Limit-Average Properties of pVASS
&
Optimal Strategies in Patrolling Games

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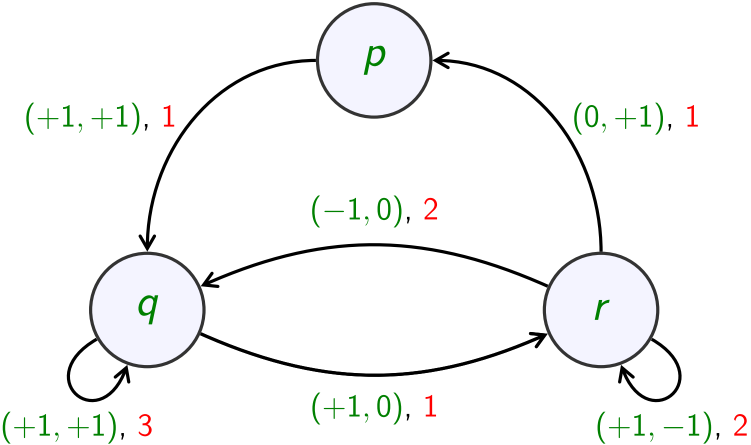
Probabilistic VASS (pVASS)



Limit-Average Properties of pVASS

- A *pattern* is a tuple of the form $p(+,0,+,0,0,+)$.
- Let w be a pVASS run.
- We define the vector $\mathcal{F}(w)$ of limit pattern frequencies:

$$\mathcal{F}(w) = \lim_{n\to\infty} \mathrm{Freq}_n(w)$$

If this limit does not exist, we put $\mathcal{F}(w) = \perp$.

The Problems

Let $p\vec{v}$ be a configuration of a d-dimensional pVASS. Then $\mathcal{F}$ is a random variable over the runs initiated in $p\vec{v}$.

- Do we have $\mathcal{P}[\mathcal{F}{=}\bot] = 0$?
- Is $\mathcal{F}$ a discrete random variable ?
- If so, is the set of admissible values finite ?
- Can we compute/approximate these values and the associated probabilities?

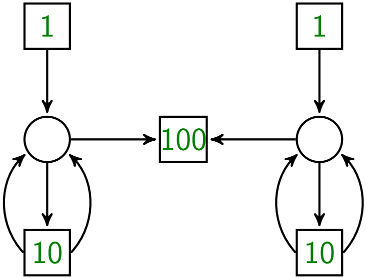
Previous Results about Stochastic Petri Nets

Theorem 1 (Florin and Natkin, 1989)

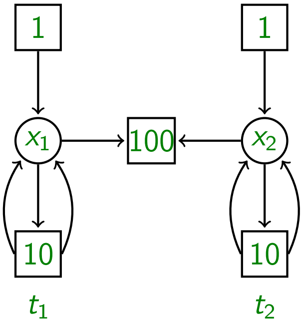
If a given SPN with an initial marking M_1 has a strongly connected graph of reachable configurations, then for every transition t there is a frequency $f \in \mathbb{R}$ such that for almost every run $M_1, t_1, M_2, t_2, M_3, \ldots$ we have that

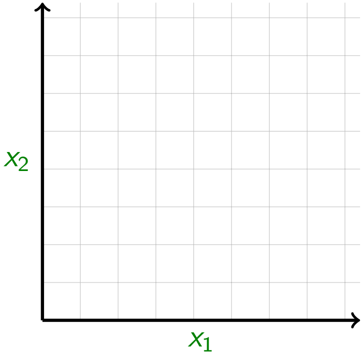
$$\lim_{n \to \infty} \frac{\#_t(t_1, \ldots, t_n)}{n} = f \ .$$

A Counterexample to Theorem 1



A Counterexample to Theorem 1 (cont.)





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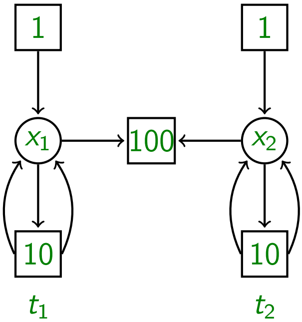

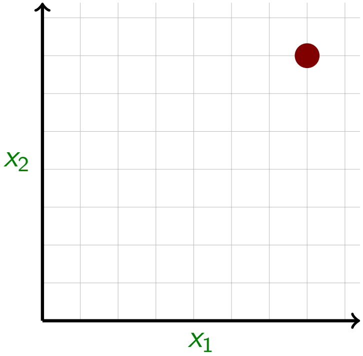

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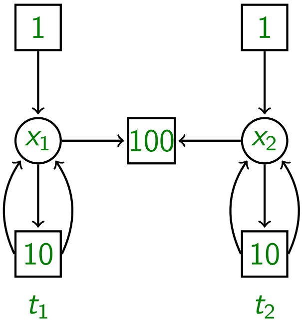

