УДК 539.12

REGULAR R-MATRIX AND WEAK HOPF ALGEBRAS

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Weak Hopf algebras as generalizations of Hopf algebras [1] were introduced in [2], where its characterizations and applications were also studied. A k -bialgebra $H=(H,\mu,\eta,\Delta,\varepsilon)$ with multiplication μ, unity η, counity ε, comultiplication Δ, is called a *weak Hopf algebra* if there exists $T\in \mathrm{Hom}_k(H,H)$ such that

$$id*T*id=id\ ,\ T*id*T=T$$

where T is called a *weak antipode* of H . One of its aims is to construct some singular solutions of the quantum Yang-Baxter equation (QYBE) [2].

We study here generalization of Hopf algebra $sl_q(2)$ by weakening the invertibility of the generator K , i.e. exchanging its invertibility $KK^{-1}=1$ to the regularity $K\overline{K}K=K$ .Here we investigate a weak Hopf algebra $wsl_q(2)$ and a J -weak Hopf algebra $vsl_q(2)$ as generalizations of $sl_q(2)$ and non-trivial examples of weak Hopf algebras [2]. A quasi-braided weak Hopf algebra $\overline{U}_q^w$ from $wsl_q(2)$ is constructed whose quasi- R -matrix is regular [3].

Let $q\in C$ and $q\neq\pm1\ ,\ 0$. The quantum enveloping algebra $U_q=U_q\left(sl_q(2)\right)$ (see [6]) is generated by four variables(Chevalley generators) E , F , K , K^{-1} with the relations

$$K^{-1}K=KK^{-1}=1\ ,$$
$$KEK^{-1}=q^2E\ ,\ KFK^{-1}=q^{-2}F\ ,$$
$$EF-FE=\frac{K-K^{-1}}{q-q^{-1}}.$$

Now we try to weaken the invertibility of K to regularity, as usually in semigroup theory [4] (see also [5] for higher regularity). It can be done in two different ways.

Define $U_q^w=wsl_q(2)$, which is called a *weak quantum algebra*, as the algebra generated by the four variables E_w , F_w , K_w , $\overline{K}_w$ with the relations:

$$K_w\overline{K}_w=\overline{K}_wK_w,\quad K_w\overline{K}_wK_w=K_w,\quad \overline{K}_wK_w\overline{K}_w=\overline{K}_w,$$
$$K_wE_w=q^2E_wK_w,\quad \overline{K}_wE_w=q^{-2}E_w\overline{K}_w,$$
$$K_wF_w=q^{-2}F_wK_w,\quad \overline{K}_wF_w=q^2F_w\overline{K}_w,$$
$$E_wF_w-F_wE_w=\frac{K_w-\overline{K}_w}{q-q^{-1}}.$$

Define $U_q^v=vsl_q(2)$, which is called a J -*weak quantum algebra*, as the algebra generated by the four variables E_v , F_w , K_v , $\overline{K}_v$ with the relations ($J_v=K_v\overline{K}_v$):

$$K_v\overline{K}_v=\overline{K}_vK_v,\quad K_v\overline{K}_vK_v=K_v,\quad \overline{K}_vK_v\overline{K}_v=\overline{K}_v,$$
$$K_vE_v\overline{K}_v=q^2E_v,\quad K_vF_v\overline{K}_v=q^{-2}F_v,\quad E_vJ_vF_v-F_vJ_vE_v=\frac{K_v-\overline{K}_v}{q-q^{-1}}.$$

Let $J_w=K_w\overline{K}_w$. List some useful properties of J_w which will be needed below. Firstly, $J_w^2=J_w$, which means that J_w is a projector. For any variable X , define “ J -conjugation” as $X_{J_w}=J_wXJ_w$, and the corresponding mapping will be written as $\mathbf{e}_w(X) : X \to X_{J_w}$. Note that the mapping $\mathbf{e}_w$ is idempotent.

