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Ternary Hopf Algebras

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Properties of ternary semigroups, groups and algebras are briefly reviewed. It is shown that there exist three types of ternary units. A ternary analog of deformation is shortly discussed. Ternary coalgebras are defined in the most general manner, their classification with respect to the property "to be derived" is made. Three types of coassociativity and three kinds of counits are given. Ternary Hopf algebras with skew and strong antipods are defined. Concrete examples of ternary Hopf algebras, including the Sweedler example (which has two ternary generalizations), are presented. A ternary analog of quasitriangular Hopf algebras is constructed, and ternary abstract quantum Yang–Baxter equation (together with its classical counterpart) is obtained. A ternary "pairing" of three Hopf algebras is built.

I would like to report about the work done in part together with Andrzej Borowiec and Wieslaw Dudek, and I am grateful to them for fruitful collaboration.

Firstly ternary algebraic operations were introduced already in the XIX-th century by A. Cayley. As the development of Cayley's ideas it were considered n-ary generalization of matrices and their determinants [1] and general theory of n-ary algebras [2, 3] and ternary rings [4] (for physical applications in Nambu mechanics, supersymmetry, Yang–Baxter equation, etc. see [5] as surveys). The notion of an n-ary group was introduced in 1928 by W. Dörnte [6]. From another side, Hopf algebras [7] and their generalizations [8, 9, 10, 11] play a basic role in the quantum group theory (also see e.g. [12, 13]). We note that the derived ternary Hopf algebras are used as an intermediate tool in obtaining the Drinfeld's quantum double [14].

Here we first present necessary material on ternary semigroups, groups and algebras [15] in the abstract arrow language. Then using systematic reversing order of arrows [7], we define ternary bialgebras and Hopf algebras, investigate their properties and give some examples[1]. Most of the constructions introduced below are valid for n-ary case as well after obvious changes.

A non-empty set G with one *ternary* operation $[\,]: G\times G\times G\to G$ is called a *ternary groupoid* and is denoted by $(G,[\,])$ or $\left(G,m^{(3)}\right)$. If on G there exists a binary operation $\odot$ (or $m^{(2)}$) such that $[xyz]=(x\odot y)\odot z$ or

$$m^{(3)}=m^{(3)}_{\mathrm{der}}=m^{(2)}\circ\left(m^{(2)}\times\mathrm{id}\right) \tag{1}$$

for all $x,y,z\in G$, then we say that $[\,]$ or $m^{(3)}_{\mathrm{der}}$ is *derived* from $\odot$ or $m^{(2)}$ and denote this fact by $(G,[\,])=\mathrm{der}(G,\odot)$. If $[xyz]=((x\odot y)\odot z)\odot b$ holds for all $x,y,z\in G$ and some fixed $b\in G$, then a groupoid $(G,[\,]$ is *b-derived* from $(G,\odot)$. In this case we write $(G,[\,])=\mathrm{der}_b(G,\odot)$ [16, 17]. A *ternary isotopy* is a set of functions $f,g,h,w:G\to G$ such that $f\left([xyz]\right)=\left[g\left(x\right),h\left(y\right),w\left(z\right)\right]$ for all $x,y,z\in G$. If $g=h=w=f$, then f is *ternary isomorphism*.

A *ternary semigroup* is $(G,[\,])$ (or $\left(G,m^{(3)}\right)$) where the operation $[\,]$ $(m^{(3)})$ is *associative* $[[xyz]\,uv]=[x\,[yzu]\,v]=[xy\,[zuv]]$ (for all $x,y,z,u,v\in G$) or

$$m^{(3)}\circ\left(m^{(3)}\times\mathrm{id}\times\mathrm{id}\right)=m^{(3)}\circ\left(\mathrm{id}\times m^{(3)}\times\mathrm{id}\right)=m^{(3)}\circ\left(\mathrm{id}\times\mathrm{id}\times m^{(3)}\right) \tag{2}$$

[1]Due to the lack of place in the Proceedings we present only important results and constructions omitting most proofs and detailed derivations which will appear elsewhere.

A ternary operation $m^{(3)}_{\text{der}}$ derived from a binary associative operation $m^{(2)}$ is also associative, but a ternary groupoid $(G,[\ ])$ b-derived (b is a cancellative element) from a semigroup $(G,\odot)$ is a ternary semigroup if and only if b lies in the center of $(G,\odot)$. Fixing in a ternary operation $m^{(3)}$ one element a we obtain a binary operation $m^{(2)}_a$. A binary groupoid $(G,\odot)$ or $\left(G,m^{(2)}_a\right)$, where $x\odot y=[xay]$ or $m^{(2)}_a=m^{(3)}\circ(\mathrm{id}\times a\times\mathrm{id})$ for some fixed $a\in G$ is called a *retract* of $(G,[\ ])$ and is denoted by $\mathrm{ret}_a(G,[\ ])$ [16, 17]. It can be shown that if there exists an element e such that for all $y\in G$ we have $[eye]=y$, then this semigroup is derived from the binary semigroup $\left(G,m^{(2)}_e\right)$, where $m^{(2)}_e=m^{(3)}\circ(\mathrm{id}\times e\times\mathrm{id})$.

An element $e_m\in G$ is called a *middle identity* of $(G,[\ ])$ if for all $x\in G$ we have $[e_mxe_m]=x$ or $m^{(3)}\circ(e_m\times\mathrm{id}\times e_m)=\mathrm{id}$. An element $e_l\in G$ satisfying the identity $[e_le_lx]=x$ or $m^{(3)}\circ(e_l\times e_l\times\mathrm{id})=\mathrm{id}$ is called a *left identity*. By analogy we define a *right identity*, satisfying $[xe_re_r]=x$ or $m^{(3)}\circ(\mathrm{id}\times e_r\times e_r)=\mathrm{id}$ for all $x\in G$. An element which is a left, middle and right identity $e=e_m=e_l=e_r$ is called a *ternary identity* (briefly: *identity*), an element which is only left and right identity is a *semi-identity* $e_{\text{semi}}=e_m=e_l$. There are ternary semigroups without left (middle, right) neutral elements, but there are also ternary semigroups in which all elements are identities [15, 18]. More general, a 2-sequence of elements $\alpha_2=e_1e_2$ is *neutral*, if $[e_1e_2x]=[xe_1e_2]=x$ for all $x\in G$ and by analogy for n-sequence. Two sequences α and β are equivalent, if there are exist another two sequences γ and δ such that $[\gamma\alpha\delta]=[\gamma\beta\delta]$.

