# Results about Asymptotic Termination Analysis for VASS Programs

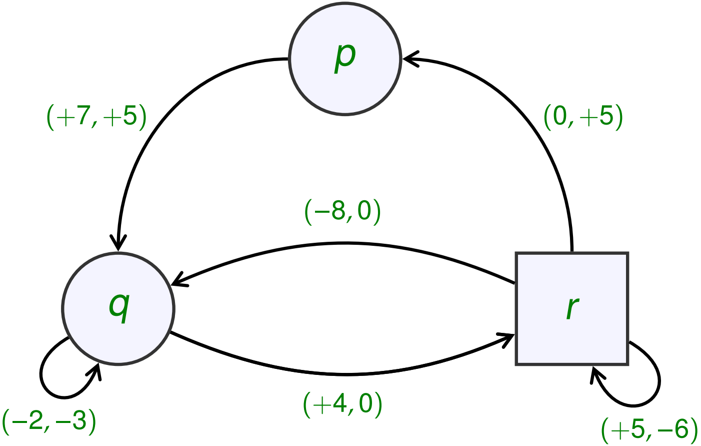
Antonín Kučera

based on a joint work with

Michal Ajdarów, Tomáš Brázdil, Krishnendu Chatterjee, Jérôme Leroux,
Petr Novotný, Dominik Velan, Florian Zuleger

IFIP WG 2.2, September 2022

# VASS Games



# VASS Games

- VASS control states are split into angelic and demonic subsets. Demon/Angel wants to prolong/shorten a computation initiated in $p\vec{v}$.
- Let $\sigma$ and $\pi$ be strategies for Angel and Demon, and let $Comp^{\sigma,\pi}(p\vec{v})$ be the maximal computation initiated in $p\vec{v}$ determined by $\sigma$ and $\pi$.
- $Tval(p\vec{v}) = \sup_{\pi} \inf_{\sigma} |Comp^{\sigma,\pi}(p\vec{v})| = \inf_{\sigma} \sup_{\pi} |Comp^{\sigma,\pi}(p\vec{v})|$
- Termination complexity: $\mathcal{L} : \mathbb{N} \to \mathbb{N}_\infty$ defined by

$$\mathcal{L}(n) = \max \left\{ Tval(p\vec{v}) \mid p \in Q, \vec{v} \le \vec{n} \right\}$$

# VASS Games

- VASS control states are split into angelic and demonic subsets. Demon/Angel wants to prolong/shorten a computation initiated in $p\vec{v}$.
- Let $\sigma$ and $\pi$ be strategies for Angel and Demon, and let $Comp^{\sigma,\pi}(p\vec{v})$ be the maximal computation initiated in $p\vec{v}$ determined by $\sigma$ and $\pi$.
- $Tval(p\vec{v}) \;=\; \sup_{\pi} \inf_{\sigma} |Comp^{\sigma,\pi}(p\vec{v})| \;=\; \inf_{\sigma} \sup_{\pi} |Comp^{\sigma,\pi}(p\vec{v})|$
- Termination complexity: $\mathcal{L} : \mathbb{N} \to \mathbb{N}_\infty$ defined by

$$\mathcal{L}(n) \quad = \quad \max\left\{ Tval(p\vec{v}) \;\mid\; p \in Q,\ \vec{v} \leq \vec{n} \right\}$$

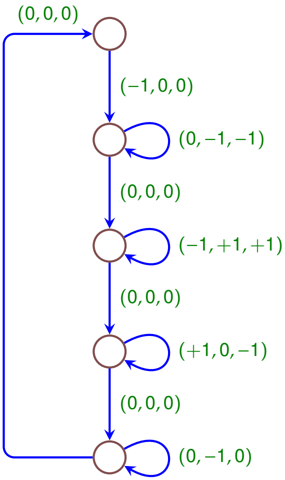
- $Cval[c](p\vec{v}) = \sup_{\pi} \inf_{\sigma} \max[c](Comp^{\sigma,\pi}(p\vec{v})) \;=\; \inf_{\sigma} \sup_{\pi} \max[c](Comp^{\sigma,\pi}(p\vec{v}))$
- Counter complexity: $\mathcal{C}[c] : \mathbb{N} \to \mathbb{N}_\infty$ defined by

$$\mathcal{C}[c](n) \quad = \quad \max\left\{ Cval[c](p\vec{v}) \;\mid\; p \in Q,\ \vec{v} \leq \vec{n} \right\}$$

