Communications in Algebra®, 34: 1651–1670, 2006

ISSN: 0092-7872 print/1532-4125 online
DOI: 10.1080/00927870500542564

BI-ELEMENT REPRESENTATIONS OF TERNARY GROUPS#

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General properties of ternary semigroups and groups are considered. The bi-element representation theory in which every representation matrix corresponds to a pair of elements is built up. Connections with the standard theory are considered and several concrete examples are constructed. For the sake of clarity and completeness the shortened versions of classical Gluskin-Hosszú and Post theorems are also presented.

Key Words: Matrix representation; Ternary group.

2000 Mathematics Subject Classification: 20N15; 20C99; 20G42; 81R05.

1. INTRODUCTION

Ternary and n-ary generalizations of algebraic structures are the most natural ways for further development and deeper understanding of their fundamental properties. Firstly, ternary algebraic operations were introduced already in the nineteenth century by A. Cayley. As the development of Cayley's ideas it were considered n-ary generalization of matrices and their determinants (Kapranov et al., 1994; Sokolov, 1972) and general theory of n-ary algebras (Carlsson, 1976; Lawrence, 1992; Pojidaev, 2003), n-group rings (Zeković and Artamonov, 1992, 1999, 2002), and ternary rings (Benz and Ghalieh, 1998; Lister, 1971). For some physical applications in Nambu mechanics, supersymmetry, the Yang–Baxter equation, etc., see e.g., Kerner (2000) and Vainerman and Kerner (1996). From another side, Hopf algebras (Abe, 1980), and their generalizations (Li and Duplij, 2002; Nikshych and Vainerman, 2002) play a basic role in the quantum group

Received April 15, 2004; Revised October 15, 2004. Communicated by V. A. Artamonov.
#Dedicated to the memory of our friends Władysław Marcinek (1953–2003) and Kazimierz Glazek (1939–2005).
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theory (see e.g., Demidov, 1998; Kassel, 1995; Shnider and Sternberg, 1993). The statement here is something different and related to our previous preliminary report (Borowiec et al., 2001), where ternary algebras and ternary Hopf algebras were considered.

The notion of an n-ary group was introduced in 1928 by Dörnte (1929) (inspired by E. Nöther) and is a natural generalization of the notion of a group and a ternary group considered by Certaine (1943) and Kasner (1904).

The (binary) group theory representations were introduced as a matrix realization of group elements and abstract group action by usual matrix multiplication, when one element was described by one matrix. In this article we propose a new bi-element approach to the representation theory of ternary groups, when one matrix parametrizes two elements of a ternary group. Our approach is motivated by theory of modules over ternary algebras (Carlsson, 1976). In this framework, we can preserve many standard notions known from representation theory of binary groups, e.g., irreducibility, unitarity, regular representations, Schur Lemma.

An alternative approach to ternary group representations was made in Wanke-Jakubowska and Wanke-Jerie (1984), where they proposed—instead of our operator-valued functions of two variables—functions of one variable taking value in a pair of operators (matrices), viz. $\Pi^{wan} : G \to (\Pi_1^{wan}(x), \Pi_2^{wan}(x)) \in GL(V) \times GL(V)$ with another analog of homomorphism. Unfortunately, in Wanke-Jakubowska and Wanke-Jerie (1984) there were given no concrete examples connections with the derived case. This explores an idea presented in the seminal article (Post, 1940).

Here, using our method, we present several concrete examples and consider connection with binary case by means of the classical Gluskin-Hosszú and Post theorems.

2. TERNARY SEMIGROUPS

In the first two chapters, which have preliminary character, we review some basic definitions and properties related to ternary groups and semigroups (cf. also Belousov, 1972; Rusakov, 1998).

Definition 1. A nonempty set G with one *ternary* operation $[\,] : G \times G \times G \to G$ is called a *ternary groupoid* and is denoted by $(G, [\,])$.

If on G there is a binary operation $\odot$ such that

$$[xyz] = (x \odot y) \odot z \tag{1}$$

for all $x, y, z \in G$, then we say that $[\,]$ is *derived* from $\odot$ and denote this fact by $(G, [\,]) = \mathrm{der}(G, \odot)$. If

$$[xyz] = ((x \odot y) \odot z) \odot b$$

holds for all $x, y, z \in G$ and some fixed $b \in G$, then a groupoid $(G, [\,]$ is *b-derived* from $(G, \odot)$. In this case we write $(G, [\,]) = \mathrm{der}_b(G, \odot)$ (cf. Dudek and Michalski, 1982, 1984).

We say that $(G, [\ ]$ is a *ternary semigroup* if the operation $[\ ]$ is *associative*, i.e., if

$$[[xyz]uv] = [x[yzu]v] = [xy[zuv]] \tag{2}$$

holds for all $x, y, z, u, v \in G$.

Obviously, a ternary operation derived from a binary associative operation is also associative in the above sense, but a ternary operation which is b-derived from an associative operation $\odot$ can be associative in the above sense only in some cases—for example, in the case when b will be in the center of $(G, \odot)$. But this condition is just sufficient. For instance, let A be a nilpotent associative algebra of index 7 (the product of any 7 elements is zero), then for any $b \in A$ (not necessarily in the center), the ternary operation $[xyz] = ((xy)z)b$ is trivially associative since $[[xyz]uv] = [x[yzu]v] = [xy[zuv]] = 0$ for any $x, y, z, u, v \in A$, as the products involve the multiplication in A of 7 elements, including two b's.

Fixing one element in a ternary operation we obtain a binary operation.

Definition 2. A binary groupoid $(G, \odot)$, where $x \odot y = [xay]$ for some fixed $a \in G$ is called a *retract* of $(G, [\ ])$ and is denoted by $\mathrm{ret}_a(G, [\ ])$.

In some special cases, described in Dudek and Michalski (1982, 1984), we have $(G, \odot) = \mathrm{ret}_a(\mathrm{der}_b(G, \odot))$ and $(G, \odot) = \mathrm{ret}_c(\mathrm{der}_d(G, \odot))$, but in general $(G, \odot)$ and $\mathrm{ret}_a(\mathrm{der}_b(G, \odot))$ are only isomorphic (Dudek and Michalski, 1984).