Lemma 1. *For any ternary semigroup $(G,[\ ])$ with a left (right) identity there exists a binary semigroup $(G,\odot)$ and its endomorphism μ such that $[xyz]=x\odot\mu(y)\odot z$ for all $x,y,z\in G$.*

Proof. Let e_l be a left identity of $(G,[\ ])$. Then the operation $x\odot y=[xe_ly]$ is associative. Moreover, for $\mu(x)=[e_lxe_l]$, we have $\mu(x)\odot\mu(y)=[[e_lxe_l]e_l[e_lye_l]]=[[e_lxe_l][e_le_ly]e_l]=[e_l[xe_ly]e_l]=\mu(x\odot y)$ and $[xyz]=[x[e_le_ly][e_le_lz]]=[[xe_l[e_lye_l]]e_lz]=x\odot\mu(y)\odot z$. In the case of right identity the proof is analogous. ■

A ternary groupoid $(G,[\ ])$ is a *left cancellative* if $[abx]=[aby]\Longrightarrow x=y$, a *middle cancellative* if $[axb]=[ayb]\Longrightarrow x=y$, a *right cancellative* if $[xab]=[yab]\Longrightarrow x=y$ hold for all $a,b\in G$. A ternary groupoid which is left, middle and right cancellative is called *cancellative*.

Definition 1. A ternary groupoid $(G,[\ ])$ is *semicommutative* if $[xyz]=[zyx]$ for all $x,y,z\in G$. If the value of $[xyz]$ is independent on the permutation of elements x,y,z, viz.

$$[x_1x_2x_3]=\left[x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}\right] \tag{3}$$

or $m^{(3)}=m^{(3)}\circ\sigma$, then $(G,[\ ])$ is a *commutative* ternary groupoid. If σ is fixed, then a ternary groupoid satisfying (3) is called σ-commutative.

The group S_3 is generated by two transpositions; (12) and (23). This means that $(G,[\ ])$ is commutative if and only if $[xyz]=[yxz]=[xzy]$ holds for all $x,y,z\in G$. Further if in a ternary semigroup $(G,[\ ])$ satisfying the identity $[xyz]=[yxz]$ there are a,b such that $[axb]=x$ for all $x\in G$, then $(G,[\ ])$ is commutative.

Mediality in the binary case $(x\odot y)\odot(z\odot u)=(x\odot z)\odot(y\odot u)$ for groups coincides with commutativity. In the ternary case they do not coincide. A ternary groupoid $(G,[\ ])$ is *medial* if it satisfies the identity

$$[[x_{11}x_{12}x_{13}][x_{21}x_{22}x_{23}][x_{31}x_{32}x_{33}]]=[[x_{11}x_{21}x_{31}][x_{12}x_{22}x_{32}][x_{13}x_{23}x_{33}]]$$

or

$$m^{(3)}\circ\left(m^{(3)}\times m^{(3)}\times m^{(3)}\right)=m^{(3)}\circ\left(m^{(3)}\times m^{(3)}\times m^{(3)}\right)\circ\sigma_{\text{medial}}, \tag{4}$$

where $\sigma_{\text{medial}}=\binom{123456789}{147258369}\in S_9$.

It is not difficult to see that a semicommutative ternary semigroup is medial. An element x such that $[xxx] = x$ is called an *idempotent*. A groupoid in which all elements are idempotents is called an *idempotent groupoid*. A left (right, middle) identity is an idempotent, also any neutral sequence $e_1 e_2$ is an idempotent.

Definition 2. A ternary semigroup $(G, [\ ])$ is a *ternary group* if for all $a, b, c \in G$ there are $x, y, z \in G$ such that $[xab] = [ayb] = [abz] = c$.

In a ternary group the equation $[xxz] = x$ has a unique solution which is denoted by $z = \overline{x}$ and called *skew element* [6], or equivalently

$$m^{(3)} \circ (\mathrm{id} \times \mathrm{id} \times \bar{\cdot}) \circ D^{(3)} = \mathrm{id},$$

where $D^{(3)}(x) = (x, x, x)$ is a ternary diagonal map.

Theorem 1. *In any ternary group $(G, [\ ])$ for all $x, y, z \in G$ the following relations take place $[xx\,\overline{x}] = [x\,\overline{x}\,x] = [\,\overline{x}\,xx] = x$, $[yx\,\overline{x}] = [y\,\overline{x}\,x] = [x\,\overline{x}\,y] = [\,\overline{x}\,xy] = y$, $\overline{[xyz]} = [\,\overline{z}\,\overline{y}\,\overline{x}]$, $\overline{\overline{x}} = x$.*

Since in an idempotent ternary group $\overline{x} = x$ for all x, an idempotent ternary group is semicommutative. From [19, 20] it follows

Theorem 2. *A ternary semigroup $(G, [\ ])$ with a unary operation $^{-} : x \to \overline{x}$ is a ternary group if and only if it satisfies identities $[yx\,\overline{x}\,] = [x\,\overline{x}\,y] = y$, or*

$$m^{(3)} \circ (\mathrm{id} \times \bar{\cdot} \times \mathrm{id}) \circ \left(D^{(2)} \times \mathrm{id}\right) = \mathrm{Pr}_2,$$

$$m^{(3)} \circ (\mathrm{id} \times \mathrm{id} \times \bar{\cdot}) \circ \left(\mathrm{id} \times D^{(2)}\right) = \mathrm{Pr}_1,$$

where $D^{(2)}(x) = (x, x)$ and $\mathrm{Pr}_1(x, y) = x$, $\mathrm{Pr}_2(x, y) = y$.

A ternary semigroup $(G, [\ ])$ is an idempotent ternary group if and only if it satisfies identities $[yxx] = [xxy] = y$. Moreover, a ternary group with an identity is derived from a binary group.

