On Higher-Order Probabilistic Computation:
Relational Reasoning, Termination, and Bayesian Programming

Ugo Dal Lago

(Based on joint work with *Michele Alberti*, *Raphaëlle Crubillé*, *Charles Grellois*, *Davide Sangiorgi*,...)

ALMA MATER STUDIORUM UNIVERSITÀ DI BOLOGNA

Inria informatiques mathématiques

IFIP WG 2.2 Annual Meeting, Brno, September 17th

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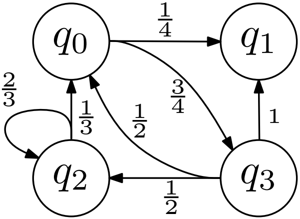

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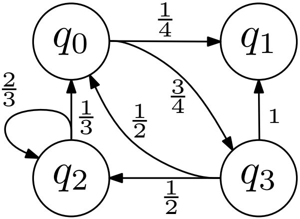
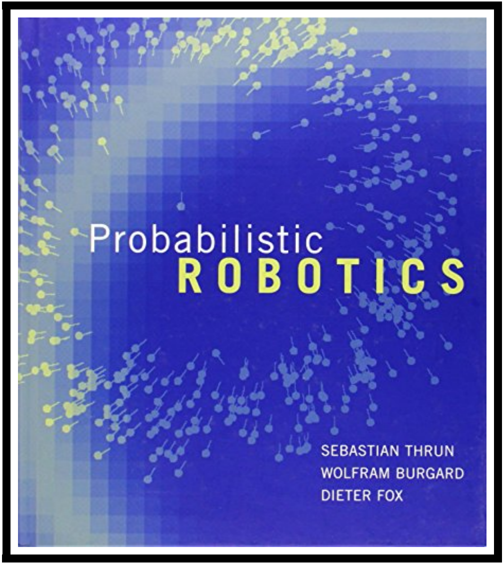


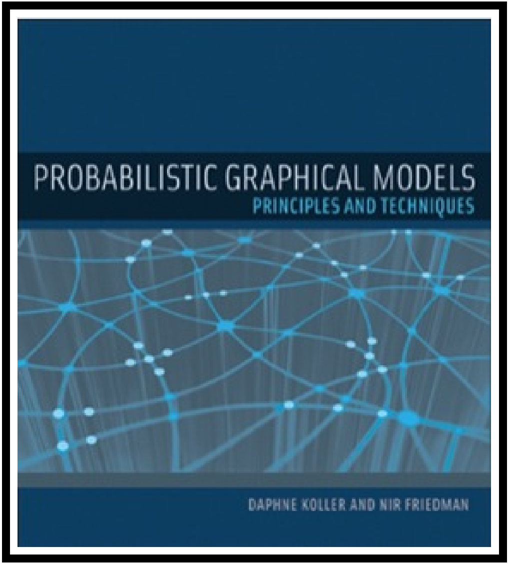
- **Abstractions:**
  - (Labelled) Markov Chains.