Lemma 3. *If in the ternary semigroup $(G, [\ ])$ there exists an element e such that for all $y \in G$ we have $[eye] = y$, then this semigroup is derived from the binary semigroup $\mathrm{ret}_e(G, [\ ])$, i.e. $(G, [\ ]) = \mathrm{der}(\mathrm{ret}_e(G, [\ ]))$.*

Proof. Indeed, if we put $x \circledast y = [xey]$, then $(x \circledast y) \circledast z = [[xey]ez] = [x[eye]z] = [xyz]$ and $x \circledast (y \circledast z) = [xe[yez]] = [x[eye]z] = [xyz]$, which completes the proof. $\square$

The same ternary semigroup $(G, [\ ])$ can be derived from two different (but isomorphic) semigroups $(G, \circledast)$ and $(G, \diamond)$. Indeed, if in G there exists $a \neq e$ such that $[aya] = y$ for all $y \in G$, then by the same argumentation we obtain $[xyz] = x \diamond y \diamond z$ for $x \diamond y = [xay]$. In this case for $\varphi(x) = x \diamond e = [xae]$ we have

$$\varphi(x \circledast y) = [[xey]ae] = [[x[aea]y]ae] = [[xae]a[yae]] = \varphi(x) \diamond \varphi(y).$$

Thus φ is a binary homomorphism such that $\varphi(e) = a$. Moreover, for $\psi(x) = [eax]$ we have

$$\psi(\varphi(x)) = [ea[xae]] = [e[axa]e] = x,$$
$$\varphi(\psi(x)) = [[eax]ae] = [e[axa]e] = x$$

and

$$\psi(x \diamond y) = [ea[xay]] = [ea[x[eae]y]] = [[eax]e[aey]] = \psi(x) \circledast \psi(y).$$

Hence semigroups $(G, \circledast)$ and $(G, \diamond)$ are isomorphic.

Definition 4. An element $e \in G$ is called a *middle identity* or a *middle neutral element* of $(G, [\ ])$, if for all $x \in G$ we have

$$[exe] = x \tag{3}$$

An element $e \in G$ satisfying the identity

$$[eex] = x \tag{4}$$

is called a *left identity* or a *left neutral element* of $(G, [\ ])$. Similarly, we define a *right identity*. An element which is a left, middle, and right identity is called a *ternary identity* (or simply *identity*).

There are ternary semigroups without left (middle, right) neutral elements, but there are also ternary semigroups in which all elements are identities.

Example 1. In a ternary semigroup derived from the symmetric group S_3, all elements of order 2 are left and right (but no middle) identities.

Example 2. In a ternary semigroup derived from the Boolean group, all elements are ternary identities, but a ternary semigroup 1-derived from the additive group $\mathbb{Z}_4$ has no left (right, middle) identities.

Lemma 5. *For any ternary semigroup $(G, [\ ])$ with a left (right) identity, there exists a binary semigroup $(G, \odot)$ and its endomorphism μ such that*

$$[xyz] = x \odot \mu(y) \odot z$$

for all $x, y, z \in G$.

Proof. Let e be a left identity of $(G, [\ ])$. It is not difficult to see that the operation $x \odot y = [xey]$ is associative. Moreover, for $\mu(x) = [exe]$, we have

$$\mu(x) \odot \mu(y) = [[exe]e[eye]] = [[exe][eey]e] = [e[xey]e] = \mu(x \odot y)$$

and

$$[xyz] = [x[eey][eez]] = [[xe[eye]]ez] = x \odot \mu(y) \odot z.$$

In the case of right identity the proof is analogous. □

Definition 6. A ternary groupoid $(G, [\ ])$ is called *σ-commutative*, if

$$[x_1x_2x_3] = [x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}] \tag{5}$$

holds for all $x_1, x_2, x_3 \in G$ and all $\sigma \in S_3$. If (5) holds for all $\sigma \in S_3$, then $(G, [\ ])$ is a *commutative* groupoid. If (5) holds only for $\sigma = (13)$, i.e., if $[x_1x_2x_3] = [x_3x_2x_1]$, then $(G, [\ ])$ is called *semicommutative*.

The group S_3 is generated by two transpositions: $(1\,2)$ and $(2\,3)$. This means that $(G, [\ ])$ is commutative if and only if $[xyz] = [yxz] = [xzy]$ holds for all $x, y, z \in G$.

Proposition 7. *If in a ternary semigroup $(G, [\ ])$ satisfying the identity $[xyz] = [yxz]$ there are a, b such that $[axb] = x$ for all $x \in G$, then $(G, [\ ])$ is commutative.*

Proof. According to the above remark, it is sufficient to prove that $[xyz] = [xzy]$. We have

$$[xyz] = [a[xyz]b] = [ax[yzb]] = [ax[zyb]] = [a[xzy]b] = [xzy]. \qquad \square$$

A groupoid $(G, \odot)$ is called *medial* if it satisfies the identity

$$(x \odot y) \odot (z \odot u) = (x \odot z) \odot (y \odot u). \tag{6}$$

This can be presented as a matrix $A^{(2)} = \left(\begin{smallmatrix} x & y \\ z & u \end{smallmatrix}\right)$, read from left by rows and from top by columns as $\left(\begin{smallmatrix} & \Downarrow & \Downarrow \\ \Rightarrow & x & y \\ \Rightarrow & z & u \end{smallmatrix}\right)$ (see Belousov, 1972). For binary groups the mediality coincides with the commutativity.

In the ternary case, instead of $A^{(2)}$ we have 3×3 matrix $A^{(3)}$, which should be read similarly.

Definition 8. A ternary groupoid $(G, [\ ])$ is *medial* if it satisfies the identity

$$[[x_{11}x_{12}x_{13}][x_{21}x_{22}x_{23}][x_{31}x_{32}x_{33}]] = [[x_{11}x_{21}x_{31}][x_{12}x_{22}x_{32}][x_{13}x_{23}x_{33}]].$$

It is not difficult to see that a semicommutative ternary semigroup is medial. Note by the way, that the definition of a medial ternary groupoid is the same as a notion of an Abelian operation in the sense of Kurosh (1974). Hence in some articles, for example in Głazek and Gleichgewicht (1977), medial ternary groupoids are called *Abelian*.

An element x such that $[xxx] = x$ is called an *idempotent*. A groupoid in which all elements are idempotents is called an *idempotent groupoid*. A left (right, middle) identity is an idempotent.

3. TERNARY GROUPS

Definition 9. A ternary semigroup $(G, [\ ])$ is a *ternary group* if for all $a, b, c \in G$, there are $x, y, z \in G$ such that

$$[xab] = [ayb] = [abz] = c. \tag{7}$$

One can prove (Post, 1940) that elements x, y, z are uniquely determined. Moreover, according to the suggestion of Post (1940) one can prove (cf. Dudek et al., 1977) that in the above definition, under the assumption of the associativity, it suffices only to postulate the existence of a solution of $[ayb] = c$, or equivalently, of $[xab] = [abz] = c$.

In a ternary group, the equation $[xxz] = x$ has a unique solution which is denoted by $z = \bar{x}$ and called the *skew element* to x (cf. Dörnte, 1929). As a consequence of results obtained in Dörnte (1929) we have the following theorem.

Theorem 10. *In any ternary group* $(G, [\,])$ *for all* $x, y, z \in G$, *the following identities take place*:

$$\begin{aligned}
[x\,x\,\bar{x}] &= [x\,\bar{x}\,x] = [\bar{x}\,x\,x] = x,\\
[y\,x\,\bar{x}] &= [y\,\bar{x}\,x] = [x\,\bar{x}\,y] = [\,\bar{x}\,x\,y] = y,\\
\overline{[x\,y\,z]} &= [\bar{z}\,\bar{y}\,\bar{x}],\\
\bar{\bar{x}} &= x.
\end{aligned}$$

Other properties of skew elements are described in Dudek (1993) and Dudek and Dudek (1981). Since in an idempotent ternary group $\bar{x} = x$ for all x, an idempotent ternary group is semicommutative. From the results obtained in Dudek et al. (1977) (see also Dudek, 1980) for $n = 3$ we have the following theorem.

