Guaranteed Bounds for Posterior Inference in Universal Probabilistic Programming

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(Joint work with Raven Beutner and Fabian Zaiser)

What is (Bayesian Statistical) Probabilistic Programming?

Bayes' Rule

$$\mathbb{P}[\theta \mid \mathcal{D}] = \frac{\mathbb{P}[\mathcal{D} \mid \theta] \; \mathbb{P}[\theta]}{\mathbb{P}[\mathcal{D}]}$$

$$\text{Posterior} \;\; \propto \;\; \text{Likelihood} \times \text{Prior}$$

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Probabilistic Programming

– a general-purpose means of expressing probabilistic models as programs, and automatically performing Bayesian inference.

- Probabilistic programming offers an elegant way of generalising graphical models, allowing a much richer representation of models, compositionally.
- Probabilistic programming systems are equipped with implementations of general-purpose inference algorithms.

Vision of Probabilistic Programming

- Expressing probabilistic models as programs: elegant, unifying, potentially benefiting from PL research (semantics and program analysis).
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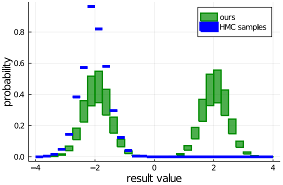
Unfortunately existing inference algorithms

✗ have few guarantees on the result, and / or

✗ only work on a restricted class of programs (models).



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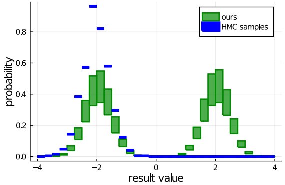

Our Contributions

Guaranteed (nonstochastic and sound) bounds on the posterior distributions.

✓ Diagnostics / (partial) correctness specification: can identify errors in inference results

✓ General applicability: works for a very broad class of probabilistic programs

✓ Basis for a new general-purpose inference algorithm (ongoing work).

A Challenging Example Model: Pedestrian

A pedestrian suddenly found himself on a long road. He knows he is no more than 3 km from his home. To get home, he does a random walk: he walks for a random distance no more than 1km, in a random direction (either away or towards his home); and he repeats this `step` until he reaches home. Upon reaching home, his old step counter tells him he has travelled 1.1km.

Question: Where was he (`start`) on the road when he started the random walk?

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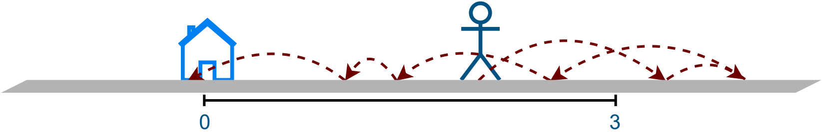

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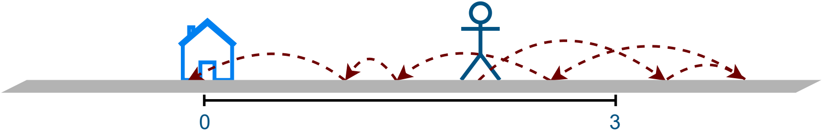

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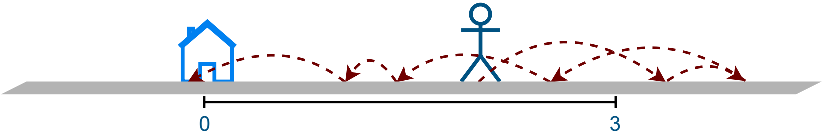

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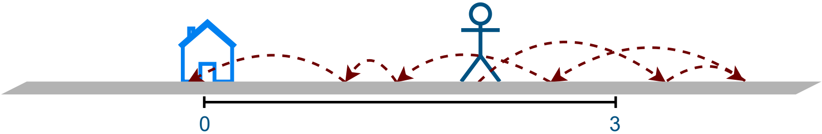

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Posterior distribution: $p(start \mid \text{observation})$?

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- Monte Carlo (particle filter; Metropolis-Hastings, Gibbs sampling, HMC, etc.): "*Through the use of random processes, Metropolis algorithm [Monte Carlo method] offers an efficient way to stumble toward answers to problems that are too complicated to solve exactly.*" (IEEE Computing award citation)
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2. **Exact methods:** posterior $= X$

– Computing a closed-form solution of the posterior inference problem using computer algebra and other forms of symbolic calculations.

PSI SOLVER HAKARU *SPPL*

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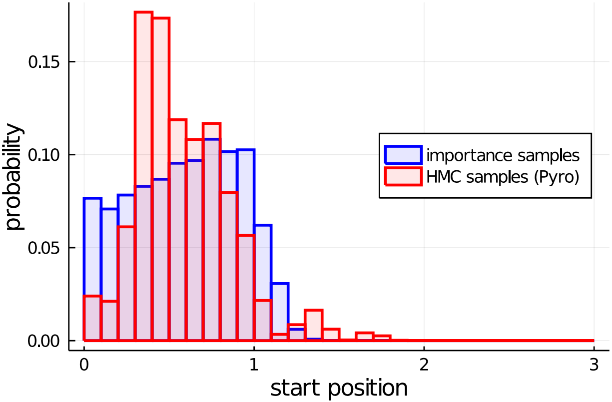
A (Real) Conundrum: Pedestrian Example



The two distributions are clearly different: at least one is wrong, but which?
(This problem actually sparked and drove the present project.)