# Example I

```
input i
while(i>0)
    i = i-1
    j = i
    while(j>0)
        j = j-1
```

```
input i
while(i>0)
    i = i-1
    j = 0; Aux = 0
    while(i>0)
        i = i-1
        j = j+1
        Aux = Aux+1
    while(Aux>0)
        i = i+1
        Aux = Aux-1
    while(j>0)
        j = j-1
```



# Example II

```
input i;
j:=0; k:=0; z:=0;

if condition    // demonic choice //
    then while (i>0) do j++; k:=k+i; i--; done
    else j:=i*i; k:=i;
         while (i>0) do j:=j+k; i--; done

choose:         // angelic choice //
    while (j>0) do j--; z++ done
or: while (k>0) do k--; z++ done
```

# Some Basic Questions (1)

- Can we decide whether $\mathcal{L}(n)$ is finite for all $n$?
  - Demonic VASS: $\mathcal{L}(n) = \infty$ for some $n$ iff there exists a "self-covering" computation $p\vec{v} \rightarrow^* p\vec{u}$ where $\vec{v} \leq \vec{u}$. Hence, the problem is in **P**.
  - VASS Games: If Angel has some strategy $\sigma^*$ such that $\sup_\pi |Comp^{\sigma^*,\pi}(p\vec{v})|$ is finite for every $p\vec{v}$, then Angel has a counterless strategy with this property. The problem is **NP**-complete.

# Some Basic Questions (2)

- How fast can $\mathcal{L}(n)$ grow?
  - $F_1(n) = 2n + 1, \quad F_2(n) = n^2, \quad F_3(n) = 2^n,$
  - $F_{k+1}(n) = \underbrace{F_k \circ \cdots \circ F_k}_{n}(n)$ for $k \geq 3$
  - $\mathcal{G}_k \quad = \quad \left\{ f : \mathbb{N} \to \mathbb{N} \;\mid\; f \leq \underbrace{F_k \circ \cdots \circ F_k}_{\mu} \text{ for some } \mu \in \mathbb{N} \right\}$
  - For every $k$, there exists a terminating demonic VASS such that $\mathcal{L}(n) \geq F_k(n)$ for all sufficiently large $n$.
    [Mayr and Meyer; JACM 1981]
  - For every terminating demonic VASS with $d$ counters, $\mathcal{L} \in \mathcal{G}_{d+1}$.
    [Schmitz; Icalp 2018]

# Grzegorczyk Complexity of Demonic VASS

**Theorem 1 ([K., Leroux, Velan; LICS 2020)**

*Let $k \in \mathbb{N}$. The problem whether $\mathcal{L} \in \mathcal{G}_k$ for a given demonic VASS is in **P**. Furthermore, if $\mathcal{L} \notin \mathcal{G}_k$, then there is a constant $c$ such that $\mathcal{L}(n) \geq F_{k+1}(\lfloor n/c \rfloor)$ for all sufficiently large $n$.*

# Polynomial Complexity of Demonic VASS

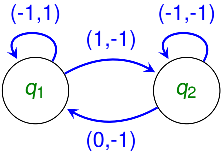
**Theorem 2 (Brázdil, Chatterjee, K., Novotný, Velan, Zuleger; LICS 2018)**

*The problem whether $\mathcal{L} \in \mathcal{O}(n)$ for a given demonic VASS is in **P**. Furthermore, if $\mathcal{L} \notin \mathcal{O}(n)$, then $\mathcal{L} \in \Omega(n^2)$.*

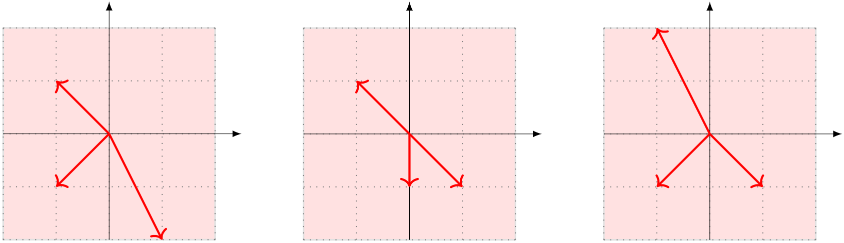
# Linear VASS termination

- Assume a strongly connected demonic VASS $\mathcal{A}$.
- Let *Inc* be the set of effect of all simple cycles.
- *Inc* is computable in time polynomial in $\mathcal{A}$ and exponential in $d$.