Proposition *(i)* $wsl_q(2)/(J_w - 1) \cong sl_q(2)$ *;* $vsl_q(2)/(J_v - 1) \cong sl_q(2)$ *; (ii) Quantum algebras* $wsl_q(2)$ *and* $vsl_q(2)$ *possess zero divisors, one of which is* $(J_{w,v} - 1)$ *which annihilates all generators.*

Lemma *(i) The idempotent* J_w *is in the center of* $wsl_q(2)$ *; (ii) There are unique algebra automorphism* ω_w *and* ω_v *(called the weak Cartan automorphisms) of* U_q^w *and* U_q^v *respectively such that* $\omega_{w,v}(K_{w,v}) = \overline{K}_{w,v}$ *,* $\omega_{w,v}(\overline{K}_{w,v}) = K_{w,v}$ *,* $\omega_{w,v}(E_{w,v}) = F_{w,v}$ *,* $\omega_{w,v}(F_{w,v}) = E_{w,v}$ *.*

Let R be an algebra over k and $\mathsf{R}[t]$ be the free left R-module consisting of all polynomials of the form $P = \sum_{i=0}^{n} a_i t^i$ with coefficients in R . If $a_n \neq 0$, define $\deg(P) = n$; say $\deg(0) = -\infty$. Let α be an algebra morphism of R . An α *-derivation* of R is a k-linear endomorphism δ of R such that $\delta(ab) = \alpha(a)\delta(b) + \delta(a)b$ for all $a, b \in \mathsf{R}$. It follows that $\delta(1) = 0$.

Theorem *(i) Assume that* $\mathsf{R}[t]$ *has an algebra structure such that the natural inclusion of* R *into* $\mathsf{R}[t]$ *is a morphism of algebras and* $\deg(PQ) \leq \deg(P) + \deg(Q)$ *for any pair* (P, Q) *of elements of* $\mathsf{R}[t]$ *. Then there exists a unique injective algebra endomorphism* α *of* R *and a unique* α *-derivation* δ *of* R *such that* $ta = \alpha(a)t + \delta(a)$ *for all* $a \in \mathsf{R}$ *.*

(ii) Conversely, given an algebra endomorphism α *of* R *and an* α *-derivation* δ *of* R *, there exists a unique algebra structure on* $\mathsf{R}[t]$ *such that the inclusion of* R *into* $\mathsf{R}[t]$ *is an algebra morphism and* $ta = \alpha(a)t + \delta(a)$ *for all* $a \in \mathsf{R}$ *.*

It is recognized as a generalization of Theorem I.7.1 in [6]. We call the algebra constructed from α and δ a *weak Ore extension* of R , denoted as $\mathsf{R}_w[t, \alpha, \delta]$. Let R be an algebra, α be an algebra automorphism and δ be an α-derivation of R . If R is a left (resp. right) Noetherian, then so is the weak Ore extension $\mathsf{R}_w[t, \alpha, \delta]$.

Theorem *The algebra* $wsl_q(2)$ *is Noetherian with the basis*

$$\mathsf{P}_w = \{E_w^i F_w^j K_w^l, E_w^i F_w^j \overline{K}_w^m, E_w^i F_w^j J_w\}$$

where i, j, l *are any non-negative integers,* m *is any positive integer.*

The similar theorem can be obtained for $vsl_q(2)$ as well. Define $U_q^{w'}$ as the algebra generated by the five variables E_w , F_w , K_w , $\overline{K}_w$, L_v with the relations:

$$K_w \overline{K}_w = \overline{K}_w K_w, \quad K_w \overline{K}_w K_w = K_w, \quad \overline{K}_w K_w \overline{K}_w = \overline{K}_w,$$
$$K_w E_w = q^2 E_w K_w, \quad \overline{K}_w E_w = q^{-2} E_w \overline{K}_w,$$
$$K_w F_w = q^{-2} F_w K_w, \quad \overline{K}_w F_w = q^2 F_w \overline{K}_w,$$
$$[L_w, E_w] = q(E_w K_w + \overline{K}_w E_w), \quad [L_w, F_w] = -q^{-1}(F_w K_w + \overline{K}_w F_w),$$
$$E_w F_w - F_w E_w = L_w, \quad (q - q^{-1}) L_w = (K_w - \overline{K}_w),$$

Then U_q^w is isomorphic to the algebra $U_q^{w'}$ with φ_w satisfying $\varphi_w(E_w) = E_w$, $\varphi_w(F_w) = F_w$, $\varphi_w(K_w) = K_w$, $\varphi_w(\overline{K}_w) = \overline{K}_w$. And, the relationship between $U_q^{w'}$ and $U(sl(2))$ is that for $q = 1$, (i) the algebra isomorphism $U(sl(2)) \cong U_1^{w'}/(K_w - 1)$ holds; (ii) there exists an injective algebra morphism π from U_1^w to $U(sl(2))[K_w]/(K_w^3 - K_w)$ satisfying $\pi(E_w) = XK_w$, $\pi(F_w) = Y$, $\pi(K_w) = K_w$, $\pi(L) = HK_w$.