Theorem 11. *A ternary semigroup* $(G, [\,])$ *is a ternary group with* $\bar{x}$ *being the skew element of* x *for any* $x \in G$, *if and only if it satisfies the identities*

$$[yx\,\bar{x}\,] = [x\,\bar{x}\,y] = y.$$

Corollary 12. *A ternary semigroup* $(G, [\,])$ *is an idempotent ternary group if and only if it satisfies the identities*

$$[yxx] = [xxy] = y.$$

Remark 1. The set $S_3 \backslash A_3$ of all odd permutations with the ternary operation $[\,]$ defined as a composition of three permutations is an example of a noncommutative ternary group which is not derived from any group (all groups with three elements are commutative and isomorphic to $\mathbb{Z}_3$).

From results proven in Dudek (1980) follows the following theorem.

Theorem 13. *A ternary group* $(G, [\,])$ *satisfying the identity*

$$[xy\bar{x}] = y \quad \text{or} \quad [\bar{x}yx] = y$$

is commutative.

The subsequent two theorems play a very important role in theory of ternary groups and shall be used later on. They interrelate ternary and binary groups, and provide constructive methods for theory of ternary groups. These theorems are also valid for arbitrary n-ary groups, but the original proofs are rather complicated (cf. Post, 1940). Here, for the sake of completeness, we present much shorter and more elegant proofs which are modifications of proofs presented already in Dudek and Michalski (1982) and Sokolov (1976) for a more general case.

Theorem 14 (Gluskin-Hosszú). *For a ternary group* $(G, [\ ])$ *and a fixed element* $a \in G$, *there exist a binary group* $(G, \circledast) = \mathrm{ret}_a(G, [\ ])$ *and its automorphism* φ *such that* $\varphi(b) = b$, $\varphi^2(x) \circledast b = b \circledast x$ *and*

$$[xyz] = x \circledast \varphi(y) \circledast \varphi^2(z) \circledast b, \tag{8}$$

where $b = [\,\bar{a}\,\bar{a}\,\bar{a}\,]$.

Proof. Let $a \in G$ be fixed. Then the binary operation $x \circledast y = [x\,a\,y]$ is associative, because

$$(x \circledast y) \circledast z = [[xay]az] = [xa[yaz]] = x \circledast (y \circledast z).$$

In $(G, \circledast)$ an element $\bar{a}$ is the identity, $[\bar{a}\,\bar{x}\,\bar{a}]$ inverse of x. $\varphi(x) = [\bar{a}\ \ x\ \ a]$ is an automorphism of $(G, \circledast)$. The easy calculation proves that the above formula holds for $b = [\bar{a}\,\bar{a}\,\bar{a}]$. □

One can prove that the group $(G, \circledast)$ is unique up to isomorphism (Dudek and Michalski, 1982). This means that a ternary groupoid b-derived from a binary group $(G, \circledast)$ is a ternary group if and only if b lies in the center of $(G, \circledast)$.

From the proof of Theorem 3 in Głazek and Gleichgewicht (1977), it follows that any medial ternary group satisfies the identity

$$\overline{[xyz]} = [\bar{x}\,\bar{y}\,\bar{z}],$$

which together with our previous results shows that in such groups we have

$$[\bar{x}\,\bar{y}\,\bar{z}] = [\bar{z}\,\bar{y}\,\bar{x}].$$

But $\bar{\bar{x}} = x$. Hence, any medial ternary group is semicommutative, thus any retract of such group is a commutative group. Moreover, for φ from the proof of Theorem 14, we have $\varphi(b) = b$ for $b = [\bar{a}\,\bar{a}\,\bar{a}]$ and

$$\varphi(\varphi(x)) = [\bar{a}[\bar{a}xa]a] = [\bar{a}a[x\bar{a}a]] = x.$$

Thus, as a consequence of Theorem 14, we obtain the following corollary.

Corollary 15. *Any medial ternary group* $(G, [\ ])$ *has the form*

$$[x\,y\,z] = x \odot \varphi(y) \odot z \odot b,$$

where $(G, \odot)$ *is a commutative group,* φ *its automorphism such that* $\varphi^2 = \mathrm{id}$ *and* $b \in G$ *is fixed.*

Corollary 16. *A ternary group is medial, if and only if it is semicommutative.*

Corollary 17. *A ternary group is semicommutative* (*medial*), *if and only if there exists* $a \in G$ *such that* $[x\,a\,y] = [y\,a\,x]$ *holds for all* $x, y \in G$.

Corollary 18. *A commutative ternary group is b-derived from some commutative group.*

Indeed, $\varphi(x) = [\bar{a}\, x\, a] = [x\, a\, \bar{a}] = x$.

Theorem 19 (Post, 1940). *For any ternary group $(G, [\ ])$ there exists a binary group $(G^*, \circledast)$ and its normal subgroup $G_0 \lhd G^*$, such that $G^*/G_0 \simeq \mathbb{Z}_2$ and*

$$[x\, y\, z] = x \circledast y \circledast z$$

for all $x, y, z \in G$.

Proof. Let c be a fixed element in G and let $G^* = G \times \mathbb{Z}_2$. In G^*, we define the binary operation $\circledast$ putting

$$\begin{aligned}
(x, 1) \circledast (y, 1) &= ([x\, y\, \bar{c}], 0) \\
(x, 1) \circledast (y, 0) &= ([x\, y\, c], 1) \\
(x, 0) \circledast (y, 1) &= ([x\, c\, y], 1) \\
(x, 0) \circledast (y, 0) &= ([x\, c\, y], 0).
\end{aligned}$$

It is not difficult to see that this operation is associative and $(\bar{c}, 0)$ is its neutral element. The inverse element (in G^*) has the form

$$(x, 1)^{-1} = (\bar{x}, 1), \qquad (x, 0)^{-1} = ([\bar{c}\, \bar{x}\, \bar{c}], 0)$$

Thus G^* is a group such that $G_0 = \{(x, 0) : x \in G\} \lhd G^*$. Obviously the set G can be identified with $G_1 = G \times \{1\}$ and

$$\begin{aligned}
x \circledast y \circledast z &= ((x, 1) \circledast (y, 1)) \circledast (z, 1) = ([x\, y\, \bar{c}], 0) \circledast (z, 1) \\
&= ([[x\, y\, \bar{c}]c\, z], 1) = ([x\, y[\bar{c}\, c\, z]], 1) = ([x\, y\, z], 1) = [x\, y\, z],
\end{aligned}$$

which completes the proof. □

The group G^* is called *covering* for the ternary group $(G, [\ ])$. In this way $(G, [\ ])$ becomes a ternary subgroup in $\mathrm{der}(G^*, \circledast)$.