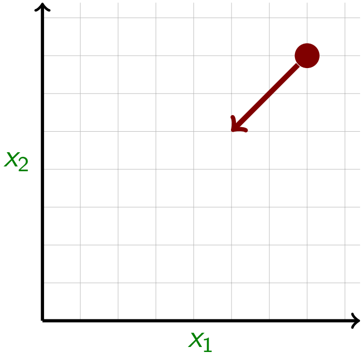

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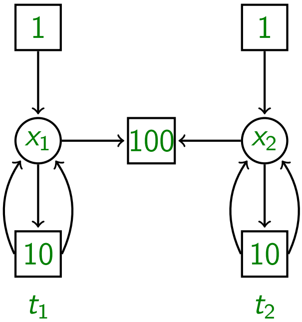

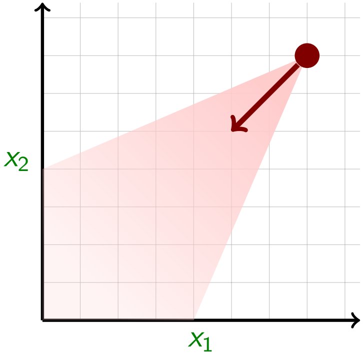

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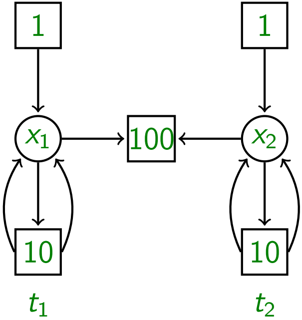

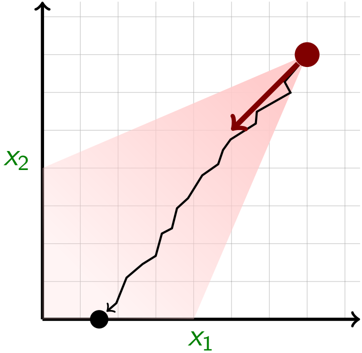

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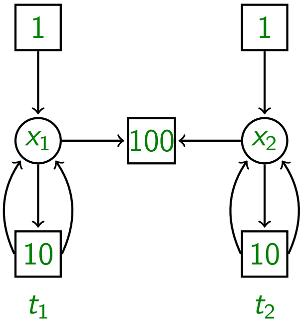

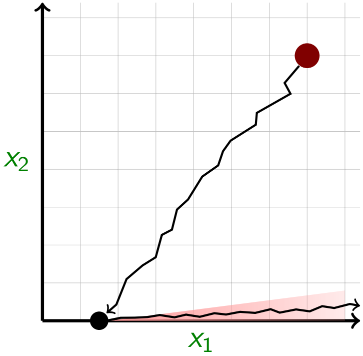

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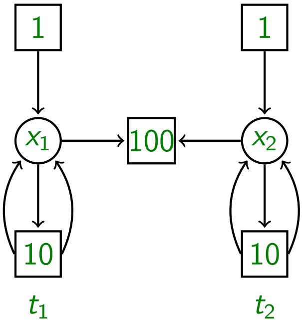

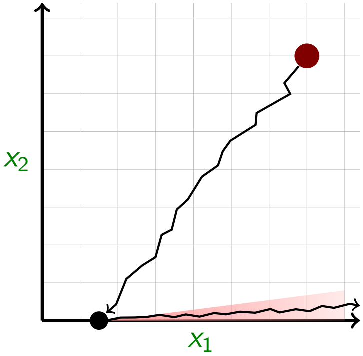

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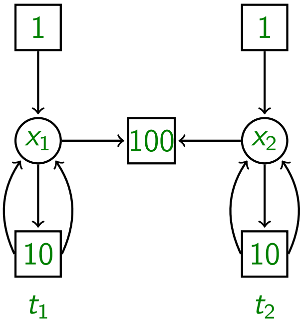

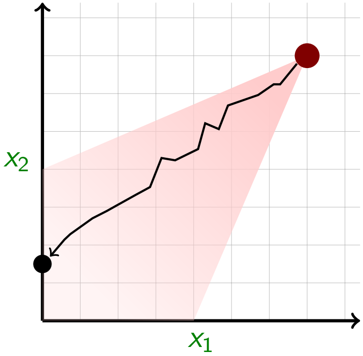

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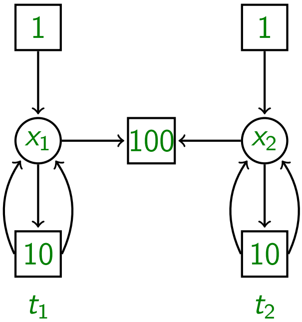
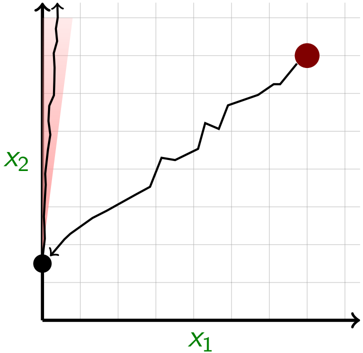