Theorem 3 (Gluskin–Hosszú). *For a ternary group $(G, [\ ])$ there exists a binary group $(G, \circledast)$, its automorphism φ and fixed element $b \in G$ such that $[xyz] = x \circledast \varphi(y) \circledast \varphi^2(z) \circledast b$.*

Proof. Let $a \in G$ be fixed. The binary operation $x \circledast y = [xay]$ ($a \in G$ fixed) is associative, because $(x \circledast y) \circledast z = [[xay]az] = [xa[yaz]] = x \circledast (y \circledast z)$ with identity $\overline{a}$ and $\varphi(x) = [\overline{a}xa]$, $b = [\overline{a}\,\overline{a}\,\overline{a}\,]$ (see [21]). ■

Theorem 4 (Post). *For any ternary group $(G, [\ ])$ there exists a binary group $(G^*, \circledast)$ and $H \lhd G^*$, such that $G^* \diagup H \simeq \mathbb{Z}_2$ and $[xyz] = x \circledast y \circledast z$ for all $x, y, z \in G$.*

Proof. Let c be a fixed element in G and let $G^* = G \times \mathbb{Z}_2$. In G^* we define binary operation $\circledast$ putting $(x, 0) \circledast (y, 0) = ([xy\overline{c}], 1)$, $(x, 0) \circledast (y, 1) = ([xyc], 0)$, $(x, 1) \circledast (y, 0) = ([xcy], 0)$, $(x, 1) \circledast (y, 1) = ([xcy], 1)$. This operation is associative and $(\overline{c}, 1)$ is its neutral element. The inverse element (in G^*) has the form $(x, 0)^{-1} = (\overline{x}, 0)$, $(x, 1)^{-1} = ([\overline{c}\,\overline{x}\,\overline{c}], 1)$. Thus G^* is a group such that $H = \{(x, 1) : x \in G\} \lhd G^*$. Obviously the set G can be identified with $G \times \{0\}$ and $[xyz] = ((x, 0) \circledast (y, 0)) \circledast (z, 0) = ([xy\overline{c}], 1) \circledast (z, 0) = ([[xy\overline{c}]cz], 0) = ([xy[\overline{c}cz]], 0) = ([xyz], 0)$, which completes the proof. ■

Let us consider ternary algebras. One can introduce autodistributivity property $[[xyz]\,ab] = [[xab]\,[yab]\,[zab]]$ (see [22]). If we take 2 ternary operations $\{\ ,\ ,\ \}$ and $[\ ,\ ,\ ]$, then distributivity is $\{[xyz]\,ab\} = [\{xab\}\,\{yab\}\,\{zab\}]$. If $(+)$ is a binary operation (addition), then *left linearity* is $[(x + z), a, b] = [xab] + [zab]$. By analogy one can define central (middle) and right linearity. *Linearity* is defined, when left, middle and right linearity hold valid simultaneously.

Definition 3. *Ternary algebra* is a triple $\left(A, m^{(3)}, \eta^{(3)}\right)$, where A is a linear space over a field $\mathbb{K}$, $m^{(3)}$ is a linear map $m^{(3)} : A \otimes A \otimes A \to A$ called *ternary multiplication* $m^{(3)}\left(a \otimes b \otimes c\right) = [abc]$ which is ternary associative $[[abc]\,de] = [a\,[bcd]\,e] = [ab\,[cde]]$ or

$$m^{(3)} \circ \left(m^{(3)} \otimes \mathrm{id} \otimes \mathrm{id}\right) = m^{(3)} \circ \left(\mathrm{id} \otimes m^{(3)} \otimes \mathrm{id}\right) = m^{(3)} \circ \left(\mathrm{id} \otimes \mathrm{id} \otimes m^{(3)}\right). \tag{5}$$

There are 3 types of ternary unit maps $\eta^{(3)} : \mathbb{K} \to A$: 1) One *strong* unit map $m^{(3)} \circ \left(\eta^{(3)} \otimes \eta^{(3)} \otimes \mathrm{id}\right) = m^{(3)} \circ \left(\eta^{(3)} \otimes \mathrm{id} \otimes \eta^{(3)}\right) = m^{(3)} \circ \left(\mathrm{id} \otimes \eta^{(3)} \otimes \eta^{(3)}\right) = \mathrm{id}$; 2) two *sequential* units $\eta_1^{(3)}$ and $\eta_2^{(3)}$ satisfying $m^{(3)} \circ \left(\eta_1^{(3)} \otimes \eta_2^{(3)} \otimes \mathrm{id}\right) = m^{(3)} \circ \left(\eta_1^{(3)} \otimes \mathrm{id} \otimes \eta_2^{(3)}\right) = m^{(3)} \circ \left(\mathrm{id} \otimes \eta_1^{(3)} \otimes \eta_2^{(3)}\right) = \mathrm{id}$; 3) Four *long* (left) ternary units

$$m^{(3)} \circ \left(\mathrm{id} \otimes \eta_1^{(3)} \otimes \eta_2^{(3)}\right) \circ \left(m^{(3)} \circ \left(\mathrm{id} \otimes \eta_3^{(3)} \otimes \eta_4^{(3)}\right)\right) = \mathrm{id}$$

which corresponds to $\left[\left[a\eta_1^{(3)}\eta_2^{(3)}\right], \eta_3^{(3)}, \eta_4^{(3)}\right] = a \in A$ (right and middle units are defined similarly). In first case the ternary analog of the binary relation $\eta^{(2)}\left(x\right) = x1$, where $x \in \mathbb{K}$, $1 \in A$, is $\eta^{(3)}\left(x\right) = [x, x, 1] = [x, 1, x] = [1, x, x]$.

Let (A, m_A, η_A), (B, m_B, η_B) and (C, m_C, η_C) be ternary algebras, then the *ternary tensor product* space $A \otimes B \otimes C$ is naturally endowed with the structure of an algebra. The multiplication $m_{A\otimes B\otimes C}$ on $A \otimes B \otimes C$ reads $[(a_1 \otimes b_1 \otimes c_1)(a_2 \otimes b_2 \otimes c_2)(a_3 \otimes b_3 \otimes c_3)] = [a_1a_2a_3] \otimes [b_1b_2b_3] \otimes [c_1c_2c_3]$, and so the set of ternary algebras is closed under taking ternary tensor products. A ternary *algebra map* (homomorphism) is a linear map between ternary algebras $f : A \to B$ which respects the ternary algebra structure $f\left([xyz]\right) = [f\left(x\right), f\left(y\right), f\left(z\right)]$ and $f\left(1_A\right) = 1_B$.