The original proof of this theorem uses equivalence relation on sequences of elements from G (see Post, 1940 and below) and is given for n-ary group ($n \geq 3$). Our proof is a new and useful modification of some general method presented in Michalski (1979). Our construction gives in the explicit form a free covering group in the sense of universal algebras and is more applicable to concrete calculations (see Section 6). Moreover, this construction gives the gradiation of G^*, i.e., $G_i \circledast G_j \subseteq G_{(j+j) \bmod 2}$, and will be used in our future work on the construction of a covering algebra for the ternary algebra.

From results obtained in Dudek and Michalski (1984) one has the following proposition.

Proposition 20. *All retracts of a ternary group* $(G, [\ ])$ *are isomorphic to the normal subgroup* G_0 *of* G^* *from the previous theorem, i.e.,*

$$\mathrm{ret}_a(G, [\ ]) \simeq G_0 \lhd G^*.$$

From the binary point of view, one has the exact sequence of groups

$$1 \to G_0 \to G^* \to \mathbb{Z}_2 \to 1.$$

This implies that covering group G^* of a ternary (n-ary) group $(G, [\ ])$ is, in fact, an extension of the cyclic group $\mathbb{Z}_2$ ($\mathbb{Z}_{n-1}$) by the retract $\mathrm{ret}_c(G, [\ ]) \simeq G_0$.

But the correspondence (up to isomorphism) between ternary groups, their retracts and covering groups is not one-to-one, e.g., the group G^* cannot be obtained, in general, from G_0 and $\mathbb{Z}_2$. For example, on the set $G = \{0, 1, 2\}$ we can define two non-isomorphic ternary groups: $(G, [\ ]) = \mathrm{der}(\mathbb{Z}_3, +)$ and $(G, \langle\ \rangle)$, where $\langle xyz \rangle = (x - y + z) \bmod 3$. Both groups (the first is commutative, the second one "only" semicommutative) have the same retract $\mathbb{Z}_3$. But the covering group of the first is $\mathbb{Z}_6$, while of the second one is S_3. Of course, $\mathbb{Z}_6 / \mathbb{Z}_3 \simeq S_3 / \mathbb{Z}_3 \simeq \mathbb{Z}_2$. More generally, one has the following corollary.

Corollary 21. *Let* $(G, [\ ]) = \mathrm{der}(G, \odot)$ *be a ternary group derived from a binary group* $(G, \odot)$. *Thus* G^* *is isomorphic to the direct product of* $(G, \odot)$ *with* $\mathbb{Z}_2$. *The converse statement is also true.*

4. REPRESENTATIONS OF TERNARY GROUPS

For a given ternary group $(G, [\ ])$ denote by $(G \times G, *)$ a semigroup with the binary multiplication

$$(x, y) * (u, v) = ([x\, y\, u], v). \tag{9}$$

Obviously, for all $x, u, v \in G$ we have $(x, \bar{x}) * (u, v) = (\bar{x}, x) * (u, v) = (u, v)$, which means that $(x, \bar{x})$ and $(\bar{x}, x)$ are left (but not right) unities in $(G \times G, *)$. Generally, $(x, \bar{x}) \neq (\bar{x}, x)$. But for all $x, y \in G$ we have also $(x, y) * (\bar{y}, y) = (\bar{y}, y) * (x, y) = (x, y)$, i.e., each element (x, y) has a "private" unit. Moreover, any element $(u, \bar{u})$, $u \in G$ is a left unit.

The semigroup $(G \times G, *)$ is left (but not right) cancellative, i.e., $(a, b) * (x, y) = (a, b) * (c, d)$ implies $(x, y) = (c, d)$. Moreover, $(G \times G, *)$ is also a right quasigroup, i.e., for every $(a, b), (c, d) \in G \times G$ there exists only one $(x, y) \in G \times G$ such that $(a, b) * (x, y) = (c, d)$. Similarly, it is not difficult to see that for each a, b, c, $d \in G$ there are uniquely determined $x, y \in G$ such that $(x, a) * (b, c) = (a, y) * (b, c) = (d, c)$.

Let V be a complex vector space and $\mathrm{End}\, V$ denote a set of $\mathbb{C}$-linear endomorphisms of V.

Definition 22. A *left representation* of a ternary group $(G, [\ ])$ in a vector space V is a map $\Pi^L : G \times G \to \mathrm{End}\, V$ such that

$$\Pi^L(x_1, x_2) \circ \Pi^L(x_3, x_4) = \Pi^L([x_1 x_2 x_3], x_4), \qquad \forall x_1, x_2, x_3, x_4 \in G, \tag{10}$$

$$\Pi^L(x, \bar{x}) = \mathrm{id}_V, \qquad \forall x \in G \tag{11}$$

Replacing in (11) x by $\bar{x}$ we obtain $\Pi^L(\bar{x}, x) = \mathrm{id}_V$, which means that in fact (11) has the form $\Pi^L(\bar{x}, x) = \Pi^L(x, \bar{x}) = \mathrm{id}_V, \forall x \in G$. Note that the axioms considered in the above definition are the natural ones satisfied by left multiplications $x \mapsto [a\,b\,x]$.

Lemma 23. *For all $x_1, x_2, x_3, x_4 \in G$ we have*

$$\Pi^L([x_1x_2x_3], x_4) = \Pi^L(x_1, [x_2x_3x_4]).$$

Proof. Indeed, we have

$$\begin{aligned}\Pi^L([x_1x_2x_3], x_4) &= \Pi^L([x_1x_2x_3], x_4) \circ \Pi^L(x, \bar{x}) \\ &= \Pi^L([[x_1x_2x_3]x_4\,x], \bar{x}) = \Pi^L([x_1[x_2x_3x_4]x], \bar{x}) \\ &= \Pi^L(x_1, [x_2x_3x_4]) \circ \Pi^L(x, \bar{x}) = \Pi^L(x_1, [x_2x_3x_4]). \qquad \square\end{aligned}$$

Note also that for all $x, y, z \in G$, we have

$$\Pi^L(x, y) = \Pi^L([x\,z\,\bar{z}], y) = \Pi^L(x, z) \circ \Pi^L(\bar{z}, y) \tag{12}$$

and

$$\Pi^L(x, z) \circ \Pi^L(\bar{z}, \bar{x}) = \Pi^L(\bar{z}, \bar{x}) \circ \Pi^L(x, z) = \mathrm{id}_V, \tag{13}$$

i.e., every $\Pi^L(x, z)$ is invertible and $(\Pi^L(x, z))^{-1} = \Pi^L(\bar{z}, \bar{x})$. This means that any left representation gives a representation of a ternary group by a binary group.

