Böhm Trees as Higher-Order Recursion Schemes

Pierre Clairambault
ENS Lyon

Andrzej Murawski
University of Warwick

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Higher-order recursion schemes (HORS)

- HORS are an abstract form of functional programs.
- They can be viewed as typed grammars generating possibly infinite trees.

Example

- TERMINALS

$$\mathrm{a} : o \to o \qquad \mathrm{b} : o \to o \to o \qquad \mathrm{c} : o$$

- NONTERMINALS

$$S : o \qquad F : (o \to o) \to o \to o \qquad G : (o \to o) \to o$$

- RULES

$$\begin{aligned} S &= G\ \mathrm{a} \\ F\ f\ x &= f\ (f\ x) \\ G\ f &= \mathrm{b}\ (f\ \mathrm{c})\ (G\ (F\ f)) \end{aligned}$$

- STARTING SYMBOL

$$S$$

Example

$$
\begin{array}{rcl}
S & = & G\ \mathrm{a} \\
F\ f\ x & = & f\ (f\ x) \\
G\ f & = & \mathrm{b}\ (f\ \mathrm{c})\ (G\ (F\ f))
\end{array}
$$

Example (Tree Generation)

S

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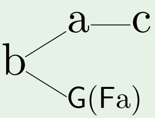
Example (Tree Generation)



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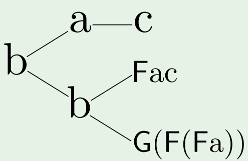

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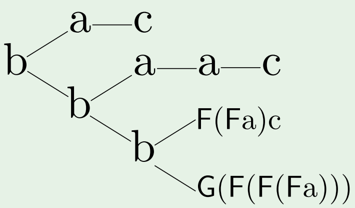

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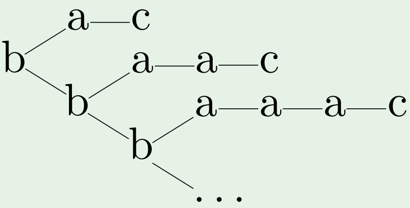

Alternative presentation : λY-calculus

Types. Simple types over one atom o.

$$\theta, \theta' ::= o \mid \theta \to \theta'$$

Terms.

$$M, N ::= x \mid \lambda x^{\theta}.M \mid M\ N \mid Y_{\theta}$$

Typing rules.

$$\frac{}{\Gamma, x : \theta \vdash x : \theta} \qquad \frac{}{\Gamma \vdash Y_{\theta} : (\theta \to \theta) \to \theta} \qquad \frac{\Gamma, x : \theta \vdash M : \theta'}{\Gamma \vdash \lambda x^{\theta}.M : \theta \to \theta'}$$

$$\frac{\Gamma \vdash M : \theta \to \theta' \qquad \Gamma \vdash N : \theta}{\Gamma \vdash M\ N : \theta'}$$

Reduction.

$$\begin{array}{rcl} (\lambda x^{\theta}.M)\ N & \to_{\beta} & M[N/x] \\ Y_{\theta}\ M & \to_{\delta} & M\ (Y_{\theta}\ M) \\ M & \to_{\eta} & \lambda x^{\theta}.M\ x \end{array}$$

(in the last, $x \notin fv(M)$ and M has type $\theta \to \theta'$)

Relationship of HORS and λY

Example

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Relationship of HORS and λY

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S &= G\ \mathrm{a} \\
F &= \lambda f.\lambda x.f\ (f\ x) \\
G &= \lambda f.\mathrm{b}\ (f\ \mathrm{c})\ (G\ (F\ f))
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$$

Relationship of HORS and λY

Example

$$S = G\ \mathrm{a}$$

$$G = \lambda f.\mathrm{b}\ (f\ \mathrm{c})\ (G\ (\lambda x.f\ (f\ x)))$$

Relationship of HORS and λY

Example

$$S = G\ \mathrm{a}$$

$$G = Y\ (\lambda G.\lambda f.\mathrm{b}\ (f\ \mathrm{c})\ (G\ (\lambda x.f\ (f\ x))))$$

Relationship of HORS and λY

Example

$$S \quad = \quad Y\ (\lambda G.\lambda f.\mathrm{b}\ (f\ \mathrm{c})\ (G\ (\lambda x.f\ (f\ x))))\ \mathrm{a}$$

Relationship of HORS and λY

Example

$$S \quad = \quad Y\ (\lambda G.\lambda f.\mathrm{b}\ (f\ \mathrm{c})\ (G\ (\lambda x.f\ (f\ x))))\ \mathrm{a}$$

This is a λY-term of type o in context

$$\mathrm{a} : o \to o \qquad \mathrm{b} : o \to o \to o \qquad \mathrm{c} : o.$$

We write $\Gamma_{\leqslant 1}$ for contexts with types of order at most 1.