A ternary (and n-ary) commutator can be obtained in different ways [23]. We will consider a simplest version called a Nambu bracket (see e.g. [24]). Let us introduce two maps $\omega_{\pm}^{(3)} : A \otimes A \otimes A \to A \otimes A \otimes A$ by

$$\omega_{+}^{(3)}\left(a \otimes b \otimes c\right) = a \otimes b \otimes c + b \otimes c \otimes a + c \otimes a \otimes b, \tag{6}$$

$$\omega_{-}^{(3)}\left(a \otimes b \otimes c\right) = b \otimes a \otimes c + c \otimes b \otimes a + a \otimes c \otimes b. \tag{7}$$

Thus obviously $m^{(3)} \circ \omega_{\pm}^{(3)} = \sigma_{\pm}^{(3)} \circ m^{(3)}$, where $\sigma_{\pm}^{(3)} \in S_3$ denotes sum of terms having even and odd permutations respectively. In the binary case $\omega_{+}^{(2)} = \mathrm{id} \otimes \mathrm{id}$ and $\omega_{-}^{(2)} = \tau$ is the twist operator $\tau : a \otimes b \to b \otimes a$, while $m^{(2)} \circ \omega_{-}^{(2)}$ is permutation $\sigma_{-}^{(2)}\left(ab\right) = ba$. So the Nambu product is $\omega_N^{(3)} = \omega_{+}^{(3)} - \omega_{-}^{(3)}$, and the ternary commutator is $[\,,\,,\,]_N = \sigma_N^{(3)} = \sigma_{+}^{(3)} - \sigma_{-}^{(3)}$, or simply $[a, b, c]_N = [abc] + [bca] + [cab] - [cba] - [acb] - [bac]$ (see [24] and refs. therein). An *abelian* ternary algebra is defined by vanishing of Nambu bracket $[a, b, c]_N = 0$ or ternary commutation relation $\sigma_{+}^{(3)} = \sigma_{-}^{(3)}$. By analogy with the binary case a *deformed* ternary algebra can be defined by

$$\sigma_{+}^{(3)} = q\sigma_{-}^{(3)} \text{ or } [abc] + [bca] + [cab] = q\left([cba] + [acb] + [bac]\right), \tag{8}$$

where multiplication by q is treated as an external operation. An opposite and more complicated possibility requires 2 deformation parameters and can be defined as $\sigma_{+}^{(3)}\left([a, b, c]\right) = \left[q, p, \sigma_{-}^{(3)}\left([a, b, c]\right)\right]$, which reminds the binary case $ab = qba$ in the following form $m^{(2)}\left(a, b\right) = m^{(2)}\left(q, \sigma_{-}^{(2)}\left(ab\right)\right)$. Here we will exploit (8).

Let C be a linear space over a field $\mathbb{K}$.

Definition 4. *Ternary comultiplication* $\Delta^{(3)}$ is a linear map over a field $\mathbb{K}$ such that

$$\Delta^{(3)} : C \to C \otimes C \otimes C. \tag{9}$$

In the standard Sweedler notations [7] $\Delta^{(3)}(a) = \sum_{i=1}^{n} a_i' \otimes a_i'' \otimes a_i''' = a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$. Consider different possible types of ternary coassociativity.

1. *Standard* ternary coassociativity

$$\left(\Delta^{(3)} \otimes \mathrm{id} \otimes \mathrm{id}\right) \circ \Delta^{(3)} = \left(\mathrm{id} \otimes \Delta^{(3)} \otimes \mathrm{id}\right) \circ \Delta^{(3)} = \left(\mathrm{id} \otimes \mathrm{id} \otimes \Delta^{(3)}\right) \circ \Delta^{(3)}. \tag{10}$$

2. *Nonstandard* ternary Σ-coassociativity (Gluskin-type — positional operatives)

$$\left(\Delta^{(3)} \otimes \mathrm{id} \otimes \mathrm{id}\right) \circ \Delta^{(3)} = \left(\mathrm{id} \otimes \left(\sigma \circ \Delta^{(3)}\right) \otimes \mathrm{id}\right) \circ \Delta^{(3)},$$

where $\sigma \circ \Delta^{(3)}(a) = \Delta_\sigma^{(3)}(a) = a_{(\sigma(1))} \otimes a_{(\sigma(2))} \otimes a_{(\sigma(3))}$ and $\sigma \in \Sigma \subset S_3$.

3. *Permutational* ternary coassociativity

$$\left(\Delta^{(3)} \otimes \mathrm{id} \otimes \mathrm{id}\right) \circ \Delta^{(3)} = \pi \circ \left(\mathrm{id} \otimes \Delta^{(3)} \otimes \mathrm{id}\right) \circ \Delta^{(3)},$$

where $\pi \in \Pi \subset S_5$.

Ternary comediality is

$$\left(\Delta^{(3)} \otimes \Delta^{(3)} \otimes \Delta^{(3)}\right) \circ \Delta^{(3)} = \sigma_{\text{medial}} \circ \left(\Delta^{(3)} \otimes \Delta^{(3)} \otimes \Delta^{(3)}\right) \circ \Delta^{(3)},$$

where σ_{medial} is defined in (4). *Ternary counit* is defined as a map $\varepsilon^{(3)} : C \to \mathbb{K}$. In general, $\varepsilon^{(3)} \neq \varepsilon^{(2)}$ satisfying one of the conditions below. If $\Delta^{(3)}$ is derived, then maybe $\varepsilon^{(3)} = \varepsilon^{(2)}$, but another counits may exist. There are 3 types of ternary counits:

1. Standard (*strong*) ternary counit

$$\left(\varepsilon^{(3)} \otimes \varepsilon^{(3)} \otimes \mathrm{id}\right) \circ \Delta^{(3)} = \left(\varepsilon^{(3)} \otimes \mathrm{id} \otimes \varepsilon^{(3)}\right) \circ \Delta^{(3)} = \left(\mathrm{id} \otimes \varepsilon^{(3)} \otimes \varepsilon^{(3)}\right) \circ \Delta^{(3)} = \mathrm{id}\,. \tag{11}$$

2. Two *sequential* (polyadic) counits $\varepsilon_1^{(3)}$ and $\varepsilon_2^{(3)}$

$$\left(\varepsilon_1^{(3)} \otimes \varepsilon_2^{(3)} \otimes \mathrm{id}\right) \circ \Delta = \left(\varepsilon_1^{(3)} \otimes \mathrm{id} \otimes \varepsilon_2^{(3)}\right) \circ \Delta = \left(\mathrm{id} \otimes \varepsilon_1^{(3)} \otimes \varepsilon_2^{(3)}\right) \circ \Delta = \mathrm{id}\,. \tag{12}$$

3. Four long ternary counits $\varepsilon_1^{(3)}$–$\varepsilon_4^{(3)}$ satisfying

$$\left(\left(\mathrm{id} \otimes \varepsilon_3^{(3)} \otimes \varepsilon_4^{(3)}\right) \circ \Delta^{(3)} \circ \left(\left(\mathrm{id} \otimes \varepsilon_1^{(3)} \otimes \varepsilon_2^{(3)}\right) \circ \Delta^{(3)}\right)\right) = \mathrm{id}\,. \tag{13}$$

Below we will consider only the first standard type of associativity (10). By analogy with (3) *σ-cocommutativity* is defined as $\sigma \circ \Delta^{(3)} = \Delta^{(3)}$.