Moreover, if a ternary group $(G, [\ ])$ is medial, then

$$\Pi^L(x_1, x_2) \circ \Pi^L(x_3, x_4) = \Pi^L(x_3, x_4) \circ \Pi^L(x_1, x_2),$$

i.e., the so obtained group is commutative. Indeed, by Corollary 17, we have

$$\begin{aligned}\Pi^L(x_1, x_2) \circ \Pi^L(x_3, x_4) &= \Pi^L(x_1, x_2) \circ \Pi^L(x_3, x_4) \circ \Pi^L(x, \bar{x}) \\ &= \Pi^L([[x_1x_2x_3]x_4\,x], \bar{x}) = \Pi^L([[x_3x_4x_1]x_2\,x],\ \bar{x}) \\ &= \Pi^L(x_3, x_4) \circ \Pi^L(x_1, x_2) \circ \Pi^L(x, \bar{x}) \\ &= \Pi^L(x_3, x_4) \circ \Pi^L(x_1, x_2).\end{aligned}$$

If $(G, [\ ])$ is commutative, then also $\Pi^L(x, y) = \Pi^L(y, x)$, because

$$\begin{aligned}\Pi^L(x, y) &= \Pi^L(x, y) \circ \Pi^L(x, \bar{x}) = \Pi^L([x\,y\,x], \bar{x}) \\ &= \Pi^L([y\,x\,x], \bar{x}) = \Pi^L(y, x) \circ \Pi^L(x, \bar{x}) = \Pi^L(y, x).\end{aligned}$$

Thus in the case of commutative and idempotent ternary groups any left representation is idempotent and, consequently, $(\Pi^L(x, y))^{-1} = \Pi^L(x, y)$. This means that commutative and idempotent ternary groups are represented by Boolean groups.

Proposition 24. *Let* $(G, [\ ]) = \mathrm{der}(G, \odot)$ *be a ternary group derived from a binary group* $(G, \odot)$. *There is one-to-one correspondence between representations of* $(G, \odot)$ *and left representations of* $(G, [\ ])$.

Proof. Because $(G, [\ ]) = \mathrm{der}(G, \odot)$, then $x \odot y = [xey]$ and $\bar{e} = e$, where e is unity of the binary group $(G, \odot)$. If $\pi \in \mathrm{Rep}(G, \odot)$, then (as it is not difficult to see) $\Pi^L(x, y) = \pi(x) \circ \pi(y)$ is a left representation of $(G, [\ ])$. Conversely, if Π^L is a left representation of $(G, [\ ])$, then $\pi(x) = \Pi^L(x, e)$ is a representation of $(G, \odot)$. Moreover, in this case $\Pi^L(x, y) = \pi(x) \circ \pi(y)$. Indeed, by Lemma 23, we have

$$\Pi^L(x, y) = \Pi^L(x, [eye]) = \Pi^L([xey], e) = \Pi^L(x, e) \circ \Pi^L(y, e) = \pi(x) \circ \pi(y)$$

for all $x, y \in G$. $\square$

In a similar manner, we can introduce right representations. Let $(G, [\ ])$ be a ternary group. On $G \times G$ we define the following binary operation

$$(x, y) \diamond (u, v) = (u, [v\, x\, y]).$$

Then $(G \times G, \diamond)$ is a binary semigroup which is isomorphic to $(G \times G, *)$. This isomorphism has the form $\varphi((x, y)) = (\bar{y}, \bar{x})$. Indeed,

$$\begin{aligned}\varphi((x, y) \diamond (u, v)) &= \varphi((u, [vxy])) = (\overline{[vxy]}, \bar{u}) \\ &= ([\bar{y}, \bar{x}, \bar{v}], \bar{u}) = (\bar{y}, \bar{x}) * (\bar{v}, \bar{u}) = \varphi((x, y)) * \varphi((u, v)).\end{aligned}$$

Based on this construction we can define the following.

Definition 25. A *right representation* of a ternary group $(G, [\ ])$ in V is a map $\Pi^R : G \times G \to \mathrm{End}\, V$ such that

$$\Pi^R(x_3, x_4) \circ \Pi^R(x_1, x_2) = \Pi^R(x_1, [x_2 x_3 x_4]), \qquad \forall x_1, x_2, x_3, x_4 \in G, \tag{14}$$

$$\Pi^R(x, \bar{x}) = \mathrm{id}_V, \qquad \forall x \in G. \tag{15}$$

From (14)–(15) it follows that

$$\Pi^R(x, y) = \Pi^R(x, [z\, \bar{z}\, y]) = \Pi^R(\bar{z}, y) \circ \Pi^R(x, z) \qquad \forall\, x, y, z \in G. \tag{16}$$

It is easy to check that $\Pi^R(x, y) = \Pi^L(\bar{y}, \bar{x}) = (\Pi^L(x, y))^{-1}$. So it is enough to consider only left representations (as in binary case).

Example 3. Let G be a ternary group and $\mathbb{C}G$ denote a vector space spanned by G. It means that any element u of $\mathbb{C}G$ can be uniquely presented in the form $u = \sum_{i=1}^{n} k_i y_i$, with $k_i \in \mathbb{C}$, $y_i \in G$, $n \in N$ (we do not assume that G has finite rank). Moreover, $\mathbb{C}G$ is a ternary (group) algebra. Then left and right regular representations can be immediately defined by means of structure

$$\Pi^L_{reg}(x_1, x_2)u = \sum_{i=1}^{n} k_i [x_1 x_2 y_i], \tag{17}$$

$$\Pi^R_{reg}(x_1, x_2)u = \sum_{i=1}^{n} k_i [y_i x_1 x_2]. \tag{18}$$

We would like to point out that the most general n-ary group rings have been systematically investigated in Zeković and Artamonov (1992, 1999, 2002). It can be easy checked that both regular representations do commute

$$\Pi^L_{reg}(x_1, y_1) \circ \Pi^R_{reg}(x_2, y_2) = \Pi^R_{reg}(x_2, y_2) \circ \Pi^L_{reg}(x_1, y_1).$$

Proposition 26. *For a finite* (*or countable*) *ternary group* $(G, [\ ])$ *left and right, regular representations are unitary.*

Proof. Take unitary scalar product $\langle\, , \rangle$ in $\mathbb{C}G$ which makes G an orthonormal basis, i.e., $\langle g, h\rangle = \delta_{g,h}$. Then the unitarity follows from the uniqueness of solutions to the group equations $[xyg] = h$ (see (7)). □

5. MIDDLE REPRESENTATIONS

Now we define another type of representations. For a given ternary group $(G, [\ ])$, we define on $G \times G^{op}$, where G^{op} is a ternary group having opposite multiplication, the following ternary operation $\langle\ \rangle$ putting

$$\langle (x_1, y_1), (x_2, y_2), (x_3, y_3)\rangle = ([x_1x_2x_3], [y_3y_2y_1]) \tag{19}$$

for all $x_1, x_2, x_3, y_1, y_2, y_3 \in G$. It is not difficult to see that $(G \times G, \langle\ \rangle)$ is a ternary group as a direct product of ternary groups. This group is commutative (medial, idempotent), if and only if $(G, [\ ])$ is commutative (respectively, medial, idempotent). It is clear that

$$\langle (x, y), (\bar{x}, \bar{y}), (a, b)\rangle = \langle (a, b), (x, y), (\bar{x}, \bar{y})\rangle = (a, b),$$
$$\langle (\bar{x}, \bar{y}), (x, y), (a, b)\rangle = \langle (a, b), (\bar{x}, \bar{y}), (x, y)\rangle = (a, b)$$

for all $x, y, a, b \in G$. This means that in the group $(G \times G, \langle\ \rangle)$, the element skew to (x, y) has the form $(\bar{x}, \bar{y})$, where $\bar{x}$ is skew to x in $(G, [\ ])$.