Relationship of HORS and λY

Example

$$S \quad = \quad Y\ (\lambda G.\lambda f.\mathrm{b}\ (f\ \mathrm{c})\ (G\ (\lambda x.f\ (f\ x))))\ \mathrm{a}$$

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$$\mathrm{a} : o \to o \qquad \mathrm{b} : o \to o \to o \qquad \mathrm{c} : o.$$

We write $\Gamma_{\leqslant 1}$ for contexts with types of order at most 1.

Proposition (Salvati, Walukiewicz)

There is a correspondence between HORS and λY-terms of the form

$$\Gamma_{\leqslant 1} \vdash M : o.$$

Higher-order program verification

Theorem (Ong)

Monadic Second-Order logic (MSO) is decidable on trees generated by HORS.

Example (Kobayashi)

Application to verification of correct resource usage.

$$\texttt{let rec g}() \;=\; \texttt{if \_ then close}() \;\texttt{else}\; (\texttt{read}(); \texttt{g}()) \;\texttt{in g}()$$

$$Y\ (\lambda G.\lambda k.\mathrm{br}\ (\mathrm{close}\ k)\ (\mathrm{read}\ (G\ k)))\ \bullet$$

with terminals:

$$\begin{aligned} \mathrm{br} &: o \to o \to o \\ \mathrm{read} &: o \to o \\ \mathrm{close} &: o \to o \\ \bullet &: o \end{aligned}$$

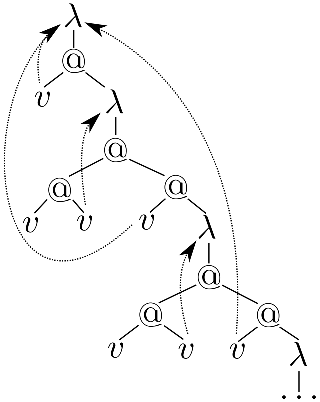
One can automatically check that all finite paths have the form $\mathrm{read}^*\mathrm{close}$.

Böhm trees (rather than trees)

Consider the term

$$g : o \to o \to o \vdash \lambda f^{(o \to o) \to o}. Y_o \ (\lambda y^o . f \ (\lambda x^o . g \ x \ y)) : ((o \to o) \to o) \to o$$

Its Böhm tree starts with



How can we generate representations of pointers within HORS?

Result

Question

Can we relate HORS and arbitrary Böhm trees?

$$\Gamma_{\leqslant 1} \vdash M : o \xrightarrow{\subseteq} \Gamma \vdash M : \theta$$

$$\Gamma \vdash M : \theta \overset{?}{\dashrightarrow} \Gamma_{\leqslant 1} \vdash M : o$$

Theorem (Clairambault, M. ; FSTTCS'13)

For any λY-term $\Gamma \vdash M : \theta$ there is a term

$$\Gamma_{rep} \vdash M_{rep} : o$$

with

$$\Gamma_{rep} = \{\, z : o,\ succ : o \to o,\ var : o \to o,\ app : o \to o \to o,\ lam : o \to o \to o \,\}$$

such that M_{rep} evaluates to a representation of M's Böhm tree, where binders are represented by **De Bruijn levels**.

We also prove the same result for terms of finitary PCF (PCF_f).

De Bruijn levels

Definition

De Bruijn levels are a variable-naming convention where

- variables are natural numbers,
- each variable is given the smallest index not yet in use.

Example

The term

$$g : o \to o \to o \vdash \lambda f.f\ (\lambda x.g\ x\ (f\ (\lambda y.g\ y\ (f\ y))$$

can be represented by

$$0 : o \to o\ \ \to o \vdash \lambda 1.1\ (\lambda 2.0\ 2\ (1\ (\lambda 3.0\ 3\ (1\ 3))))$$

Proposition

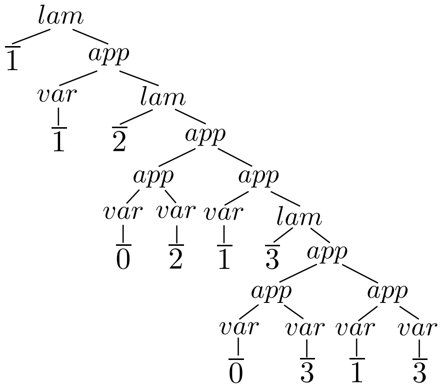
Two terms M and M' have the same De Bruijn levels representation iff they are α-equivalent.