Definition 5. *Ternary coalgebra* is a triple $\left(C, \Delta^{(3)}, \varepsilon^{(3)}\right)$, where C is a linear space and $\Delta^{(3)}$ is a ternary comultiplication (9) which is coassociative in one of the above senses and $\varepsilon^{(3)}$ is one of the above counits.

Let $\left(A, m^{(3)}\right)$ be a ternary algebra and $\left(C, \Delta^{(3)}\right)$ be a ternary coalgebra and $f, g, h \in \mathrm{Hom}_{\mathbb{K}}(C, A)$. *Ternary convolution product* is

$$[f, g, h]_* = m^{(3)} \circ (f \otimes g \otimes h) \circ \Delta^{(3)} \tag{14}$$

or in the Sweedler notation $[f, g, h]_*(a) = \left[f\left(a_{(1)}\right) g\left(a_{(2)}\right) h\left(a_{(3)}\right)\right]$.

Definition 6. Ternary coalgebra is called *derived*, if there exists a binary (usual, see e.g. [7]) coalgebra $\Delta^{(2)} : C \to C \otimes C$ such that (cf. 1))

$$\Delta^{(3)}_{\mathrm{der}} = \left(\mathrm{id} \otimes \Delta^{(2)}\right) \otimes \Delta^{(2)}. \tag{15}$$

Definition 7. Ternary bialgebra B is $\left(B, m^{(3)}, \eta^{(3)}, \Delta^{(3)}, \varepsilon^{(3)}\right)$ for which $\left(B, m^{(3)}, \eta^{(3)}\right)$ is a ternary algebra and $\left(B, \Delta^{(3)}, \varepsilon^{(3)}\right)$ is a ternary coalgebra and they are compatible

$$\Delta^{(3)} \circ m^{(3)} = m^{(3)} \circ \Delta^{(3)}. \tag{16}$$

One can distinguish four kinds of ternary bialgebras with respect to a "being derived" property:

1. Δ-*derived* ternary bialgebra

$$\Delta^{(3)} = \Delta^{(3)}_{\mathrm{der}} = \left(\mathrm{id} \otimes \Delta^{(2)}\right) \circ \Delta^{(2)}. \tag{17}$$

2. m-*derived* ternary bialgebra

$$m^{(3)}_{\mathrm{der}} = m^{(3)}_{\mathrm{der}} = m^{(2)} \circ \left(m^{(2)} \otimes \mathrm{id}\right). \tag{18}$$

3. *Derived* ternary bialgebra is simultaneously m-derived and Δ-derived ternary bialgebra.
4. *Non-derived* ternary bialgebra which does not satisfy (17) and (18).

Let us consider a ternary analog of the Woronowicz example of a bialgebra construction, which in the binary case has two generators satisfying $xy = qyx$ (or $\sigma^{(2)}_{+}(xy) = q\sigma^{(2)}_{-}(xy)$), then the following coproducts $\Delta^{(2)}(x) = x \otimes x$, $\Delta^{(2)}(x) = y \otimes x + 1 \otimes y$ are algebra maps. In the derived ternary case using (8) we have $\sigma^{(3)}_{+}([xey]) = q\sigma^{(3)}_{-}([xey])$, where e is the ternary unit and ternary coproducts are $\Delta^{(3)}(e) = e \otimes e \otimes e$, $\Delta^{(3)}(x) = x \otimes x \otimes x$, $\Delta^{(3)}(x) = y \otimes x \otimes x + e \otimes y \otimes x + e \otimes e \otimes y$, which are ternary algebra maps, i.e. they satisfy $\sigma^{(3)}_{+}\left(\left[\Delta^{(3)}(x)\,\Delta^{(3)}(e)\,\Delta^{(3)}(y)\right]\right) = q\sigma^{(3)}_{-}\left(\left[\Delta^{(3)}(x)\,\Delta^{(3)}(e)\,\Delta^{(3)}(y)\right]\right)$.

Possible types of ternary antipodes can be defined using analogy with binary coalgebras.

Definition 8. *Skew ternary antipod* is

$$\begin{aligned} & m^{(3)} \circ \left(S^{(3)}_{\mathrm{skew}} \otimes \mathrm{id} \otimes \mathrm{id}\right) \circ \Delta^{(3)} \\ & \quad = m^{(3)} \circ \left(\mathrm{id} \otimes S^{(3)}_{\mathrm{skew}} \otimes \mathrm{id}\right) \circ \Delta^{(3)} = m^{(3)} \circ \left(\mathrm{id} \otimes \mathrm{id} \otimes S^{(3)}_{\mathrm{skew}}\right) \circ \Delta^{(3)} = \mathrm{id}. \end{aligned} \tag{19}$$

If only one equality from (19) is satisfied, the corresponding skew antipod is called *left*, *middle* or *right*.

Definition 9. *Strong ternary antipod* is

$$\left(m^{(2)} \otimes \mathrm{id}\right) \circ \left(\mathrm{id} \otimes S^{(3)}_{\mathrm{strong}} \otimes \mathrm{id}\right) \circ \Delta^{(3)} = 1 \otimes \mathrm{id},$$

$$\left(\mathrm{id} \otimes m^{(2)}\right) \circ \left(\mathrm{id} \otimes \mathrm{id} \otimes S^{(3)}_{\mathrm{strong}}\right) \circ \Delta^{(3)} = \mathrm{id} \otimes 1,$$

where 1 is a unit of algebra.

If in a ternary coalgebra $\Delta^{(3)} \circ S = \tau_{13} \circ (S \otimes S \otimes S) \circ \Delta^{(3)}$, where $\tau_{13} = \binom{123}{321}$, then it is called *skew-involutive*.

Definition 10. Ternary Hopf algebra $\left(H, m^{(3)}, \eta^{(3)}, \Delta^{(3)}, \varepsilon^{(3)}, S^{(3)}\right)$ is a ternary bialgebra with a ternary antipod $S^{(3)}$ of the type corresponding to the above.