Using (19) we construct the middle representations as follows.

Definition 27. A *middle representation* of a ternary group $(G, [\ ])$ in V is a map $\Pi^M : G \times G \to \operatorname{End} V$ such that

$$\Pi^M(x_3, y_3) \circ \Pi^M(x_2, y_2) \circ \Pi^M(x_1, y_1)$$
$$= \Pi^M([x_3x_2x_1], [y_1y_2y_3]), \qquad \forall x_1, x_2, x_3, y_1, y_2, y_3 \in G, \tag{20}$$
$$\Pi^M(x, y) \circ \Pi^M(\bar{x}, \bar{y}) = \Pi^M(\bar{x}, \bar{y}) \circ \Pi^M(x, y) = \mathrm{id}_V \qquad \forall x, y \in G. \tag{21}$$

It is seen that a middle representation is a ternary group homomorphism $\Pi^M : G \times G^{op} \to \operatorname{der}(\operatorname{End} V)$. Note that instead of (21) one can use $\Pi^M(x, \bar{y}) \circ \Pi^M(\bar{x}, y) = id_V$ after changing x to $\bar{x}$ and taking into account that $x = \bar{\bar{x}}$.

Remark 2. In the case of the idempotent elements x and y, we have $\Pi^M(x, y) \circ \Pi^M(x, y) = \mathrm{id}_V$, which means that the matrices Π^M are Boolean. Thus all middle representation matrices of idempotent ternary groups are Boolean.

In general, the composition $\Pi^M(x_1, y_1) \circ \Pi^M(x_2, y_2)$ is not a middle representation, but the following proposition holds.

Proposition 28. *Let Π^M be a middle representation of a ternary group $(G, [\ ])$, then for any fixed $z \in G$ the following holds true*:

1. $\Pi^L_z(x, y) = \Pi^M(x, z) \circ \Pi^M(y, \bar{z})$ *is a left representation of* $(G, [\ ])$;
2. $\Pi^R_z(x, y) = \Pi^M(z, y) \circ \Pi^M(\bar{z}, x)$ *is a right representation of* $(G, [\ ])$.

Proof. The proof is a verification of the corresponding axioms. □

Corollary 29. *If a middle representation Π^M of a ternary group $(G, [\ ])$ satisfies the condition $\Pi^M(x, \bar{x}) = \mathrm{id}_V$ for all $x \in G$, then Π^M is at the same time a left (and right) representation. Moreover, $\Pi^M(x, y) = \Pi^M(y, x)$ for all $x, y \in G$.*

Proof. We consider the case of left representation only. First we notice

$$\begin{aligned}\Pi^M(x, y) &= \Pi^M([xy\bar{y}], [y\bar{z}z]) \\ &= \Pi^M(x, z) \circ \Pi^M(y, \bar{z}) \circ \Pi^M(\bar{y}, y) = \Pi^M(x, z) \circ \Pi^M(y, \bar{z}).\end{aligned}$$

Thus

$$\begin{aligned}\Pi^M(x, y) \circ \Pi^M(u, v) &= \Pi^M(x, z) \circ \Pi^M(y, \bar{z}) \circ \Pi^M(u, z) \circ \Pi^M(v, \bar{z}) \\ &= \Pi^M([xyu], z) \circ \Pi^M(v, \bar{z}) = \Pi^M([xyu], v).\end{aligned}$$

Finally,

$$\Pi^M(x, y) = \Pi^M(\,[xy\bar{y}], [yx\bar{x}]\,) = \Pi^M(x, \bar{x}) \circ \Pi^M(y, x) \circ \Pi^M(\bar{y}, y) = \Pi^M(y, x),$$

which completes the proof. □

Observe, however, that $\Pi^M_{reg}(x, \bar{x}) \neq \mathrm{id}$ in general, where

$$\Pi^M_{reg}(x_1, x_2)u = \sum_{i=1}^{n} k_i [x_1 y_i x_2]\ .$$

6. RELATIONS BETWEEN REPRESENTATIONS

Let $(G, [\ ])$ be a ternary group and let $(G \times G, \langle\ \rangle)$ denote a ternary group applied to the construction of middle representation. In $(G \times G, \langle\ \rangle)$, we can define the relation

$$(a, b) \sim (c, d) \Longleftrightarrow [azb] = [czd]$$

for all $z \in G$. It is not difficult to see that this relation is a congruence in a ternary group $(G \times G, \langle\ \rangle)$. Thus middle regular representation takes constant values on the corresponding equivalence classes, i.e.,

$$(a, b) \sim (c, d) \Rightarrow \Pi^M_{reg}(a, b) = \Pi^M_{reg}(c, d).$$

Let us then define another relation

$$(a,b) \approx (c,d) \Longleftrightarrow \begin{cases} c = [xa\bar{x}] \quad \text{and} \quad d = [yb\bar{y}] \\ \text{for some } (x,y) \in G \times G. \end{cases}$$

It turns out to be an equivalence relation on $G \times G$ too. (Notice that the first line defines an equivalence relation in G.) Moreover, if $(G, [\,])$ is medial, then this relation is also a congruence in $(G \times G, \langle\,\rangle)$. Unfortunately, it is a weak relation. In a ternary group $\mathbb{Z}_3$, where $[xyz] = (x - y + z)(\bmod 3)$ we have only one class, i.e., all elements are equivalent. In $\mathbb{Z}_4$ with the operation $[xyz] = (x + y + z + 1)(\bmod 4)$ we have $a \approx a' \Longleftrightarrow a = a'$. Nevertheless, the following lemma holds.

Lemma 30. *If $(a,b) \approx (c,d)$, then for any middle representation Π^M one has*

$$\operatorname{tr} \Pi^M(a,b) = \operatorname{tr} \Pi^M(c,d).$$

Proof. Indeed,

$$\begin{aligned} \operatorname{tr} \Pi^M(a,b) &= \operatorname{tr} \Pi^M([xc\bar{x}],[yd\bar{y}]) = \operatorname{tr}(\Pi^M(x,\bar{y}) \circ \Pi^M(c,d) \circ \Pi^M(\bar{x},y)) \\ &= \operatorname{tr}(\Pi^M(x,\bar{y}) \circ \Pi^M(\bar{x},y) \circ \Pi^M(c,d)) = \operatorname{tr} \Pi^M(c,d). \end{aligned}$$ □

Let $(G, [\ ])$ and $(G \times G, \langle\,\rangle)$ be as above. In what follows, we shall denote by $((G \times G)^*, \circledast)$ the corresponding covering group and $(G \times G, \diamond) = \operatorname{ret}_{(a,b)}(G \times G, \langle\,\rangle)$. For Π^M being a middle representation of $(G, [\ ])$, we can introduce

$$\rho(x,y) = \Pi^M(x,y) \circ \Pi^M(a,b).$$

Thus ρ turns out to be representation of the retract $(G \times G, \diamond)$ induced by (a,b). Indeed, $(\bar{a},\bar{b})$ is the identity of this retract and $\rho(\bar{a},\bar{b}) = \Pi^M(\bar{a},\bar{b}) \circ \Pi^M(a,b) = \mathrm{id}_V$. Similarly,

$$\begin{aligned} \rho((x,y) \diamond (z,u)) &= \rho(\langle (x,y),(a,b),(z,u)\rangle) = \rho([xaz],[uby]) \\ &= \Pi^M([xaz],[uby])) \circ \Pi^M(a,b) \\ &= \Pi^M(x,y) \circ \Pi^M(a,b) \circ \Pi^M(z,u) \circ \Pi^M(a,b) \\ &= \rho(x,y) \circ \rho(z,u). \end{aligned}$$