(not to be confused with **De Bruijn indices**)

Representation of De Bruijn levels in λY

We represent terms with binders as Böhm trees of type o in the context

$$\Gamma_{rep} = \{\, z : o,\ succ : o \to o,\ var : o \to o,\ app : o \to o \to o,\ lam : o \to o \to o \,\}$$



$\overline{n} = succ\ (succ\ \ldots\ (succ\ z)\ldots)$

Formal statement

Theorem

Let $\Gamma \vdash M : \theta$ be a λY-term.

There exists a λY-term $\Gamma_{rep} \vdash M_{rep} : o$ (a HORS) such that

$$BT(M_{rep}) = \mathrm{rep}(BT(M)).$$

Write θ^* for $\theta[o \to o/o]$ and $M^* = M[o \to o/o]$.

There exists a λ-term

$$\Gamma_{rep} \vdash \downarrow_\theta : \theta^* \to o \to o$$

such that, for $\vdash M : \theta$, setting

$$M_{rep} = \downarrow_\theta \ M^* \ \overline{0}$$

validates the above theorem.

Normalization by evaluation for the simply-typed λ-calculus

Step 1: Interpretation. Let E be a set containing representations of terms.

$$
\begin{array}{rcl}
[\![o]\!] & = & E \\
[\![x]\!]_\rho & = & \rho(x) \\
[\![M\ N]\!]_\rho & = & [\![M]\!]_\rho([\![N]\!]_\rho)
\end{array}
\qquad
\begin{array}{rcl}
[\![\theta \to \theta']\!] & = & [\![\theta]\!] \to [\![\theta']\!] \\
[\![\lambda x^\theta.M]\!]_\rho & = & \boldsymbol{\lambda} a^{[\![\theta]\!]}.[\![M]\!]_{\rho \oplus \{x \mapsto a\}}
\end{array}
$$

All the right-hand-side operations are operations on sets and functions.

Step 2: Reification. The normal form of $\vdash M : \theta$ can be **extracted** from $[\![M]\!]$ by setting $\mathrm{nf(M)} = \Downarrow_\theta [\![M]\!]$.

$$
\begin{array}{rcl}
\Downarrow_\theta & : & [\![\theta]\!] \to E \\
\Downarrow_o x & = & x \\
\Downarrow_{\theta_1 \to \theta_2} x & = & lam\ n \Downarrow_{\theta_2}(x\ (\Uparrow_{\theta_1}(var\ n)))\quad (n \text{ fresh})
\end{array}
$$

$$
\begin{array}{rcl}
\Uparrow_\theta & : & E \to [\![\theta]\!] \\
\Uparrow_o e & = & e \\
\Uparrow_{\theta_1 \to \theta_2} e & = & \boldsymbol{\lambda} x^{[\![\theta_1]\!]}.\Uparrow_{\theta_2} app\ e\ (\Downarrow_{\theta_2} x)
\end{array}
$$

Example

$$
\begin{aligned}
\Downarrow_{o\to o\to o} [\![\lambda x^o.\lambda y^o.x]\!] &= \mathit{lam}\ 0\ (\Downarrow_{o\to o} [\![\lambda x^o.\lambda y^o.x]\!]\ (\Uparrow_o (\mathit{var}\ 0)))\\
&= \mathit{lam}\ 0\ (\Downarrow_{o\to o} [\![\lambda x^o.\lambda y^o.x]\!]\ (\mathit{var}\ 0))\\
&= \mathit{lam}\ 0\ (\mathit{lam}\ 1\ (\Downarrow_o [\![\lambda x^o.\lambda y^o.x]\!]\ (\mathit{var}\ 0)\ (\Uparrow_o (\mathit{var}\ 1))))\\
&= \mathit{lam}\ 0\ (\mathit{lam}\ 1\ ([\![\lambda x^o.\lambda y^o.x]\!]\ (\mathit{var}\ 0)\ (\mathit{var}\ 1)))\\
&= \mathit{lam}\ 0\ (\mathit{lam}\ 1\ ((\boldsymbol{\lambda} a^E.\boldsymbol{\lambda} b^E.a)\ (\mathit{var}\ 0)\ (\mathit{var}\ 1)))\\
&= \mathit{lam}\ 0\ (\mathit{lam}\ 1\ (\mathit{var}\ 0))
\end{aligned}
$$