Guaranteed Bounds on the Posterior: posterior$_P(E) \in [a, b]$

Desiderata

- A middle ground between exact and approximate methods.
- Given arbitrary program P of a universal PPL with continuous distributions and **`observe`**, and error tolerance, infer guaranteed (nonstochastic and sound) bounds of the posterior: $a \leq \text{posterior}_P(E) \leq b$

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Unnormalized posterior of E (joint distribution of latent and observed variables):

$$[\![P]\!](E) := \int_{\{s \mid \mathsf{val}(s) \in E\}} \mathsf{weight}(s)\,\mathrm{d}s = \text{“}\,\mathbb{P}(\mathtt{start} \in E, \mathsf{obs})\,\text{”}$$

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By Bayes' rule, **normalized posterior** (conditional probability):

$$\mathbb{P}(\mathtt{start} \in E \mid \mathsf{obs}) = \frac{\mathbb{P}(\mathtt{start} \in E, \mathsf{obs})}{\mathbb{P}(\mathsf{obs})} = \frac{[\![P]\!](E)}{[\![P]\!](\mathbb{R})}$$

N.B. Normalising constant, $\mathbb{P}(\mathsf{obs})$, is a special case of unnormalised posterior.

Interval Traces

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An interval trace is just a sequence of intervals of the reals; e.g. $\langle [0.1, 0.3], [0.7, 1] \rangle$.

So a set of interval traces summarises a set of traces. E.g. interval trace $\langle [0.1, 0.3], [0.7, 1] \rangle$ contains (or is refined by) traces $\langle 0.2, 0.9 \rangle$ and $\langle 0.23, 0.75 \rangle$.

Idea behind upper bounding $[\![P]\!](E)$:

- Given event E, find a summary (i.e. covering set) $\mathcal{T}$ of interval traces: every $\boldsymbol{s}$ s.t. $\mathsf{val}(\boldsymbol{s}) \in E$ is contained in some interval trace in $\mathcal{T}$.

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- Then

$$[\![P]\!](E) \leq \sum_{\boldsymbol{t} \in \mathcal{T}} (\max \mathsf{weight}(\boldsymbol{t})) \, \mathrm{vol}(\boldsymbol{t})$$

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`start`		
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weight weight$(\boldsymbol{s})$	1	$[1, 1]$
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Robustness, Precision and Effectivity

Soundness

For all non-overlapping and exhaustive set of interval traces $\mathcal{T}$:

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$\mathsf{lowerBd}_P^{\mathcal{T}}$ and $\mathsf{upperBd}_P^{\mathcal{T}}$ are super-/sub-additive measures.

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For all intervals I and $\epsilon > 0$, there is a countable set $\mathcal{T}$ of (non-overlapping and exhaustive) interval traces s.t.

$$\mathsf{upperBd}_P^{\mathcal{T}}(I) - \epsilon \leq [\![P]\!](I) \leq \mathsf{lowerBd}_P^{\mathcal{T}}(I) + \epsilon$$

under the assumptions:

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For all intervals I and $\epsilon > 0$, there is a finite set $\mathcal{T}$ of (non-overlapping and exhaustive) interval traces s.t.

$$[\![P]\!](I) \leq \mathsf{lowerBd}_P^{\mathcal{T}}(I) + \epsilon$$

under the assumptions:

- the primitive functions are continuous*
- each **`sample`**d value is used at most once in each conditional, **`observe`** statement, and in the return value.

Empirical Evaluation

- Implementation: GuBPI (Guaranteed Bounds for Posterior inference)
  gubpi-tool.github.io

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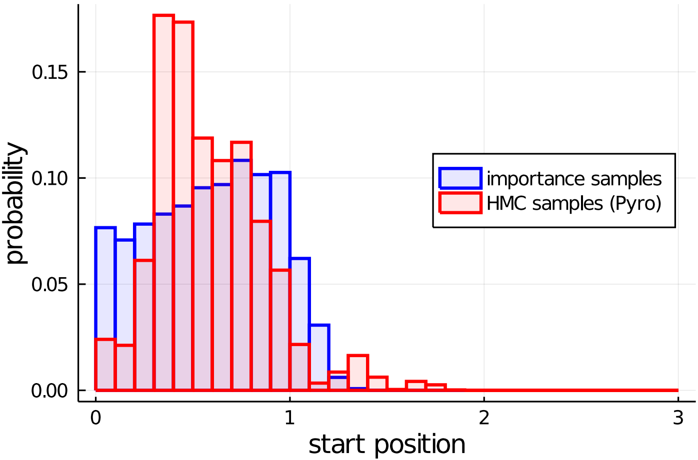
Pedestrian example:



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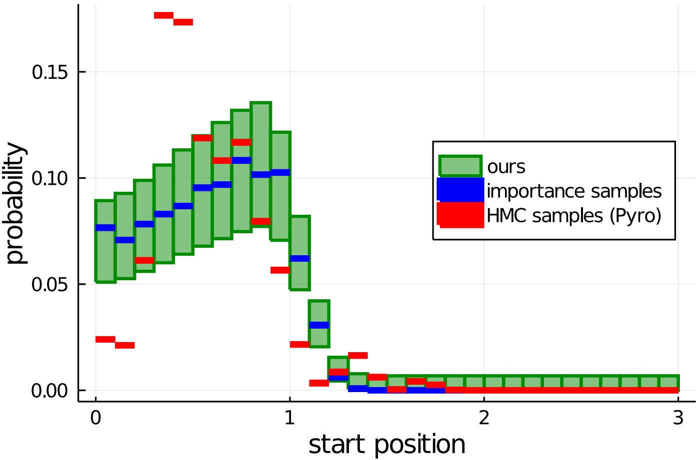
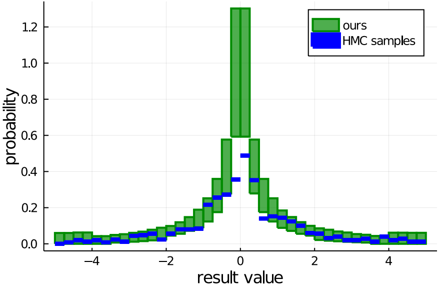
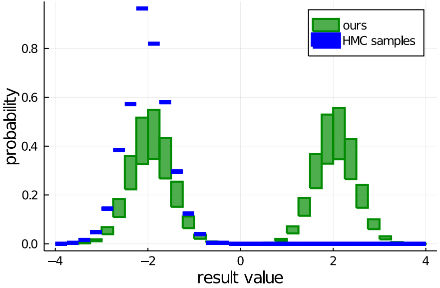


Examples that Trip Up MCMC!



Neal's funnel



Mixture model

Comparison with Previous Work

Sankaranarayanan et al. (PLDI2013)

- framework for bounding probabilities of events definable by score-free programs
- ours is more general, hence usually slower, but often finds tighter bounds

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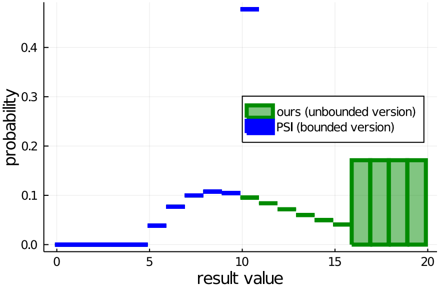
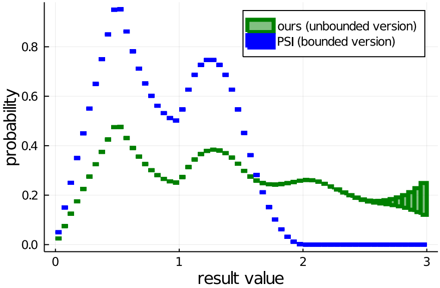
PSI solver (CAV2016)

- consistency check: benchmarks from the PSI repository
- we can handle unbounded loops, contrary to PSI. Warning: Artificially bounding recursive programs can yield different (hence wrong) posterior distribution!





Also in our PLDI22 paper[1]

- **Constraint-based interval type system:** approximates unbounded loops and recursion *soundly*
- **Symbolic execution & linear programming:** optimization for linear guards
- **Comparison with statistical validation methods:** simulation-based calibration

[1] Raven Beutner, C.-H. Luke Ong, Fabian Zaiser: Guaranteed bounds for posterior inference in universal probabilistic programming. PLDI 2022: 536-551

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Limitations: GuBPi struggles if

- program has lots of branching: path explosion problem
- model is high-dimensional (i.e. has many `sample`s)

[1] Raven Beutner, C.-H. Luke Ong, Fabian Zaiser: Guaranteed bounds for posterior inference in universal probabilistic programming. PLDI 2022: 536-551

Guaranteed Bounds for Posterior Inference in Universal Probabilistic Programming

Raven Beutner Luke Ong Fabian Zaiser

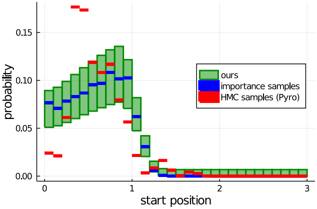
. . . are a **middle ground** between *approximate* and *exact*:

- guaranteed correct (vs. approximate inference)
- supports many language features (vs. exact inference)

Theory: soundness & completeness

Practice:

- detect issues with inference results
- competitive on existing benchmarks
- guaranteed partial correctness specifications for programs that other tools cannot handle



Future work

- better heuristics for finding a "good" set of interval traces $\mathcal{T}$
- basis for a new approximate inference algorithm.