Observe that ρ automatically extends to the representation π of the entire covering group $((G \times G)^*, \circledast)$ by

$$\pi(x,y,0) = \rho(x,y) \qquad \text{and} \qquad \pi(x,y,1) = \Pi^M(x,y).$$

Similarly, μ defined by $\mu(x,0) = \Pi^M(x,\bar{x}) \circ \Pi^M(a,\bar{a})$ and $\mu(x,1) = \Pi^M(x,\bar{x})$ is a representation of the covering group G^* for $(G, [\,])$ (see Post's Theorem). On the other hand, $\beta(x) = \Pi^M(x,\bar{x}) \circ \Pi^M(a,\bar{a})$ is a representation of the retract $(G, \cdot) = \operatorname{ret}_a(G, [\,])$. Thus β can induce some middle representation of $(G, [\,])$ (by the

Gluskin-Hosszú Theorem). One has $\mu = \pi \circ \tau$, where $\tau(x) = (x, \bar{x})$ is an embedding of $(G, [\ ])$ into $(G \times G, \langle\ \rangle)$.

Note that in a ternary group of the quaternions $(\mathbb{K}, [\ ])$ (with norm 1), where $[xyz] = xyz(-1) = -xyz$ and xy is a multiplication of quaternions (-1 is a central element) we have $\bar{1} = -1$, $\overline{-1} = 1$ and $\bar{x} = x$ for others. In $(K \times K, \langle\ \rangle)$ we have $(a, b) \sim (-a, -b)$ and $(a, -b) \sim (-a, b)$, which gives 32 two-elements equivalence classes. The embedding $\tau(x) = (x, \bar{x})$ suggest that $\Pi^M(i, i) = \pi(i) \neq \pi(-i) = \Pi^M(-i, -i)$. Generally $\Pi^M(a, b) \neq \Pi^M(-a, -b)$ and $\Pi^M(a, -b) \neq \Pi^M(-a, b)$.

In the so-called *derived* case the connection between binary and ternary group representations is instead established by the following proposition.

Proposition 31. *Let* $(G, [\ \ ]) = \mathrm{der}(G, \odot)$. *There is one-to-one correspondence between a pairwise commuting binary group representations and a middle ternary representation of derived group.*

Proof. Let $\pi, \rho \in Rep(G, \odot)$ such that $\pi(x) \circ \rho(y) = \rho(y) \circ \pi(x)$. Thus $\Pi^M(x, y) = \pi(x) \circ \rho(y^{-1})$. Conversely, for given $\Pi^M \in Rep(G, [\ ])$ one defines $\pi(x) = \Pi^M(x, e)$ and $\rho(x) = \Pi^M(e, \bar{x})$. All necessary properties can be directly checked out by means of (20). □

In the case of left ternary representations, to which we switch on now, we obtain the following theorem.

Theorem 32. *There is one-to-one correspondence between left ternary representations of* $(G, [\ ])$ *and binary representations of the retract* $\mathrm{ret}_a(G, [\ ])$.

Proof. Let $\Pi^L(x, y)$ be given, then $\rho(x) = \Pi^L(x, a)$ is a representation of the retract $\mathrm{ret}_a(G, [\ ])$, which can be directly shown. Conversely, assume that $\rho(x)$ is a representation of the retract $(G, \circledast) = \mathrm{ret}_a(G, [\ ])$. Define $\Pi^L(x, y) = \rho(x) \circ \rho(\bar{y})^{-1}$, then $\Pi^L(x, y) \circ \Pi^L(z, u) = \rho(x) \circ \rho(\bar{y})^{-1} \circ \rho(z) \circ \rho(\bar{u})^{-1} = \rho(x \circledast (\bar{y})^{-1} \circledast z) \circ \rho(\bar{u})^{-1} = \rho([[x\, a\, [\bar{a}\, y\, \bar{a}]]\, a\, z]) \circ \rho(\bar{u})^{-1} = \rho([x\, y\, x]) \circ \rho(\bar{u})^{-1} = \Pi^L([x\, y\, z], u)$, which completes the proof. □

Remark 3. One can see that Proposition 24 is a direct consequence of this theorem.

Let $(G, [\ ])$ be a ternary group and $(G \times G, *)$ be a semigroup used to the construction of left representations. According to Post (1940) one says that two pairs (a, b), (c, d) of elements of G are equivalent, i.e., $(a, b) \asymp (c, d)$, if there exists an element $x \in G$ such that $[abx] = [cdx]$. Using a covering group we can see that if this equation holds for some $x \in G$, then it holds also for all $x \in G$. The importance of this relation is due to the fact that for any left representation Π^L of $(G, [\ ])$ one has

$$\Pi^L(a, b) = \Pi^L(c, d) \Longleftrightarrow (a, b) \asymp (c, d).$$

Indeed, if $[abx] = [cdx]$ holds for some $x \in G$, then

$$\begin{aligned}\Pi^L(a,b) &= \Pi^L(a,b)\circ\Pi^L(x,\bar{x}) = \Pi^L([abx],\bar{x})\\ &= \Pi^L([cdx],\bar{x}) = \Pi^L(c,d)\circ\Pi^L(x,\bar{x}) = \Pi^L(c,d).\end{aligned}$$

The converse is obvious. In fact, this relation is a congruence on $(G\times G,*)$ and the corresponding quotient semigroup is isomorphic to the group G_0. Thus any left representation takes constant values on equivalence classes $(a,b)_{\asymp}$. Moreover, this classes are uniquely labelled by elements from G_0. This gives deeper insight into construction of left ternary representation of $(G,[\ ])$ from a given (binary) representation of G_0.

As an application of the above formalism one can also get the following proposition.

Proposition 33. *Every left representation of a commutative group $(G,[\ ])$ is a middle representation.*

Proof. For regular representations the statement is trivial. In general case one has to check

$$\Pi^L(x,y)\circ\Pi^L(\bar{x},\bar{y}) = \Pi^L([x\,y\,\bar{x}],\bar{y}) = \Pi^L([x\,\bar{x}\,y],\bar{y}) = \Pi^L(y,\bar{y}) = \mathrm{id}_V$$

and

$$\begin{aligned}\Pi^L(x_1,x_2)\circ\Pi^L(x_3,x_4)\circ\Pi^L(x_5,x_6) &= \Pi^L([[x_1x_2x_3]x_4x_5],x_6)\\ &= \Pi^L([[x_1x_3x_2]x_4x_5],x_6)\\ &= \Pi^L([x_1x_3[x_2x_4x_5]],x_6)\\ &= \Pi^L([x_1x_3[x_5x_4x_2]],x_6)\\ &= \Pi^L([x_1x_3x_5],[x_4x_2x_6])\\ &= \Pi^L([x_1x_3x_5],[x_6x_4x_2]).\end{aligned}$$ □

Note that the converse holds only for middle representations such that $\Pi^M(x,\bar{x}) = \mathrm{id}_V$ (cf. Corollary 29).