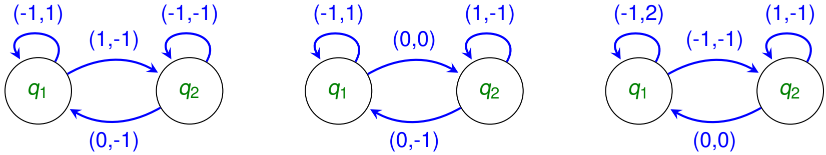
# Linear VASS termination (2)



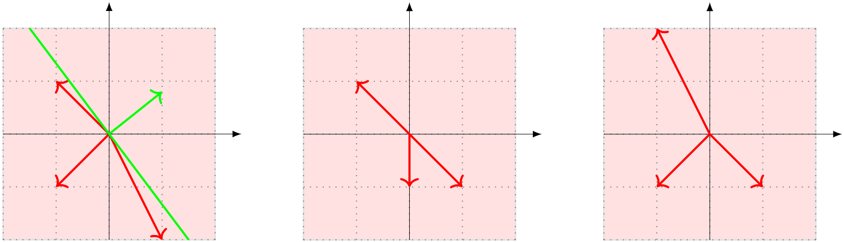

$(-1,1),(-1,-1),(1,-2)$ $(-1,1),(1,-1),(0,-1)$ $(-1,2),(1,-1),(-1,-1)$



# Linear VASS termination (2)




$(-1,1),(-1,-1),(1,-2)$ $\quad$ $(-1,1),(1,-1),(0,-1)$ $\quad$ $(-1,2),(1,-1),(-1,-1)$



# Linear VASS termination (3)

- Let $\mathcal{A}$ be a VASS of dimension $d$ where $Q$ is the set of states.
- Every $\mathbf{c} \in \mathbb{R}^d$ and $h : Q \to \mathbb{R}$ determine the corresponding weighted linear map defined by $\mu(p\vec{v}) = \mathbf{c} \cdot \vec{v} + h_p$
- A weighted linear map $\mu$ is a weighted linear ranking function for $\mathcal{A}$ if $\mathbf{c} \geq 0$ and there is $\varepsilon > 0$ s.t., for every $p\vec{v}$ and $p\vec{v} \to q\vec{u}$,

$$\mu(p\vec{v}) \geq \mu(q\vec{u}) + \varepsilon$$

# Linear VASS termination (4)

**Theorem 3**

*Let $\mathcal{A}$ be a VASS. We have that $\mathcal{L}(n) \in \mathcal{O}(n)$ iff there exists a weighted linear ranking function for $\mathcal{A}$. Further, the existence of a weighted linear ranking function for $\mathcal{A}$ can be decided in time polynomial in $||\mathcal{A}||$.*

- Establishing the existence of a weighted linear ranking function for $\mathcal{A}$ is a sound and complete method for proving linear VASS termination.

# Deciding "Polynomiality" of Demonic VASS

**Theorem 4 (Leroux; Icalp 2018)**

*The problem whether there exist $k \in \mathbb{N}$ such that $\mathcal{L} \in \mathcal{O}(n^k)$ for a given demonic VASS is in **P**. Furthermore, if $\mathcal{L} \notin \mathcal{O}(n^k)$ for any $k$, then $\mathcal{L} \in 2^{\Omega(n)}$.*

- An iteration scheme of $\mathcal{A}$ is a sequence of cycles $\varrho_1, \ldots, \varrho_m$ such that every counter $c$ decreased by some $\varrho_i$ is strictly increased by the total effect of the sequence.
- $\mathcal{L}(n)$ is not polynomial iff there is an iteration scheme of $\mathcal{A}$.
- $\mathcal{C}[c](n)$ is not polynomial iff there is an iteration scheme whose total effect increases $c$.

# Classifying Polynomial Demonic VASS

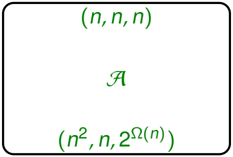
**Theorem 5 (Zuleger, FoSSaCS 2020)**

*Let $\mathcal{A}$ be a strongly connected demonic VASS with $d$ counters. For every counter $c$, we have that either $\mathcal{C}[c] \in \Theta(n^k)$ for some $1 \leq k \leq 2^d$, or $\mathcal{C}[c] \in 2^{\Omega(n)}$. It is decidable in polynomial time which of the two possibilities holds. In the first case, the $k$ is computable in polynomial time.*

# Classifying Polynomial Demonic VASS

**Theorem 5 (Zuleger, FoSSaCS 2020)**

*Let $\mathcal{A}$ be a strongly connected demonic VASS with $d$ counters. For every counter $c$, we have that either $\mathcal{C}[c] \in \Theta(n^k)$ for some $1 \leq k \leq 2^d$, or $\mathcal{C}[c] \in 2^{\Omega(n)}$. It is decidable in polynomial time which of the two possibilities holds. In the first case, the $k$ is computable in polynomial time.*