For $wsl_q(2)$, define the maps $\Delta_w : wsl_q(2) \to wsl_q(2) \otimes wsl_q(2)$, $\varepsilon_w : wsl_q(2) \to k$ and $T_w : wsl_q(2) \to wsl_q(2)$ satisfying respectively

$$\begin{aligned}&\Delta_w(E_w)=1\otimes E_w+E_w\otimes K_w,\ \ \Delta(F_w)=F_w\otimes 1+\overline{K}_w\otimes F_w,\\&\Delta_w(K_w)=K_w\otimes K_w,\ \ \Delta_w(\overline{K}_w)=\overline{K}_w\otimes\overline{K}_w,\\&\varepsilon_w(E_w)=\varepsilon_w(F_w)=0,\ \ \varepsilon_w(K_w)=\varepsilon_w(\overline{K}_w)=1,\\&T_w(E_w)=-E_w\overline{K}_w,\ \ T_w(F_w)=-K_wF_w,\\&T(K_w)=\overline{K}_w,\ \ T_w(\overline{K}_w)=K_w\end{aligned}.$$

Proposition *The relations above endow $wsl_q(2)$ with a bialgebra structure possessing a weak antipode T_w .*

Proposition *T_w^2 is an inner endomorphism of the algebra $wsl_q(2)$ satisfying $T_w^2\left(X\right)=K_wX\overline{K}_w$ for any $X\in wsl_q(2)$.*

It can be shown that about the operations above, it is not possible $wsl_q(2)$ would possess an antipode S so as to become a Hopf algebra. Hence, $wsl_q(2)$ is an example for a non-commutative and non-cocommutative weak Hopf algebra which is *not a Hopf algebra*. For J -weak quantum algebra $vsl_q(2)$, a thorough analysis gives the following nontrivial definitions

$$\begin{aligned}&\Delta_v(E_v)=J_v\otimes J_vE_vJ_v+J_vE_vJ_v\otimes K_v,\\&\Delta_v(F_v)=J_vF_vJ_v\otimes J_v+\overline{K}_v\otimes J_vF_vJ_v,\\&\Delta_v(K_v)=K_v\otimes K_v,\ \ \Delta_v(\overline{K}_v)=\overline{K}_v\otimes\overline{K}_v,\\&\varepsilon_v(E_v)=\varepsilon_v(F_v)=0,\ \ \varepsilon_v(K_v)=\varepsilon_v(\overline{K}_v)=1,\\&T_v(E_v)=-J_vE_v\overline{K}_v,\ \ T_v(F_v)=-K_vF_vJ_v,\\&T_v(K_v)=\overline{K}_v,\ \ T_v(\overline{K}_v)=K_v\end{aligned}.$$

These relations endow $vsl_q(2)$ with a bialgebra structure with a J -weak antipode T_v , i.e. satisfying the regularity conditions

$$(\mathbf{e}_v *_v T_v *_v \mathbf{e}_v)(X)=\mathbf{e}_v\left(X\right),\ \ (T_v *_v \mathbf{e}_v *_v T_v)(X)=T_v(X),$$

for any X in $vsl_q(2)$. From the difference between id and $\mathbf{e}_v$, $vsl_q(2)$ is not a weak Hopf algebra in the definition of [2]. So we will call it J *-weak Hopf algebra* and T_v a J *-weak antipode*. Remark the variable $\mathbf{e}_v$ can be treated as $n=2$ example of the ``tower identity" $e_{\alpha\beta}^{(n)}$ introduced for semisupermanifolds or the ``obstructor" $\mathbf{e}_X^{(n)}$ for general mappings, categories and Yang-Baxter equation in [5].

Now, we discuss the set $G(wsl_q(2))$ of all group-like elements of $wsl_q(2)$. The concept of *inverse monoid* can be found in [4].