7. MATRIX REPRESENTATIONS

As an illustrative example we shall consider few matrix representations for concrete ternary groups. Let us observe before that many classical notions known from the representation theory can be extended in a natural way to the ternary case. This includes direct sum and tensor product of representations, characters, irreducibility (Schur lemma), equivalence of representations etc.

Example 4. Let $G = \mathbb{Z}_3 \ni \{0, 1, 2\}$ and the ternary multiplication is $[xyz] = x - y + z$. Then $[xyz] = [zyx]$ and $\bar{0} = 0$, $\bar{1} = 1$, $\bar{2} = 2$, therefore $(G, [\ ])$ is an idempotent medial ternary group. Thus $\Pi^L(x, y) = \Pi^R(y, x)$ and

$$\Pi^L(a, b) = \Pi^L(c, d) \Longleftrightarrow (a - b) = (c - d) \bmod 3. \tag{22}$$

Straightforward calculations give the left regular representation in the manifest matrix form

$$\Pi^L_{reg}(0,0) = \Pi^L_{reg}(2,2) = \Pi^L_{reg}(1,1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\Pi^L_{reg}(2,0) = \Pi^L_{reg}(1,2) = \Pi^L_{reg}(0,1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\Pi^L_{reg}(2,1) = \Pi^L_{reg}(1,0) = \Pi^L_{reg}(0,2) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

As we already mentioned, in this case $G^* = S_3$ and $G_0 = \mathbb{Z}_3$. Therefore Π^L_{reg}, as being at the same time representation of the commutative group $\mathbb{Z}_3$, decomposes into one-dimensional irreducible pieces, namely

$$\Pi^L_{reg} = \pi_0 \oplus \pi_1 \oplus \pi_2,$$

where for any $n \in \mathbb{Z}_3$ one has $\pi_k(n) = \epsilon^{kn}$; $k \in \{0, 1, 2\}$. Here $n \equiv (0, n)_{\asymp}$ and $\epsilon = \exp \frac{2\pi}{3}$ is primitive third order root of the unit.

Example 5. For the same ternary group as above one can also calculate matrices of middle regular representation to be

$$\Pi^M_{reg}(0,0) = \Pi^M_{reg}(1,2) = \Pi^M_{reg}(2,1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = [1] \oplus \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\Pi^M_{reg}(0,1) = \Pi^M_{reg}(1,0) = \Pi^M_{reg}(2,2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = [1] \oplus \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix},$$

$$\Pi^M_{reg}(0,2) = \Pi^M_{reg}(2,0) = \Pi^M_{reg}(1,1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = [1] \oplus \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}.$$

This representation Π^M_{reg} is equivalent to the orthogonal direct sum of two irreducible representations, i.e., one-dimensional trivial and two-dimensional.

Remark 4. In this example for all x we have $\Pi^M(x, \bar{x}) = \Pi^M(x, x) \neq \mathrm{id}_V$, but $\Pi^M(x, y) \circ \Pi^M(x, y) = \mathrm{id}_V$, and so matrices Π^M are of the second degree.

We can "algebralize" both regular representations from the Examples 3, 4 in the following way. From (10) we have for the left representation $\Pi^L_{reg}(i, j) \circ \Pi^L_{reg}(k, l) = \Pi^L_{reg}([ijk], l)$, where $[ijk] = i - j + kl$, $i, j, k, l \in \mathbb{Z}_3$. Denote $\gamma^L_i = \Pi^L_{reg}(0, i)$, $i \in \mathbb{Z}_3$, then we obtain the algebra with the relations $\gamma^L_i \gamma^L_j = \gamma^L_{i+j}$. Conversely, any matrix representation of $\gamma_i \gamma_j = \gamma_{i+j}$ leads to the left representation by $\Pi^L(i, j) = \gamma_{j-i}$. In the case of the middle regular representation we introduce $\gamma^M_{k+l} = \Pi^M_{reg}(k, l)$, $k, l \in \mathbb{Z}_3$, then we obtain

$$\gamma^M_i \gamma^M_j \gamma^M_k = \gamma^M_{[ijk]}, \qquad i, j, k \in \mathbb{Z}_3. \tag{23}$$

In some sense (23) can be treated as a *ternary analog of Clifford algebra*. As before, any matrix representation of (23) gives rise to the middle representation $\Pi^M(k, l) = \gamma_{k+l}$.

Example 6. As a final example let us consider $G = \mathbb{Z}_4 \ni \{0, 1, 2, 3\}$ with ternary multiplication $[xyz] = (x + y + z + 1) \bmod 4$. Of course, it is commutative and non-idempotent. Thus it is sufficient to consider the left representation only. Since $\mathrm{ret}_3(G, [\,]) = (\mathbb{Z}_4, +)$, from the proof of Theorem 32 it follows that $\rho(x) = \Pi^L(x, 3)$ is a representation of the group $(\mathbb{Z}_4, +)$, if only $\Pi^L(x, y)$ is a left representation of $(G, [\,])$. Moreover, for left representations of this ternary group we have

$$\Pi^L(a, b) = \Pi^L(c, d) \Longleftrightarrow (a + b) = (c + d) \bmod 4.$$

So

$$\begin{aligned}
\Pi^L_{reg}(0, 0) &= \Pi^L_{reg}(1, 3) = \Pi^L_{reg}(2, 2) = \Pi^L_{reg}(3, 1) = \rho(1), \\
\Pi^L_{reg}(0, 1) &= \Pi^L_{reg}(1, 0) = \Pi^L_{reg}(2, 3) = \Pi^L_{reg}(3, 2) = \rho(2), \\
\Pi^L_{reg}(0, 2) &= \Pi^L_{reg}(1, 1) = \Pi^L_{reg}(2, 0) = \Pi^L_{reg}(3, 3) = \rho(3), \\
\Pi^L_{reg}(0, 3) &= \Pi^L_{reg}(1, 2) = \Pi^L_{reg}(2, 1) = \Pi^L_{reg}(3, 0) = \rho(0).
\end{aligned}$$

The representation ρ of $\mathbb{Z}_4$ decomposes into sum $\rho = \rho_0 \oplus \rho_1 \oplus \rho_2 \oplus \rho_3$, where $\rho_k(n) = i^{kn}$, and $i = \sqrt{-1}$.

ACKNOWLEDGMENTS

Andrez Borowiec is grateful to K. Głazek and Z. Oziewicz for interesting discussions. Steven Duplij would like to thank J. Lukierski for his kind hospitality at the University of Wrocław, where joint work was initiated (June, 2001). We are indebted to A.V. Artamonov for critical remarks improving our manuscript.

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