Probabilistic Models



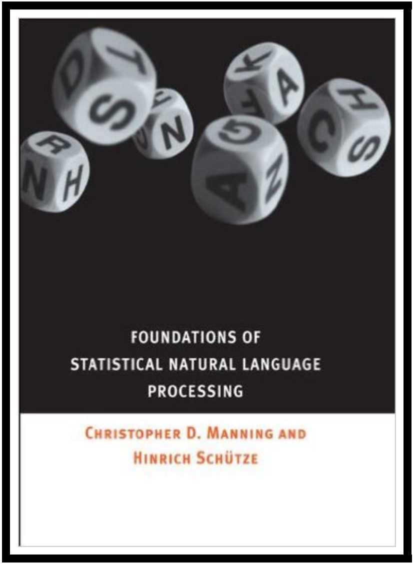
ROBOTICS

Probabilistic Models



ARTIFICIAL INTELLIGENCE

Probabilistic Models



NATURAL LANGUAGE
PROCESSING

Randomized Computation

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```
Input: n > 3, an odd integer to be tested for primality;
Input: k, a parameter that determines the accuracy of the test
Output: composite if n is composite, otherwise probably prime
write n − 1 as 2^s·d with d odd by factoring powers of 2 from n − 1
WitnessLoop: repeat k times:
   pick a random integer a in the range [2, n − 2]
   x ← a^d mod n
   if x = 1 or x = n − 1 then do next WitnessLoop
   repeat s − 1 times:
      x ← x^2 mod n
      if x = 1 then return composite
      if x = n − 1 then do next WitnessLoop
   return composite
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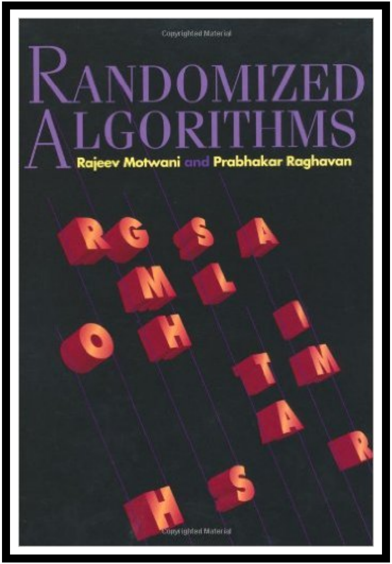
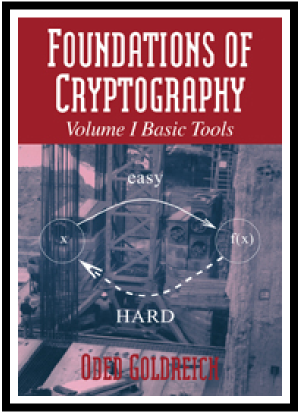
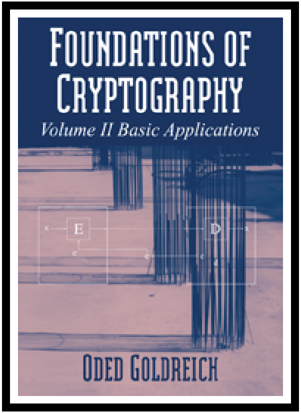
- **Abstractions:**
  - Randomized algorithms;
  - Probabilistic Turing machines.
  - Labelled Markov chains.

Randomized Computation



ALGORITHMICS





CRYPTOGRAPHY

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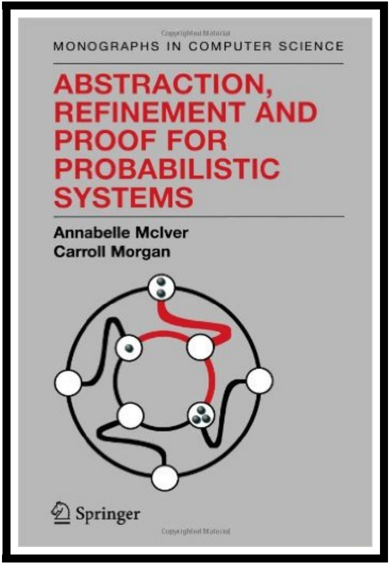

PROGRAM VERIFICATION

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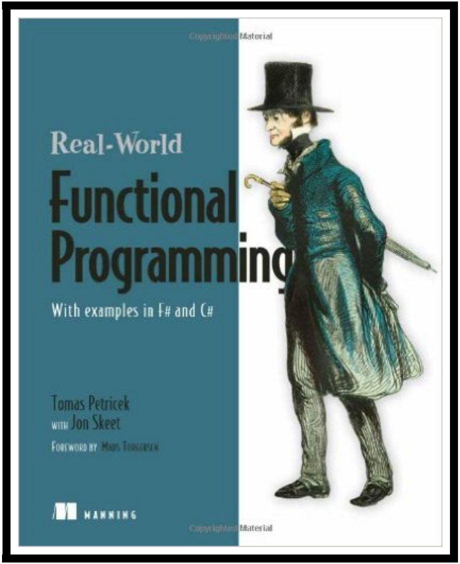
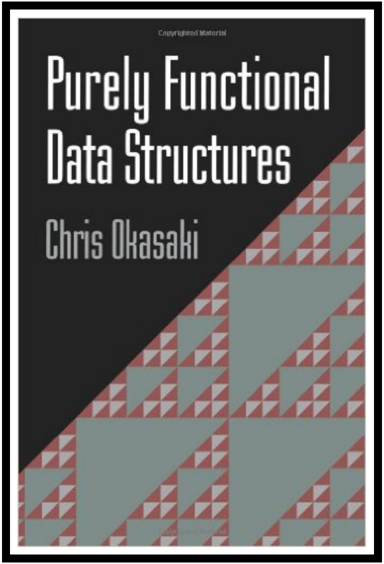
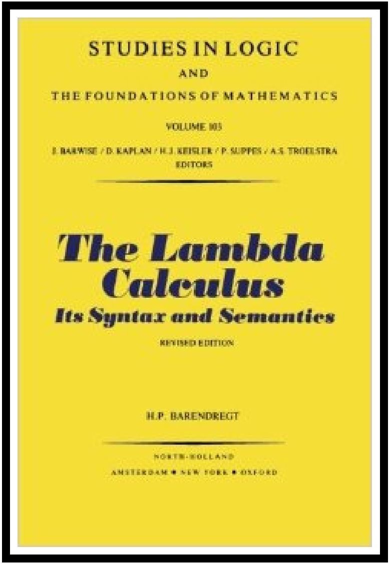
- **Models**:
  - λ-calculus



FUNCTIONAL PROGRAMMING



FUNCTIONAL
DATA STRUCTURES



λ-CALCULUS

Does it Make Sense?