There are 8 types of associative ternary Hopf algebras and 4 types of medial Hopf algebras. Also it can happen that there are several ternary units $\eta_i^{(3)}$ and several ternary counits $\varepsilon_i^{(3)}$ (see (11)–(13)), as well as different skew antipodes (see (19) and below), which makes number of ternary Hopf algebras enormous.

Let us consider concrete constructions of ternary comultiplications, bialgebras and Hopf algebras. A *ternary group-like element* can be defined by $\Delta^{(3)}(g) = g \otimes g \otimes g$, and for 3 such elements we have $\Delta^{(3)}([g_1 g_2 g_3]) = \Delta^{(3)}(g_1)\,\Delta^{(3)}(g_2)\,\Delta^{(3)}(g_3)$. But an analog of the binary primitive element (satisfying $\Delta^{(2)}(x) = x \otimes 1 + 1 \otimes x$) cannot be chosen simply as $\Delta^{(3)}(x) = x \otimes e \otimes e + e \otimes x \otimes e + e \otimes e \otimes x$, since the algebra structure is not preserved. Nevertheless, if we introduce *two* idempotent units e_1, e_2 satisfying "semiorthogonality" $[e_1 e_1 e_2] = 0$, $[e_2 e_2 e_1] = 0$, then

$$\Delta^{(3)}(x) = x \otimes e_1 \otimes e_2 + e_2 \otimes x \otimes e_1 + e_1 \otimes e_2 \otimes x \tag{20}$$

and now $\Delta^{(3)}([x_1 x_2 x_3]) = \left[\Delta^{(3)}(x_1)\,\Delta^{(3)}(x_2)\,\Delta^{(3)}(x_3)\right]$. Using (20) $\varepsilon(x) = 0$, $\varepsilon(e_{1,2}) = 1$, and $S^{(3)}(x) = -x$, $S^{(3)}(e_{1,2}) = e_{1,2}$, one can construct a ternary universal enveloping algebra in full analogy with the binary case (see e.g. [12]).

One of the most important examples of noncommutative Hopf algebras is the well known Sweedler Hopf algebra [7] which in the binary case has two generators x and y satisfying (in the "arrow language") $m^{(2)}(x,x) = 1$, $m^{(2)}(y,y) = 0$, $\sigma_+^{(2)}(xy) = -\sigma_-^{(2)}(xy)$. It has the following comultiplication $\Delta^{(2)}(x) = x \otimes x$, $\Delta^{(2)}(y) = y \otimes x + 1 \otimes y$, unit $\varepsilon^{(2)}(x) = 1$, $\varepsilon^{(2)}(y) = 0$, and antipod $S^{(2)}(x) = x$, $S^{(2)}(y) = -y$, which respect to the algebra structure. In the derived case a *ternary Sweedler algebra* is generated also by two generators x and y obeying $m^{(3)}(x,e,x) = m^{(3)}(e,x,x) = m^{(3)}(x,x,e) = e$, $\sigma_+^{(3)}([yey]) = 0$, $\sigma_+^{(3)}([xey]) = -\sigma_-^{(3)}([xey])$. The derived Hopf algebra structure is given by

$$\Delta^{(3)}(x) = x \otimes x \otimes x, \qquad \Delta^{(3)}(y) = y \otimes x \otimes x + e \otimes y \otimes x + e \otimes e \otimes y, \tag{21}$$

$$\varepsilon^{(3)}(x) = \varepsilon^{(2)}(x) = 1, \qquad \varepsilon^{(3)}(y) = \varepsilon^{(2)}(y) = 0, \tag{22}$$

$$S^{(3)}(x) = S^{(2)}(x) = x, \qquad S^{(3)}(y) = S^{(2)}(y) = -y, \tag{23}$$

and it can be checked that (21)–(22) are algebra maps, while (23) is antialgebra maps. To obtain a non-derived ternary Sweedler example we have the possibilities: 1) one "even" generator x, two "odd" generators $y_{1,2}$ and one ternary unit e; 2) two "even" generators $x_{1,2}$, one "odd" generator y and two ternary units $e_{1,2}$. In the first case the ternary algebra structure is (no summation, $i = 1,2$)

$$\begin{gathered}
[xxx] = e, \qquad [y_i y_i y_i] = 0, \quad \sigma_+^{(3)}([y_i x y_i]) = 0, \qquad \sigma_+^{(3)}([x y_i x]) = 0, \\
[x e y_i] = -[x y_i e], \qquad [e x y_i] = -[y_i x e], \qquad [e y_i x] = -[y_i e x], \\
\sigma_+^{(3)}([y_1 x y_2]) = -\sigma_-^{(3)}([y_1 x y_2]).
\end{gathered} \tag{24}$$

The corresponding ternary Hopf algebra structure is

$$\begin{gathered}
\Delta^{(3)}(x) = x \otimes x \otimes x, \qquad \Delta^{(3)}(y_{1,2}) = y_{1,2} \otimes x \otimes x + e_{1,2} \otimes y_{2,1} \otimes x + e_{1,2} \otimes e_{2,1} \otimes y_{2,1}, \\
\varepsilon^{(3)}(x) = 1, \qquad \varepsilon^{(3)}(y_i) = 0, \qquad S^{(3)}(x) = x, \qquad S^{(3)}(y_i) = -y_i.
\end{gathered} \tag{25}$$

In the second case we have for the algebra structure

$$\begin{gathered}
[x_i x_j x_k] = \delta_{ij}\delta_{ik}\delta_{jk} e_i, \qquad [yyy] = 0, \qquad \sigma_+^{(3)}([y x_i y]) = 0, \qquad \sigma_+^{(3)}([x_i y x_i]) = 0, \\
\sigma_+^{(3)}([y_1 x y_2]) = 0, \qquad \sigma_-^{(3)}([y_1 x y_2]) = 0,
\end{gathered} \tag{26}$$

and the ternary Hopf algebra structure is

$$\Delta^{(3)}(x_i) = x_i \otimes x_i \otimes x_i, \qquad \Delta^{(3)}(y) = y \otimes x_1 \otimes x_1 + e_1 \otimes y \otimes x_2 + e_1 \otimes e_2 \otimes y,$$
$$\varepsilon^{(3)}(x_i) = 1, \qquad \varepsilon^{(3)}(y) = 0, \qquad S^{(3)}(x_i) = x_i, \qquad S^{(3)}(y) = -y. \tag{27}$$