Remarks

- Normal form obtained by **evaluation** in the model
- Need for generation of fresh variable indices

Generating De Bruijn levels (Berger, Schwichtenberg)

Expressions replaced with **indexed expressions** $\widehat{E} = \mathbb{N} \to E$.

Step 1: Interpretation. Let E be a set containing representations of terms.

$$
\begin{array}{rclcrcl}
[\![o]\!] & = & \mathbb{N} \to E & \qquad & [\![\theta \to \theta']\!] & = & [\![\theta]\!] \to [\![\theta']\!] \\
[\![x]\!]_\rho & = & \rho(x) & & [\![\lambda x^\theta . M]\!]_\rho & = & \boldsymbol{\lambda} a^{[\![\theta]\!]} . [\![M]\!]_{\rho \oplus \{x \mapsto a\}} \\
[\![M\ N]\!]_\rho & = & [\![M]\!]_\rho([\![N]\!]_\rho) & & & &
\end{array}
$$

All the right-hand-side operations are operations on sets and functions.

Step 2: Reification. The normal form of $\vdash M : \theta$ can be **extracted** from $[\![M]\!]$ by setting $\mathrm{nf(M)} = \Downarrow_\theta [\![M]\!]$.

$$
\begin{array}{rclcrcl}
\Downarrow_o x & = & x & \qquad & \Downarrow_{\theta_1 \to \theta_2} x & = & \widehat{lam}\ (\boldsymbol{\lambda} n^N . \Downarrow_{\theta_2} (x\ (\Uparrow_{\theta_1} \widehat{var}\ n))) \\
\Uparrow_o e & = & e & & \Uparrow_{\theta_1 \to \theta_2} e & = & \boldsymbol{\lambda} x^{[\![\theta_1]\!]} . \Uparrow_{\theta_2} \widehat{app}\ e\ (\Downarrow_{\theta_2} x)
\end{array}
$$

Generalized constructors

Constructors. *var*, *lam*, *app* are replaced with compositional variants.

$$\widehat{var} = \boldsymbol{\lambda} v^{N}.\boldsymbol{\lambda} n^{N}.var\ v : N \to \widehat{E}$$

$$\widehat{app} = \boldsymbol{\lambda} e_1^{\widehat{E}}.\boldsymbol{\lambda} e_2^{\widehat{E}}.\boldsymbol{\lambda} n^{N}.app\ (e_1\ n)\ (e_2\ n) : \widehat{E} \to \widehat{E} \to \widehat{E}$$

$$\widehat{lam} = \boldsymbol{\lambda} f^{N \to \widehat{E}}.\boldsymbol{\lambda} n^{N}.lam\ n\ (f\ n\ (succ\ n)) : (N \to \widehat{E}) \to \widehat{E}$$

The semantic ingredients used in NBE for λY can be expressed within the λY-calculus!

Internalization

Expressions are λY-terms $\Gamma_{rep} \vdash M : o$.

Indexed expressions have the type $\widehat{E} = o \to o$.

Interpretation is the substitution $\theta^* = \theta[o \to o/o]$ and $M^* = M[o \to o/o]$.

Term formers

$$
\begin{aligned}
\widehat{var} &= \lambda v^o.\lambda n^o.var\ v \\
\widehat{lam} &= \lambda f^{o \to o}.\lambda n^o.lam\ n\ (f\ \ n\ (succ\ n)) \\
\widehat{app} &= \lambda e_1^o.\lambda e_2^o.\lambda n^o.app\ (e_1\ \ n)\ (e_2\ \ n)
\end{aligned}
$$

Reify/reflect are now terms of the λY-calculus.