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What Kind of Metatheory Does it Have?

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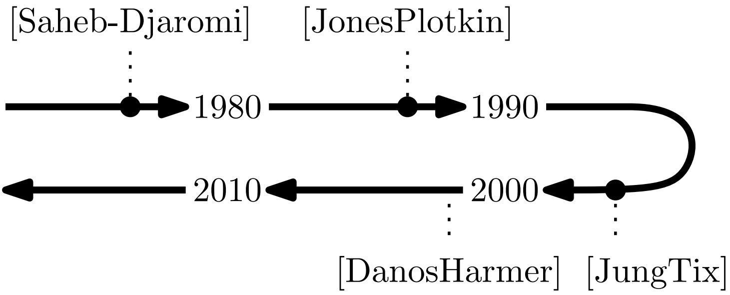
Applications?


[Saheb-Djaromi]
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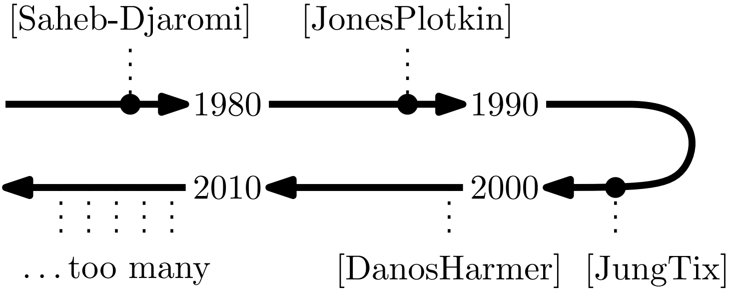
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Outline

Part I

Relational Reasoning

Syntax and Operational Semantics of $\Lambda_\oplus$

- **Terms**: $M ::= x \;\Big|\; \lambda x.M \;\Big|\; MM \;\Big|\; M \oplus M;$

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$$V \xrightarrow{\mathcal{D}} \mathcal{D}(V) \in \mathbb{R}_{[0,1]} \qquad\qquad \sum \mathcal{D} = \sum_V \mathcal{D}(V) \leq 1.$$

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- **Semantics**: $[\![M]\!] = \sup_{M \Downarrow \mathcal{D}} \mathcal{D}$;

Syntax and Operational Semantics of $\Lambda_\oplus$

$$\frac{}{M \Downarrow \emptyset} \qquad \frac{}{V \Downarrow \{V^1\}} \qquad \frac{M \Downarrow \mathcal{D} \qquad N \Downarrow \mathcal{E}}{M \oplus N \Downarrow \frac{1}{2}\mathcal{D} + \frac{1}{2}\mathcal{E}}$$

$$\frac{M \Downarrow \mathscr{K} \qquad \{P[N/x] \Downarrow \mathscr{E}_P\}_{\lambda x.P \in \mathsf{S}\mathscr{K}}}{MN \Downarrow \sum_{\lambda x.P \in \mathsf{S}\mathscr{K}} \mathscr{K}(\lambda x.P) \cdot \mathscr{E}_P}$$

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$C ::= [\cdot] \mid \lambda x.C \mid CM \mid MC \mid C \oplus M \mid M \oplus C$

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- **Context Distance**:
  $\delta^C(M, N) = \sup_C |\sum [\![C[M]]\!] - \sum [\![C[N]]\!]|$.

Examples

$$I \oplus \Omega \qquad \text{vs.} \qquad I$$

Examples

$\lambda x.x$

$$I \oplus \Omega \qquad \text{vs.} \qquad I$$

Examples

$\Delta\Delta = (\lambda x.xx)(\lambda x.xx)$

$$I \oplus \Omega \qquad \text{vs.} \qquad I$$

Exam

Not Context Equivalent: $C = [\cdot]$.

Context Distance? Consider $C_n = (\lambda x.\ \underbrace{x \ldots x}_{n \text{ times}})[\cdot]$.

