), we obtain the general solution of the above system as

$$F_h^{g,g'} = \Phi(h^{-1}g, h^{-1}g') - \Phi(gh^{-1}, g'h^{-1}) \tag{33}$$

with an arbitrary function $\Phi : G \times G \to \mathbb{C}$.

Bicovariant maps $\underbrace{\Omega^1 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \Omega^1}_{r} \to \mathcal{A}$. Here we consider bicovariant maps $\delta_\Delta^{r|0} : \underbrace{\Omega^1 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \Omega^1}_{r} \to \mathcal{A}$ where Ω^1 is the space of 1-forms of a bicovariant differential calculus on a finite group. First we need the left and right coactions on the r-fold tensor product $\underbrace{\Omega^1 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \Omega^1}_{r}$. These can be obtained from the coactions on Ω^1 according to formulas (3.3) and (3.4) in [20]

$$\Delta_L(e^{g_1,\ldots,g_r}) = \sum_{h\in G} e^{h^{-1}} \otimes e^{hg_1,\ldots,hg_r} \tag{34}$$

$$\Delta_R(e^{g_1,\ldots,g_r}) = \sum_{h\in G} e^{g_1h,\ldots,g_rh} \otimes e^{h^{-1}} \ . \tag{35}$$

Let us expand

$$\delta_\Delta^{r|0}(e^{g_1,\ldots,g_r}) = \sum_{h\in G} F_h^{g_1,\ldots,g_r}\, e^h$$

with coefficients $F_h^{g_1,\ldots,g_r} \in \mathbb{C}$. Evaluation of the bicovariance condition (31) then yields

$$F_{hh'}^{g_1,\ldots,g_r} = F_{h'}^{h^{-1}g_1,\ldots,h^{-1}g_r} + F_h^{g_1h'^{-1},\ldots,g_rh'^{-1}}, \qquad \forall h,h' \in G.$$

By full analogy with (26) and (33) we find the general solution

$$F_h^{g_1,\ldots,g_r} = \Phi(h^{-1}g_1,\ldots,h^{-1}g_r) - \Phi(g_1h^{-1},\ldots,g_rh^{-1})$$

with an arbitrary function $\Phi : \underbrace{G \times \cdots \times G}_{r} \to \mathbb{C}$. Again for commutative and finite G we have the only solution $F_h^{g_1,\ldots,g_r} = 0$ and so $\delta_\Delta^{r|0} = 0$.

Covariant maps between tensor products of 1-forms. Let Ω^1 be the space of 1-forms, and we consider a linear map

$$\delta_\Delta^{r|s} : \underbrace{\Omega^1 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \Omega^1}_{r \text{ times}} \to \underbrace{\Omega^1 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \Omega^1}_{s \text{ times}} \ .$$

We expand

$$\delta_\Delta^{r|s}(e^{g_1,\ldots,g_r}) = \sum_{h_1,\ldots,h_s\in G} F_{h_1,\ldots,h_s}^{g_1,\ldots,g_r}\, e^{h_1,\ldots,h_s} \ .$$

First we explore the bicovariance condition (14) with $\Delta' = \Delta_L + \Delta_R$ where the left and right coactions are given by (34)–(35). The calculation shows that in this case the bicovariance condition splits into the separate conditions of left and right covariance, i.e. (11) and (12) with $\Delta'_L = \Delta_L$ and $\Delta'_R = \Delta_R$. The left and right covariance condition take the form

$$a_{h_1,\ldots,h_s}^{g_1,\ldots,g_r} = a_{hh_1,\ldots,hh_s}^{hg_1,\ldots,hg_r}, \qquad a_{h_1,\ldots,h_s}^{g_1,\ldots,g_r} = a_{h_1h,\ldots,h_sh}^{g_1h,\ldots,g_rh}, \qquad \forall h \in G,$$

respectively. This means that the coefficients $a_{h_1,\ldots,h_s}^{g_1,\ldots,g_r}$ are constant on left and right orbits.

Another possibility is the covariance condition (8) with $\Delta_{\mathcal{D}} = \Delta_{\mathcal{C}} = \Delta = \Delta_L + \Delta_R$.

Covariant maps $\mathcal{A} \to \Omega^1 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \Omega^1$. Let us consider a linear map

$$\delta_\Delta^{0|s} : \mathcal{A} \to \underbrace{\Omega^1 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \Omega^1}_{s}.$$

with expansion

$$\delta_\Delta^{0|s}(e^g) = \sum_{h_1,\ldots,h_s\in G} F_{h_1,\ldots,h_s}^{g}\, e^{h_1,\ldots,h_s} \ .$$

Now we explore the covariance condition (14) with $\Delta' = \Delta_L + \Delta_R$ where the left and right coactions are given by (11) and (12). The calculation shows that in this case the bicovariance condition splits into the separate conditions of left and right covariance. The left covariance condition takes the form

$$F^g_{h_1,\dots,h_s} = F^{hg}_{hh_1,\dots,hh_s}, \qquad \forall h \in G,$$

whereas the right covariance condition reads

$$F^g_{h_1,\dots,h_s} = F^{gh}_{h_1h,\dots,h_sh} \qquad \forall h \in G\ .$$

This means that the coefficients $F^g_{h_1,\dots,h_s}$ are constant on left and right orbits.