$$
\begin{aligned}
\downarrow_o &= \lambda x^o.x & \downarrow_{\theta_1 \to \theta_2} &= \lambda x^{\theta_1^* \to \theta_2^*}.\widehat{lam}\ (\boldsymbol{\lambda} n^N.\downarrow_{\theta_2}\ (x\ (\uparrow_{\theta_1}\ \widehat{var}\ \ n))) \\
\uparrow_o &= \lambda e^o.e & \uparrow_{\theta_1 \to \theta_2} &= \lambda e^o.\lambda x^{\theta_1^*}.\uparrow_{\theta_2}\ \widehat{app}\ e\ (\downarrow_{\theta_2}\ x)
\end{aligned}
$$

Internalization

Theorem

If $\vdash M : \theta$ *is a* λY*-term then the term* M_{rep} *defined as*

$$\Gamma_{rep} \vdash\downarrow_\theta \; M^* \; \overline{0} : o$$

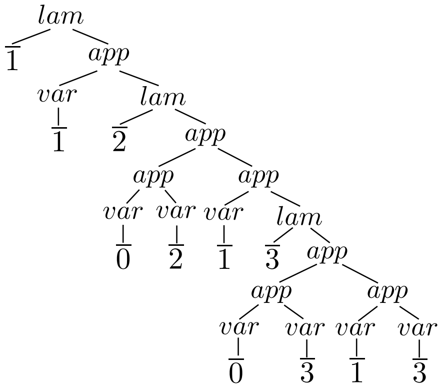
satisfies

$$BT(M_{rep}) = \mathrm{rep}(BT(M)).$$

Outcome

We represent terms with binders as Böhm trees of type o in the context

$$\Gamma_{rep} = \{\, z : o,\ succ : o \to o,\ var : o \to o,\ app : o \to o \to o,\ lam : o \to o \to o \,\}$$



$\overline{n} = succ\ (succ\ \ldots\ (succ\ z)\ldots)$

Extension to PCF_f

Definition

The types and terms of PCF_f are defined as follows.

$$\begin{array}{rcl} \theta, \theta' & ::= & \mathrm{B} \mid \theta \to \theta' \\ M, N & ::= & x \mid \lambda x^{\theta}.M \mid M\ N \mid Y_\theta \\ & & tt \mid ff \mid \mathrm{if}\ M\ \mathrm{then}\ N\ \mathrm{else}\ N' \end{array}$$

equipped with the standard operational semantics.

Definition (PCF Böhm trees)

The notion of (infinite) normal forms

$$\frac{}{\Gamma \vdash \bot : \mathrm{B}} \qquad \frac{}{\Gamma \vdash tt : \mathrm{B}} \qquad \frac{}{\Gamma \vdash ff : \mathrm{B}} \qquad \frac{\Gamma, x_1 : A_1, \ldots, x_n : A_n \vdash M : \mathrm{B}}{\Gamma \vdash \lambda \vec{x}.M : \vec{A} \to \mathrm{B}}$$

$$\frac{\Gamma \vdash M_i : \theta_i \quad (1 \leqslant i \leqslant n) \qquad \Gamma \vdash N_1 : \mathrm{B} \qquad \Gamma \vdash N_2 : \mathrm{B} \qquad (x : \vec{\theta} \to \mathrm{B}) \in \Gamma}{\Gamma \vdash \mathrm{if}\ x\ \vec{M}\ \mathrm{then}\ N_1\ \mathrm{else}\ N_2 : \mathrm{B}}$$

The NBE translation for PCF_f

Representation. In the ω-cpo E of infinitary terms $\Gamma_{pcf} \vdash M : o$, with:

$$\Gamma_{pcf} = \Gamma_{rep} \cup \{tt : o, ff : o, if : o \to o \to o \to o\}$$

Semantics. Standard domain semantics of PCF, based on:

$$[\![\mathrm{B}]\!] = \widehat{E} \to \widehat{E} \to \widehat{E}$$

Reflect and reify. Adaptations of those for λY.

Internalization. Follows the same lines as for λY.

Normal forms. The normal forms generated are **infinitary** PCF **Böhm trees**, or equivalently, **innocent strategies**.

Consequences

Corollary

The following problems are recursively equivalent.

(1) *Equivalence of HORS*

(2) *Language equivalence of deterministic collapsible pushdown automata*

(3) *Böhm tree equivalence for* λY

(4) *Contextual equivalence for* PCF_f *(wrt contexts with state and control operators)*

By MSO model-checking on HORS

Corollary

The following problems are decidable for PCF_f *and* λY *terms:*

(1) *Normalizability*

(2) *Finiteness*

(3) *Finite prefix*

...

Thank you!