Pattern Frequency in 2-dimensional pVASS

Theorem 2

Let $p\vec{v}$ be an initial configuration in a 2-dimensional pVASS. Then

- $\mathcal{P}[\mathcal{F}{=}\bot] = 0$;
- $\mathcal{F}$ *is a discrete random variable;*
- $\mathcal{F}$ *may take infinitely many values with a positive probability, and it is decidable whether this set is finite or infinite;*
- *these values and the associated probabilities can be approximated up to an arbitrarily small given* $\varepsilon > 0$.

- The complexity bounds employ the complexity results about 2-dimensional (non-probabilistic) VASS.

Pattern Frequency in 3-dimensional pVASS

Theorem 3

There exists a 3-dimensional pVASS $\mathcal{A}$ and a initial configuration $p\vec{v}$ such that the graph of reachable configurations is strongly connected and $\mathcal{P}[\mathcal{F}{=}\bot] = 1$. Further, this property is preserved in ε-perturbations of $\mathcal{A}$ for some $\varepsilon > 0$.

Pattern Frequency in 3-dimensional pVASS (cont.)

The idea behind the construction of $\mathcal{A}$:

$$(k,0,0) \xrightarrow{(+,+,0)} (0,2k,0) \xrightarrow{(0,+,+)} (0,0,4k) \xrightarrow{(+,0,+)}$$
$$(8k,0,0) \xrightarrow{(+,+,0)} (0,16k,0) \xrightarrow{(0,+,+)} (0,0,32k) \xrightarrow{(+,0,+)}$$
$$(64k,0,0) \xrightarrow{(+,+,0)} (0,128k,0) \xrightarrow{(0,+,+)} (0,0,256k) \xrightarrow{(+,0,+)}$$

Conclusions

- For pVASS with at least three counters, the limit-average behaviour is generally undefined and this feature can be robust.
- The remaining challenge is to identify reasonable sufficient conditions for eVASS with $d \geq 3$ counters under which the limit-average behaviour exists and can be effectively analyzed.

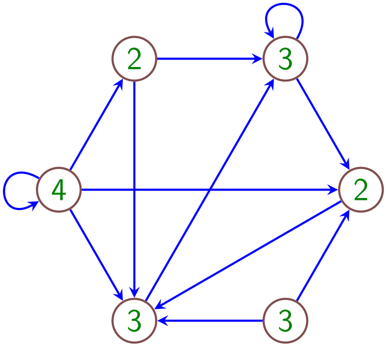
Patrolling Problem (Informally)

- One of the basic problems in operations research.
- Design the best possible strategy for a patroller who travels among a given set of vulnerable targets and aims at detecting possible intrusions.
- Many technical variants: the number of patrollers/attackers, attacker's abilities, various levels of target improtance/vulnerability, etc.

Patrolling Problem (Informally)

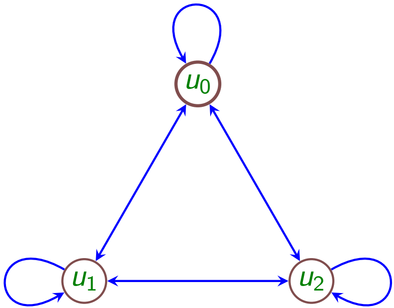
- Existing results.
    - Main task: Compute as good strategy as possible.
    - Main tools: Mathematical programming (scalability problems).
    - Some basic "game-theoretic" questions are not studied in greater detail (and sometimes answered incorrectly).
- This contribution.
    - We study adversarial patrolling games in unrestricted environment where all targets are equally important.
    - We yield a compositional method for computing (sub)optimal strategies (no scalability problems).

Patrolling Problem (Formally)



- Defender's strategy: $\sigma : V^+ \to \Delta(V)$
- Attacker's strategy: $\pi : V^+ \to V \cup \{*\}$ (must be "prefix free")
- $\mathcal{P}^{\sigma,\pi}(DRuns)$
- $val = \sup_\sigma \inf_\pi \mathcal{P}^{\sigma,\pi}(DRuns)$

Example 1



Attack length $= 2$

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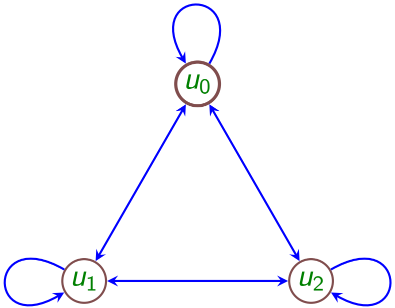

