The Satisfiablility Problem for Probabilistic CTL

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IFIP WG 2.2, Tallinn, July 2024

The Satisfiability Problem

- Given a formula φ of some logic $\mathcal{L}$, does φ have a model?
- Closely related to the validity problem: φ is not satisfiable iff $\neg\varphi$ is a tautology.

Satisfiability for First-Order Logic

- Non-satisfiability is semidecidable: a sentence φ is not satisfiable iff $\models \neg\varphi$ iff $\vdash \neg\varphi$.
- Non-satisfiability is not decidable: For a given Minsky machine $\mathcal{M}$, there is an effectively constructible sentence $\varphi_{\mathcal{M}}$ over the language $\{0, \mathit{Succ}, \mathit{Reach}\}$ such that $\mathcal{M}$ halts iff $\models \varphi$ iff $\neg\varphi$ is not satisfiable.

$$\varphi_{\mathcal{M}} \equiv (\mathit{Reach}([1],[0],[0]) \wedge \mathit{Closed}) \Rightarrow \exists c, d.\mathit{Reach}([m+1], c, d)\}$$

$$\mathit{Closed} \equiv \text{“whenever } \mathit{Reach}([i], c, d)\text{, then } \mathit{Reach}([i'], c', d') \text{ where } (i, c, d) \mapsto (i', c', d')\text{”}$$

- Consequently, satisfiability is not even semidecidable.

Finite Satisfiability for First-Order Logic

- Finite satisfiability is semidecidable.
- Finite satisfiability is not decidable: For a given Minsky machine $\mathcal{M}$, there is an effectively constructible sentence $\psi_{\mathcal{M}}$ over the language $\{0, Succ, Reach\}$ such that $\mathcal{M}$ halts iff $\psi_{\mathcal{M}}$ has a finite-state model.

$$\begin{aligned}\psi_{\mathcal{M}} &\equiv \mathit{Reach}([1],[0],[0],[0]) \wedge \mathit{Closed} \\ &\wedge \mathit{Reach}(x_1,x_2,x_3,y) \Rightarrow (S(y)\neq 0 \wedge \forall z(y\neq z \Rightarrow S(y)\neq S(z)))\end{aligned}$$

- Consequently, the finite non-satisfiability (and hence also finite validity) is not semi-decidable. In particular, there is no complete deductive system satisfying $\models_f \varphi$ iff $\vdash \varphi$.

Satisfiability for Temporal Logics

- The satisfiability problem for the modal μ-calculus is EXPTIME-complete.
- Small model property: every satisfiable φ has a model whose size is at most exponential in $|\varphi|$.
- There is a complete deductive system (satisfying $\models \varphi$ iff $\vdash \varphi$).

Satisfiability for Probabilistic Temporal Logics (1)

- Probabilistic CTL:

$$\begin{array}{lcl} \varphi & ::= & a \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid P(\Phi) \bowtie r \\ \Phi & ::= & \mathbf{X}\,\varphi \mid \varphi_1\,\mathbf{U}\,\varphi_2 \mid \varphi_1\,\mathbf{U}^k\,\varphi_2 \end{array}$$

  Here, $a \in AP$, $\bowtie \in \{\geq, >, \leq, <, =\}$, $r \in [0,1]$ is a rational constant, and $k \in \mathbb{N}$.
- $\mathbf{F}\,\varphi$ and $\mathbf{F}^k\,\varphi$ abbreviate $\mathit{true}\,\mathbf{U}\,\varphi$ and $\mathit{true}\,\mathbf{U}^k\,\varphi$.
- We write $\mathbf{X}_{=1}\,\varphi$ instead of $P(\mathbf{X}\,\varphi) = 1$, $\mathbf{G}_{=1}\,\varphi$ instead of $\mathbf{F}_{=0}\,\neg\varphi$, etc.
- The qualitative fragment of PCTL is obtained by restricting r to 0 and 1.
- PCTL formulae are interpreted over Markov chains.

Satisfiability for Probabilistic Temporal Logics (2)

- PCTL does not have the small model property. There are satisfiable PCTL formulae with only infinite-state models.

  $\mathbf{G}_{>0}(\neg a \wedge \mathbf{F}_{>0}\, a)$

- The satisfiability problem has been studied in two basic variants:
  - general satisfiability, i.e., the existence of an unrestricted model;
  - finite satisfiability, i.e., the existence of a finite-state model.

Satisfiability for Probabilistic Temporal Logics (3)

- General/finite PCTL satisfiability has been first studied for the qualitative PCTL fragment. Both problems are EXPTIME-complete, and a (finite representation of) a model for a satisfiable qPCTL formula is effectively constructible in exponential time.
- Proof techniques are similar to non-probabilistic logics (filtration, tableaux,...)

Satisfiability for Probabilistic Temporal Logics (4)

- The decidability of general/finite PCTL satisfiability has been open for about 30 years, despite numerous research attempts.
- There are positive decidability results about finite PCTL satisfiability obtained for various PCTL fragments.
- For a given PCTL formula φ and a given $n \geq 1$, the existence of model for φ with precisely n states is decidable (by encoding the question in first-order arithmetic of the reals).
- Hence, the finite PCTL satisfiability is semidecidable. The decidability can be obtained by establishing any computable upper bound on the number of states of a model for a finite satisfiable PCTL formula.

Satisfiability for Probabilistic Temporal Logics (5)

Theorem 1 (Chodil, K., 2024)

The finite PCTL satisfiability is undecidable. The general PCTL satisfiability is even highly undecidable (beyond the arithmetical hierarchy). Consequently, there is no complete deductive system proving all valid (or all finitely valid) PCTL formulae.