Let us consider the group $G = SL(n,\mathbb{K})$. Then the algebra generated by $a^i_j \in SL(n,\mathbb{K})$ can be endowed by the structure of ternary Hopf algebra (see e.g. [25] for binary case) by choosing the ternary coproduct, counit and antipod as (here summation is implied)

$$\Delta^{(3)}\left(a^i_j\right) = a^i_k \otimes a^k_l \otimes a^l_j, \qquad \varepsilon\left(a^i_j\right) = \delta^i_j, \qquad S^{(3)}\left(a^i_j\right) = \left(a^{-1}\right)^i_j. \tag{28}$$

This antipod is a skew one since from (19) it follows $m^{(3)} \circ (S^{(3)} \otimes \mathrm{id} \otimes \mathrm{id}) \circ \Delta^{(3)}\left(a^i_j\right) = S^{(3)}\left(a^i_k\right) a^k_l a^l_j = \left(a^{-1}\right)^i_k a^k_l a^l_j = \delta^i_l a^l_j = a^i_j$. This ternary Hopf algebra is derived since for $\Delta^{(2)} = a^i_j \otimes a^j_k$ we have $\Delta^{(3)} = \left(\mathrm{id} \otimes \Delta^{(2)}\right) \otimes \Delta^{(2)}\left(a^i_j\right) = \left(\mathrm{id} \otimes \Delta^{(2)}\right)\left(a^i_k \otimes a^k_j\right) = a^i_k \otimes \Delta^{(2)}\left(a^k_j\right) = a^i_k \otimes a^k_l \otimes a^l_j$. In the most important case $n = 2$ we can obtain the manifest action of the ternary coproduct $\Delta^{(3)}$ on components. Possible non-derived matrix representations of the ternary product can be done only by four-rank $n \times n \times n \times n$ twice covariant and twice contravariant tensors $\left\{a^{ij}_{kl}\right\}$. Among all products the non-derived ones are only the following $a^{oi}_{jk} b^{jl}_{oo} c^{ko}_{il}$ and $a^{ij}_{ok} b^{ol}_{io} c^{ko}_{il}$ (where o is any index). So using e.g. the first choice we can define the non-derived Hopf algebra structure by $\Delta^{(3)}\left(a^{ij}_{kl}\right) = a^{i\mu}_{v\rho} \otimes a^{v\sigma}_{kl} \otimes a^{\rho j}_{\mu\sigma}$, $\varepsilon\left(a^{ij}_{kl}\right) = \frac{1}{2}\left(\delta^i_k \delta^j_l + \delta^i_l \delta^j_k\right)$, and the skew antipod $s^{ij}_{kl} = S^{(3)}\left(a^{ij}_{kl}\right)$ which is a solution of the equation $s^{i\mu}_{v\rho} a^{v\sigma}_{kl} = \delta^i_\rho \delta^\mu_k \delta^\sigma_l$.

Next consider ternary dual pair $k(G)$ (push-forward) and $\mathcal{F}(G)$ (pull-back) which are related by $k^*(G) \cong \mathcal{F}(G)$ (see e.g. [26]). Here $k(G) = \mathrm{span}\,(G)$ is a ternary group algebra (G has a ternary product $[\,]_G$ or $m^{(3)}_G$) over a field k. If $u \in k(G)$ $(u = u^i x_i, x_i \in G)$, then $[uvw]_k = u^i v^j w^l [x_i x_j x_l]_G$ is associative, and so $(k(G), [\,]_k)$ becomes a ternary algebra. Define a ternary coproduct $\Delta^{(3)}_k : k(G) \to k(G) \otimes k(G) \otimes k(G)$ by $\Delta^{(3)}_k(u) = u^i x_i \otimes x_i \otimes x_i$ (derived and associative), then $\Delta^{(3)}_k([uvw]_k) = \left[\Delta^{(3)}_k(u)\, \Delta^{(3)}_k(v)\, \Delta^{(3)}_k(w)\right]_k$, and $k(G)$ is a ternary bialgebra. If we define a ternary antipod by $S^{(3)}_k = u^i \bar{x}_i$, where $\bar{x}_i$ is a skew element of x_i, then $k(G)$ becomes a ternary Hopf algebra. In the dual case of functions $\mathcal{F}(G) : \{\varphi : G \to k\}$ a ternary product $[\,]_{\mathcal{F}}$ or $m^{(3)}_{\mathcal{F}}$ (derived and associative) acts on $\psi(x,y,z)$ as $\left(m^{(3)}_{\mathcal{F}} \psi\right)(x) = \psi(x,x,x)$, and so $\mathcal{F}(G)$ is a ternary algebra. Let $\mathcal{F}(G) \otimes \mathcal{F}(G) \otimes \mathcal{F}(G) \cong \mathcal{F}(G \times G \times G)$, then we define a ternary coproduct $\Delta^{(3)}_{\mathcal{F}} : \mathcal{F}(G) \to \mathcal{F}(G) \otimes \mathcal{F}(G) \otimes \mathcal{F}(G)$ as $\left(\Delta^{(3)}_{\mathcal{F}} \varphi\right)(x,y,z) = \varphi([xyz]_{\mathcal{F}})$, which is derive and associative. Thus we can obtain $\Delta^{(3)}_{\mathcal{F}}([\varphi_1 \varphi_2 \varphi_3]_{\mathcal{F}}) = \left[\Delta^{(3)}_{\mathcal{F}}(\varphi_1)\, \Delta^{(3)}_{\mathcal{F}}(\varphi_2)\, \Delta^{(3)}_{\mathcal{F}}(\varphi_3)\right]_{\mathcal{F}}$, and therefore $\mathcal{F}(G)$ is a ternary bialgebra. If we define a ternary antipod by $S^{(3)}_{\mathcal{F}}(\varphi) = \varphi(\bar{x})$, where $\bar{x}$ is a skew element of x, then $\mathcal{F}(G)$ becomes a ternary Hopf algebra.

Let us introduce a ternary analog of R-matrix. For a ternary Hopf algebra H we consider a linear map $R^{(3)} : H \otimes H \otimes H \to H \otimes H \otimes H$.