Bicovariant maps of the space of 2-forms. There is a bimodule isomorphism $\sigma : \Omega^1 \otimes_{\mathcal{A}} \Omega^1 \to \Omega^1 \otimes_{\mathcal{A}} \Omega^1$ given by

$$\sigma(e^{g_1,g_2} \otimes_{\mathcal{A}} e^{g_2,g_3}) = e^{g_1,g_3g_2^{-1}g_1} \otimes_{\mathcal{A}} e^{g_3g_2^{-1}g_1,g_3}$$

(see [20]). If $\delta^{2|0}_\Delta$ is a bicovariant map $\Omega^1 \otimes_{\mathcal{A}} \Omega^1 \to \mathcal{A}$, then also $\delta'_\Delta = \delta^{2|0}_\Delta \circ \sigma$. Indeed, from [20] we have

$$\Delta_L \circ \sigma = (\mathrm{id} \otimes \sigma) \circ \Delta_L, \qquad \Delta_R \circ \sigma = (\sigma \otimes \mathrm{id}) \circ \Delta_R\ .$$

If $\delta^{2|0}_\Delta$ is bicovariant and satisfies (14), we can directly find

$$\Delta \circ \delta^{2|0}_\Delta \circ \sigma = (\mathrm{id} \otimes \delta^{2|0}_\Delta \circ \sigma) \circ \Delta_L + (\delta \circ \sigma^{2|0}_\Delta \otimes \mathrm{id}) \circ \Delta_R$$

so that $\delta'_\Delta = \delta^{2|0}_\Delta \circ \sigma$ is also bicovariant. In particular, if $\delta^{2|0}_\Delta$ is a bicovariant map, then also $\delta''_\Delta = \delta^{2|0}_\Delta \circ (\mathrm{id} - \sigma)$. This induces a bicovariant map on the space Ω^2_{Wor} of the Woronowicz 2-forms [20].

Bicomplexes. If $\delta^2_\Delta = 0$, then $(\mathcal{A}, \Delta, C, \Delta_L, \Delta_R, \delta_\Delta)$ is a *complex* . We should check that the last condition is compatible with (14). Indeed, we have $0 = \Delta \circ \delta^2_\Delta = (\mathrm{id} \otimes \delta + \delta \otimes \mathrm{id}) \circ \Delta \circ \delta_\Delta$ and the last expression indeed vanishes using (14) and $\delta^2_\Delta = 0$. The set $(C, \Delta, \delta_\Delta, \mathrm{d})$ is a *bicomplex* [37] if $(C, \Delta, \delta_\Delta)$ and (C, Δ, d) are both complexes and $\delta_\Delta \mathrm{d} = -\mathrm{d}\delta_\Delta$. The latter condition is indeed compatible with (14) since also the right side of $\Delta \circ (\mathrm{d}\delta_\Delta + \delta_\Delta \mathrm{d}) = (\Delta \circ \mathrm{d}) \circ \delta_\Delta + (\Delta \circ \delta_\Delta) \circ \mathrm{d}$ vanishes as a consequence of these conditions. A bicomplex generalizes the concept of a bicovariant differential calculus.

CONCLUSIONS

Thus, we have studied bicovariant differential and codifferential calculi in general using the language of Hopf algebras and in application to finite groups, which provides a bridge between noncommutative geometry and various treatments of field theories on discrete spaces (like lattice gauge theory). In discrete (field) theories, discrete groups may appear as gauge groups, as isometry groups, and as structures underlying discrete space-time models. One may view a field theory on a discrete set as an approximation of a continuum theory, e.g., for the purpose of numerical simulations, and in this context the idea that a discrete space-time could actually be more fundamental than the continuum.

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КОПРОИЗВОДНЫЕ И КОДИФФЕРЕНЦИАЛЬНЫЕ ИСЧИСЛЕНИЯ НА КОНЕЧНЫХ ГРУППАХ

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Рассмотрены различные аспекты некоммутативной геометрии, являющейся в последнее время важным инструментом для построения теоретических моделей физики элементарных частиц в некоммутативном пространстве-времени. Указан путь последовательного построения кодифференциального исчисления, дуального аналога дифференциального исчисления, представляющего собой основу любой модели частиц. Введены в рассмотрение дуальный аналог производной, копроизводная, и ковариантные отображения, представлены соответствующие уравнения согласованности для коалгебр и комодулей. Явно исследован конкретный пример для конечных групп, который важен для моделей частиц в дискретном пространстве-времени. Построено кодифференциальное исчисление, при этом условия ковариантности отображений приводят к специальной системе функциональных уравнений, которые решаются в различных конкретных случаях. Отмечается, что дальнейшие исследования могут быть связаны с построением соответствующих комплексов и бикомплексов.

КЛЮЧЕВЫЕ СЛОВА: некоммутативная геометрия, кодифференциальное исчисление, конечная группа, комплекс, бикомплекс, коалгебра, комодуль