Th Undecidability Proof

- If the finite PCTL satisfiability is undecidable, then the intuition about the existence of a bounded-size model must be wrong.
- We show that there exists a fixed parameterized PCTL formula $\psi(x, y)$ enforcing arbitrarily large finite models just be changing the numerical probability constraints x, y.
- Intuitively, the vector (p, q) substituted for (x, y) encodes an (arbitrarily large) non-negative integer value n, and the formula $\psi(x/p, y/q)$ enforces the existence of states representing all counter values ranging from 0 to n by "implementing the decrement operation".
- Then, we show how to implement the increment (test for zero is trivial due to the chosen encoding). Finally, we show how to encode two counters simultaneously, and how simulate a Minsky machine.

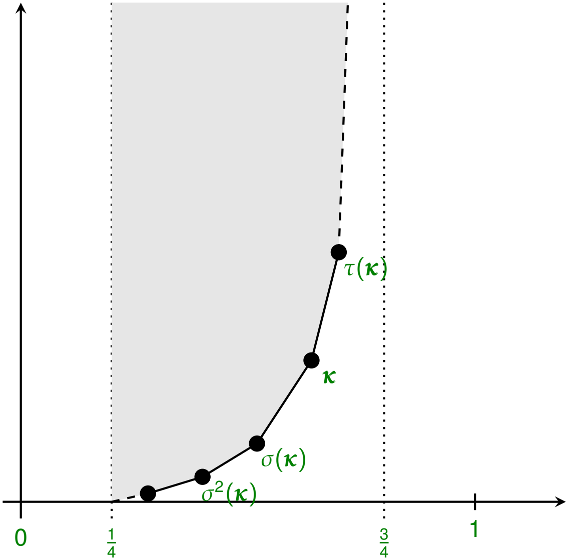
Encoding a non-negative counter in PCTL formulae

- Let $q = \frac{13}{16}$, $I = (\frac{1}{4}, \frac{3}{4})$, $\boldsymbol{\kappa} = (\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2)$ where $\boldsymbol{\kappa}_1 \in I$, $\boldsymbol{\kappa}_2 > 0$, and $\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_2 \leq 1$.
- Let $W = I \times (0, \infty)$, and let $\tau, \sigma : W \to W$ be functions defined as follows:

$$\tau(\boldsymbol{v}) = \left(\frac{q - 1 + \boldsymbol{v}_1}{\boldsymbol{v}_1}, \frac{\boldsymbol{v}_2}{\boldsymbol{v}_1}\right), \qquad \sigma(\boldsymbol{v}) = \left(\frac{1 - q}{1 - \boldsymbol{v}_1}, \frac{\boldsymbol{v}_2(1 - q)}{1 - \boldsymbol{v}_1}\right).$$

- Intuitively $0, 1, 2, \ldots$ are represented by vectors $\boldsymbol{\kappa}$, $\sigma(\boldsymbol{\kappa})$, $\sigma(\sigma(\boldsymbol{\kappa}))$, …
- A state t of a Markov chain represents a given $n \in \mathbb{N}$ iff the path formulae $\mathbf{X}\, a$ and $\mathbf{X}\, b$ are satisfied in t with the probabilities $\sigma^n(\boldsymbol{\kappa})_1$ and $\sigma^n(\boldsymbol{\kappa})_2$.

Encoding a non-negative counter in PCTL formulae



Constructing the Formula $\psi(x, y)$

Let $A = \{a, b, c, h, r_0, r_1, r_2, r_3, r_4\}$. We put

$$\psi(x, y) \equiv Init(x, y) \wedge \mathbf{G}_{=1}\, Invariant$$

where

$$\begin{aligned} Init(x, y) &\equiv \langle a, r_0 \rangle \wedge \mathbf{X}_{=x}\, a \wedge \mathbf{X}_{=y}\, b \\ Invariant &\equiv Fin \vee Trans \vee Free \end{aligned}$$

where

$$\begin{aligned} Free &\equiv h \wedge \bigvee_{B \subseteq A} (\langle B \rangle \wedge \mathbf{X}_{=1} \langle B \rangle) \\ Fin &\equiv \bigvee_{i \in \{0,\ldots,4\}} \langle a, r_i \rangle \wedge FSuc_i \wedge Zero \end{aligned}$$

where

$$\begin{aligned} FSuc_i &\equiv \mathbf{X}_{=1}(\langle h, a, S(r_i) \rangle \vee \langle h, b, S^2(r_i) \rangle \vee \langle h, c, S^2(r_i) \rangle), \\ Zero &\equiv \mathbf{X}_{=\kappa_1}\, a \wedge \mathbf{X}_{=\kappa_2}\, b\,. \end{aligned}$$

Constructing the Formula $\psi(x, y)$

Finally, we put

$$\mathit{Trans} \equiv \bigvee_{i \in \{0,\ldots,4\}} \langle a, r_i \rangle \wedge \mathit{Suc}_i \wedge \mathit{Interval} \wedge \mathit{Eq}_i$$

where

$$\begin{aligned}
\mathit{Suc}_i &\equiv \mathbf{X}_{=1}(\langle a, S(r_i)\rangle \vee \langle h, b, S^2(r_i)\rangle \vee \langle h, c, S^2(r_i)\rangle) \\
\mathit{Interval} &\equiv \mathbf{X}_{>\frac{1}{4}}\, a \ \wedge\ \mathbf{X}_{<\frac{3}{4}}\, a \ \wedge\ \mathbf{X}_{>0}\, b \\
\mathit{Eq}_i &\equiv \mathbf{F}^2_{=q}\, S^2(r_i) \ \wedge\ \mathbf{F}^2_{=q}((S^2(r_i) \wedge \neg b) \vee (S^3(r_i) \wedge b))
\end{aligned}$$