Definition 11. A ternary Hopf algebra $\left(H, m^{(3)}, \eta^{(3)}, \Delta^{(3)}, \varepsilon^{(3)}, S^{(3)}\right)$ is called *quasifiveangular* (the reason of such notation is clear from (32)) if it satisfies

$$\left(\Delta^{(3)} \otimes \mathrm{id} \otimes \mathrm{id}\right) = R^{(3)}_{145} R^{(3)}_{245} R^{(3)}_{345}, \tag{29}$$
$$\left(\mathrm{id} \otimes \Delta^{(3)} \otimes \mathrm{id}\right) = R^{(3)}_{125} R^{(3)}_{145} R^{(3)}_{135}, \tag{30}$$
$$\left(\mathrm{id} \otimes \mathrm{id} \otimes \Delta^{(3)}\right) = R^{(3)}_{125} R^{(3)}_{124} R^{(3)}_{123}, \tag{31}$$

where as usual index of R denotes action component positions.

Using the standard procedure (see e.g. [12, 27, 13]), we obtain set of abstract *ternary quantum Yang–Baxter equations*, one of which has the form

$$R_{243}^{(3)}R_{342}^{(3)}R_{125}^{(3)}R_{145}^{(3)}R_{135}^{(3)} = R_{123}^{(3)}R_{132}^{(3)}R_{145}^{(3)}R_{245}^{(3)}R_{345}^{(3)}, \tag{32}$$

and others can be obtained by corresponding permutations. The classical ternary Yang–Baxter equations for one parameter family of solutions $R(t)$ can be obtained by the expansion $R^{(3)}(t) = e\otimes e\otimes e + rt + \mathcal{O}\left(t^2\right)$, where r is a ternary classical R-matrix, then e.g. for (32) we have

$$\begin{aligned} &r_{342}r_{125}r_{145}r_{135} + r_{243}r_{125}r_{145}r_{135} + r_{243}r_{342}r_{145}r_{135} + r_{243}r_{342}r_{125}r_{135} + r_{243}r_{342}r_{125}r_{145} \\ &\qquad = r_{132}r_{145}r_{245}r_{345} + r_{123}r_{145}r_{245}r_{345} + r_{123}r_{132}r_{245}r_{345} \\ &\qquad + r_{123}r_{132}r_{145}r_{345} + r_{123}r_{132}r_{145}r_{245}. \end{aligned}$$

For *three* ternary Hopf algebras

$$\left(H_A, m_A^{(3)}, \eta_A^{(3)}, \Delta_A^{(3)}, \varepsilon_A^{(3)}, S_A^{(3)}\right), \qquad \left(H_B, m_B^{(3)}, \eta_B^{(3)}, \Delta_B^{(3)}, \varepsilon_B^{(3)}, S_B^{(3)}\right) \quad \text{and}$$
$$\left(H_C, m_C^{(3)}, \eta_C^{(3)}, \Delta_C^{(3)}, \varepsilon_C^{(3)}, S_C^{(3)}\right)$$

we can introduce a non-degenerate ternary “pairing” (see e.g. [27] for binary case) $\langle\, , \, , \,\rangle^{(3)} : H_A\times H_B\times H_C \to \mathbb{K}$, trilinear over $\mathbb{K}$, satisfying

$$\begin{aligned}
&\left\langle \eta_A^{(3)}(a), b, c\right\rangle^{(3)} = \left\langle a, \varepsilon_B^{(3)}(b), c\right\rangle^{(3)}, \qquad \left\langle a, \eta_B^{(3)}(b), c\right\rangle^{(3)} = \left\langle \varepsilon_A^{(3)}(a), b, c\right\rangle^{(3)}, \\
&\left\langle b, \eta_B^{(3)}(b), c\right\rangle^{(3)} = \left\langle a, b, \varepsilon_C^{(3)}(c)\right\rangle^{(3)}, \qquad \left\langle a, b, \eta_C^{(3)}(c)\right\rangle^{(3)} = \left\langle a, \varepsilon_B^{(3)}(b), c\right\rangle^{(3)}, \\
&\left\langle a, b, \eta_C^{(3)}(c)\right\rangle^{(3)} = \left\langle \varepsilon_A^{(3)}(a), b, c\right\rangle^{(3)}, \qquad \left\langle \eta_A^{(3)}(a), b, c\right\rangle^{(3)} = \left\langle a, b, \varepsilon_C^{(3)}(c)\right\rangle^{(3)}, \\
&\left\langle m_A^{(3)}(a_1\otimes a_2\otimes a_3), b, c\right\rangle^{(3)} = \left\langle a_1\otimes a_2\otimes a_3, \Delta_B^{(3)}(b), c\right\rangle^{(3)}, \\
&\left\langle \Delta_A^{(3)}(a), b_1\otimes b_2\otimes b_3, c\right\rangle^{(3)} = \left\langle a, m_B^{(3)}(b_1\otimes b_2\otimes b_3), c\right\rangle^{(3)}, \\
&\left\langle a, m_B^{(3)}(b_1\otimes b_2\otimes b_3), c\right\rangle^{(3)} = \left\langle a, b_1\otimes b_2\otimes b_3, \Delta_C^{(3)}(c)\right\rangle^{(3)}, \\
&\left\langle a, \Delta_B^{(3)}(b), c_1\otimes c_2\otimes c_3\right\rangle^{(3)} = \left\langle a, b, m_C^{(3)}(c_1\otimes c_2\otimes c_3)\right\rangle^{(3)}, \\
&\left\langle a, b, m_C^{(3)}(c_1\otimes c_2\otimes c_3)\right\rangle^{(3)} = \left\langle \Delta_A^{(3)}(a), b, c_1\otimes c_2\otimes c_3\right\rangle^{(3)}, \\
&\left\langle a_1\otimes a_2\otimes a_3, b, \Delta_C^{(3)}(c)\right\rangle^{(3)} = \left\langle m_A^{(3)}(a_1\otimes a_2\otimes a_3), b, c\right\rangle^{(3)}, \\
&\left\langle S_A^{(3)}(a), b, c\right\rangle^{(3)} = \left\langle a, S_B^{(3)}(b), c\right\rangle^{(3)} = \left\langle a, b, S_C^{(3)}(c)\right\rangle^{(3)},
\end{aligned}$$

where $a, a_i \in H_A$, $b, b_i \in H_B$. The ternary “paring” between $H_A\otimes H_A\otimes H_A$ and $H_B\otimes H_B\otimes H_B$ is given by $\langle a_1\otimes a_2\otimes a_3, b_1\otimes b_2\otimes b_3\rangle^{(3)} = \langle a_1, b_1\rangle^{(3)}\langle a_2, b_2\rangle^{(3)}\langle a_3, b_3\rangle^{(3)}$. These constructions can naturally lead to ternary generalization of duality concept and quantum double [14, 12, 13].

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