Attack length $= 2$

$\sigma(h) = \mu_\ell, \;\; \ell = |h| \; mod \; 2$

$\mu_0(u_0) = 0,$
$\mu_0(u_1) = \kappa,$
$\mu_0(u_2) = 1 - \kappa$

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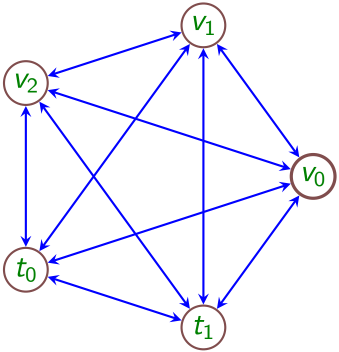
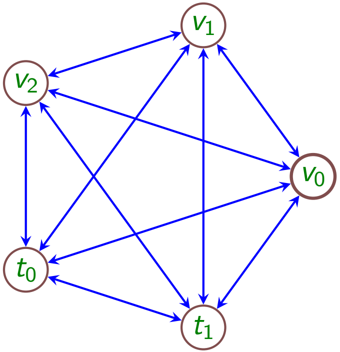
$\kappa = (\sqrt{5} - 1)/2 = 0.618\ldots$

Example 2



$d(t_i)=2, d(v_i)=3$

Example 2



$d(t_i)=2, d(v_i)=3$

$\sigma(h)$ selects uniformly between $v_{|h|+1 \, mod \, 3}$ and $t_{|h|+1 \, mod \, 2}$

$val^\sigma = 1/2$

$val = 1/2$

The Existence of Optimal Strategies

- Recall $val = \sup_\sigma \inf_\pi \mathcal{P}^{\sigma,\pi}(DRuns)$
- A strategy σ^* is *optimal* if $\inf_\pi \mathcal{P}^{\sigma^*,\pi}(DRuns) = val$

The Existence of Optimal Strategies

- Recall $val = \sup_\sigma \inf_\pi \mathcal{P}^{\sigma,\pi}(DRuns)$
- A strategy σ^* is *optimal* if $\inf_\pi \mathcal{P}^{\sigma^*,\pi}(DRuns) = val$

Theorem 4

There exists an optimal strategy σ^* for the defender.

An Upper Bound on the Value

- A *signature* of a game $\mathcal{G}$ is a function $S : \mathbb{N} \to \mathbb{N}_0$ where $S(k)$ is the total number of nodes u where $d(u) = k$.

An Upper Bound on the Value

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Theorem 5

Let $\mathcal{G}$ be a game with signature S. Then

$$val \leq \left(\sum_{k \in supp(S)} \frac{S(k)}{k} \right)^{-1}$$

where $supp(S)$ is the set of all $k \in \mathbb{N}$ such that $S(k) > 0$.

Compositionality in Patrolling Games (1)

- A strategy σ is *modular* if $\sigma(h)$ depends only on $|h| \mod c$ for some $c \in \mathbb{N}$.
- Let $\mathcal{G}$ be a game, and let $U = U_1 \uplus \ldots \uplus U_n$.
- Let $\sigma_1, \ldots, \sigma_n$ be modular strategies for $\mathcal{G}/U_1, \ldots, \mathcal{G}/U_n$.
- Let $\mu \in \Delta\{1, \ldots, n\}$. A μ-*composition* of $\sigma_1, \ldots, \sigma_n$ is a modular strategy σ for $\mathcal{G}$ defined by $\sigma(h) = \sum_{i=1}^{n} \mu(i) \cdot \sigma_i(h)$

Theorem 6

We have that $val^{\sigma} \geq \min_{i=1}^{n} \mu(i) \cdot val^{\sigma_i}$

Compositionality in Patrolling Games (2)

- A signature S is *well-formed* if k divides $S(k)$ for every $k \in \mathbb{N}$.

Theorem 7

Let $\mathcal{G}$ be a patrolling game with a well-formed signature S where the graph of $\mathcal{G}$ is complete. Then there exist an optimal modular strategy for the defender constructible in polynomial time.

Compositionality in Patrolling Games (3)

Theorem 8

For every well formed signature S there exists a graph $\mathcal{H}_S$ computable in polynomial time such that for every patrolling game $\mathcal{G}$ with signature S we have the following: the defender has a strategy achieving the upper bound $\left(\sum_{k \in supp(S)} \frac{S(k)}{k}\right)^{-1}$ iff $\mathcal{H}_S$ is a subgraph of $\mathcal{G}$.

The problem whether the defender has a strategy achieving the bound in a given $\mathcal{G}$ is **NP***-complete.*

Open problems

- Classification/construction of optimal strategies for games with non-well-formed signatures.
- Extending the obtained results to more general models.
- Meta-theorems for security games.