Proposition *The set of all group-like elements $G(wsl_q(2))=\{J^{(ij)}=K_w^i\overline{K}_w^j\ :\ i,j$ run over all non-negative integers $\}$, which forms a regular monoid under the multiplication of $wsl_q(2)$.*

For $vsl_q(2)$ we can get a similar statement.

Theorem *$wsl_q(2)$ possesses an ideal W and a sub-algebra Y satisfying $wsl_q(2)=Y\oplus W$ and $W\cong sl_q(2)$ as Hopf algebras.*

Let us assume here that q is a root of unity of order d in the field k where d is an odd integer and $d>1$. Set $I=(E_w^d,F_w^d,K_w^d-J_w)$ the two-sided ideal of U_q^w and the algebra $\overline{U}_q^w=U_q^w/I$. I is also a coideal of U_q and $T_w(I)\subseteq I$. Then I is a weak Hopf ideal and $\overline{U}_q^w$ has a unique weak Hopf algebra structure with the same operations of U_q^w .

We have $\overline{U}_q^w = U_q^w / I = Y/I \oplus W/I \cong Y/(E_w^d, F_w^d) \oplus \widetilde{U}_q$ where $\widetilde{U}_q = sl_q(2)/(E_w^d, F_w^d, K^d - 1)$ is a finite dimensional Hopf algebra. The sub-algebra $\widehat{B}_q$ of $\widetilde{U}_q$ generated by $\{E_w^m K_w^n \,:\, 0 \le m, n \le d-1\}$ is a finite dimensional Hopf sub-algebra and $\widetilde{U}_q$ is a braided Hopf algebra as a quotient of the quantum double of $\widehat{B}_q$ [6]. The R-matrix of $\widetilde{U}_q$ is

$$\widehat{R} = \tfrac{1}{d} \textstyle\sum_{0 \le i,j,k \le d-1} \frac{(q-q^{-1})^k}{[k]!} q^{k(k-1)/2 + 2k(i-j) - 2ij} E_w^k K_w^i \otimes F_w^k K_w^j.$$

Since $sl_q(2) \overset{\rho}{\cong} W$ and $(E^d, F^d, K^d - 1) \overset{\rho}{\cong} I$, we get $\widetilde{U}_q \cong W/I$ under the induced morphism of ρ . Then W/I possesses also a R-matrix

$$R^w = \tfrac{1}{d} \textstyle\sum_{0 \le k \le d-1; 1 \le i,j \le d} \frac{(q-q^{-1})^k}{[k]!} q^{k(k-1)/2 + 2k(i-j) - 2ij} E_w^k K_w^i \otimes F_w^k K_w^j \ .$$

So, we get

Theorem $\overline{U}_q$ *is a quasi-braided weak Hopf algebra with*

$$R^w = \frac{1}{d} \sum_{0 \le k \le d-1; 1 \le i,j \le d} \frac{(q-q^{-1})^k}{[k]!} q^{k(k-1)/2 + 2k(i-j) - 2ij} E_w^k K_w^i \otimes F_w^k K_w^j$$

as its quasi- R *-matrix, which is von Neumann's regular.*

Because the identity of W/I is J_w , there exists the inverse $\widetilde{R}^w$ of R^w such that $\widetilde{R}^w R^w = R^w \widetilde{R}^w = J_w$ (the identity). Then we have

$$R^w \widetilde{R}^w R^w = R^w,$$
$$\widetilde{R}^w R^w \widetilde{R}^w = \widetilde{R}^w,$$

which shows that this R-matrix is regular in $\overline{U}_q$. It obeys the following relations

$$\Delta_w^{op}(x) R^w = R^w \Delta_w(x)$$

for any $x \in W/I$ and

$$(\Delta_w \otimes \mathrm{id})(R^w) = R_{13}^w R_{23}^w$$
$$(\mathrm{id} \otimes \Delta_w)(R^w) = R_{13}^w R_{12}^w$$

which are also satisfied in $\overline{U}_q$. Therefore R^w is a von Neumann's regular quasi- R-matrix of $\overline{U}_q$.

A further interesting work is to study our weak Hopf algebras through the similar objects and methods for the non-unital weak Hopf algebras [7] (their class and the class of weak Hopf algebras [2,3] are not included each other) and to find applications in the theory of quantum chain models and other relative areas.

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