# Classifying Polynomial Demonic VASS (2)

## Theorem 6 (Ajdarów, K.; CONCUR 2021)

*Let $k \geq 1$. For every demonic VASS $\mathcal{A}$ we have that $\mathcal{L}$ is either in $\mathcal{O}(n^k)$ or in $\Omega(n^{k+1})$. Furthermore, the problem whether*

- $\mathcal{L} \in \mathcal{O}(n^k)$ *is in* ***P*** *for $k = 1$, and* ***coNP****-complete for $k \geq 2$;*
- $\mathcal{L} \in \Omega(n^k)$ *is in* ***P*** *for $k \leq 2$, and* ***NP****-complete for $k \geq 3$;*
- $\mathcal{L} \in \Theta(n^k)$ *is in* ***P*** *for $k = 1$,* ***coNP****-complete for $k = 2$, and* ***DP****-complete for $k \geq 3$.*

*Similar results hold also for $C[c]$.*

# Classifying Polynomial Demonic VASS (3)

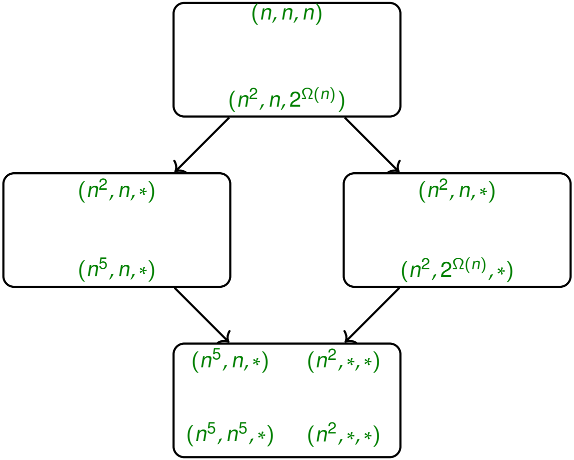
- For every $\vec{v} = (k_1, \ldots, k_d) \in \mathbb{N}^d$, let $\mathcal{C}[c, \vec{v}] : \mathbb{N} \to \mathbb{N}_\infty$ where

$$\mathcal{C}[c, \vec{v}](n) \quad = \quad \max\left\{ Cval[c](p\vec{u}) \mid p \in Q,\ \vec{u} = (n^{k_1}, \ldots, n^{k_d}) \right\}$$

## Theorem 7

*Let $\mathcal{A}$ be a strongly connected demonic VASS with $d$ counters, and let $\vec{v} \in \mathbb{N}^d$ such that $\vec{v}(i) \leq 2^{j \cdot d}$ for every $i \leq d$, where $j < |Q|$. For every counter $c$, we have that either $\mathcal{C}[c, \vec{v}] \in \Theta(n^k)$ for some $1 \leq k \leq 2^{(j+1) \cdot d}$, or $\mathcal{C}[c, \vec{v}] \in 2^{\Omega(n)}$. It is decidable in polynomial time which of the two possibilities holds. In the first case, the $k$ is computable in polynomial time.*

# Classifying Polynomial Demonic VASS (4)



# Results about VASS Games

**Theorem 8 (Ajdarów, K.; CONCUR 2021)**

*Let $k \geq 1$. For every VASS game $\mathcal{A}$ we have that $\mathcal{L}$ is either in $\mathcal{O}(n^k)$ or in $\Omega(n^{k+1})$. Furthermore, the problem whether*

- $\mathcal{L} \in \mathcal{O}(n^k)$ *is **NP**-complete for $k{=}1$ and **PSPACE**-complete for $k{\geq}2$;*
- $\mathcal{L} \in \Omega(n^k)$ *is in **P** for $k{=}1$, **coNP**-complete for $k{=}2$, and **PSPACE**-complete for $k{\geq}3$;*
- $\mathcal{L} \in \Theta(n^k)$ *is **NP**-complete for $k{=}1$ and **PSPACE**-complete for $k{\geq}2$.*

*The same results hold also for $\mathcal{C}[c]$.*

# Polynomial termination for VASS games

- Angel must take into account the asymptotic growth of the counters along the computational history.
- Key new concepts:
  - locking strategy
  - locking decomposition

# Future work

- The analysis of termination complexity for VASS MDPs (and VASS stochastic games).

A recent summary:

A. Kučera. *Algorithmic Analysis of Termination and Counter Complexity in Vector Addition Systems with States: A Survey of Recent Results.* ACM SIGLOG News. 8(4):4-21, 2021.