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Examples

Not Context Equivalent: $C = [\cdot]$.
Context Distance? Cannot Easily Amplify.

$$I \oplus \Omega \qquad \text{vs.} \qquad \Omega$$

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$$(\lambda x.I) \oplus (\lambda x.\Omega) \qquad \text{vs.} \qquad \lambda x.I \oplus \Omega$$

Examples

$$I \oplus \Omega \qquad \text{vs.} \qquad I$$

Not Context Equivalent in CBV: $C = (\lambda x.x(xI))[\cdot]$
Apparently Context Equivalent in CBN.

$$(\lambda x.I) \oplus (\lambda x.\Omega) \qquad \text{vs.} \qquad \lambda x.I \oplus \Omega$$

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$$Y_1 \qquad \text{vs.} \qquad Y_2$$

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$$Y_1 M \to^* M(Y_2 M) \oplus M(Y_3 M)$$
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$$Y_3 M \to^* M(Y_1 M) \oplus M(Y_2 M)$$

$$Y_1 \qquad \text{vs.} \qquad Y_2$$

Terms

A Labelled Markov Chain for $\Lambda_{\oplus}$

Terms **Values**

A Labelled Markov Chain for $\Lambda_{\oplus}$

Terms	**Values**
M	

A Labelled Markov Chain for $\Lambda_\oplus$

Terms **Values**

$$\begin{tikzcd}
M \arrow[r, "{eval,\ \llbracket M \rrbracket(V)}"] \arrow[dr, "{eval,\ \llbracket M \rrbracket(W)}"] \arrow[ddr, "{eval,\ \llbracket M \rrbracket(Z)}"'] & V \\
& W \\
& Z \\
& \vdots
\end{tikzcd}$$

A Labelled Markov Chain for $\Lambda_{\oplus}$

Terms	**Values**
	$\lambda x.N$

A Labelled Markov Chain for $\Lambda_{\oplus}$

Terms		**Values**
$N\{W/x\}$	$\xleftarrow{W,\ 1}$	$\lambda x.N$

Probabilistic Applicative Bisimulation

$$\lambda x.M \;\; \mathcal{R} \;\; \lambda x.N$$

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$$\lambda x.M \;\; \mathcal{R} \;\; \lambda x.N \qquad\qquad M\{L/x\}$$

Probabilistic Applicative Bisimulation

$$\begin{tikzcd}
\lambda x.M \arrow[rrr, bend left, "L"] & \mathcal{R} & \lambda x.N \arrow[rr, bend right, "L"'] & M\{L/x\} & N\{L/x\}
\end{tikzcd}$$

Probabilistic Applicative Bisimulation

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\begin{tikzcd}
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$$M \quad \mathcal{R} \quad N$$

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$$\begin{tikzcd}
 & & & \llbracket M \rrbracket \\
M \arrow[urrr, bend left, "eval"] & \mathcal{R} & N &
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 & & & \llbracket N \rrbracket \arrow[urr, bend right] & &
\end{tikzcd}$$

Probabilistic Applicative Bisimulation

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\begin{tikzcd}
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M \arrow[urr, bend left, "eval"] & \mathcal{R} & N \arrow[d, bend right, "eval"'] & \llbracket M \rrbracket(E) = \llbracket N \rrbracket(E) \\
 & & \llbracket N \rrbracket \arrow[ur, bend right] &
\end{tikzcd}
$$

Applicative Bisimilarity vs. Context Equivalence

- **Bisimilarity**: the union $\sim$ of all bisimulation relations.
- Is it that $\sim$ is included in $\equiv$? How to prove it?
- Natural strategy: is $\sim$ a congruence?
  - If this is the case:

$$
\begin{aligned}
M \sim N \implies C[M] \sim C[N] &\implies \sum [\![C[M]]\!] = \sum [\![C[N]]\!] \\
&\implies M \equiv N.
\end{aligned}
$$

  - This is a necessary sanity check anyway.
- The naïve proof by induction **fails**, due to application: from $M \sim N$, one cannot directly conclude that $LM \sim LN$.

Howe's Technique

$\mathcal{R}$ $\mathcal{R}^H$

$$\begin{tikzcd} \mathcal{R} \arrow[rr, bend left, "\subseteq"] & & \mathcal{R}^H \end{tikzcd}$$

Howe's Technique

$\mathcal{R}^H$ is a
Congruence
whenever $\mathcal{R}$ is
an equivalence

$$\begin{tikzcd} \mathcal{R} \arrow[r, bend left, "\subseteq"] & \mathcal{R}^H \end{tikzcd}$$

Howe's Technique

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Howe's Technique

$\sim^H$ is a **Congruence**

$$\begin{tikzcd} \sim \arrow[r, bend left, "\subseteq"] & \sim^H \arrow[l, bend left, "\supseteq"] \end{tikzcd}$$

Key Lemma

Our Neighborhood

- Λ, where we observe **convergence**

	$\sim\ \subseteq\ \equiv$	$\equiv\ \subseteq\ \sim$
CBN	✓	✓
CBV	✓	✓

[Abramsky1990, Howe1993]

- $\Lambda_{\oplus}$ with nondeterministic semantics, where we observe **convergence**, in its **may** or **must** flavors.

	$\sim\ \subseteq\ \equiv$	$\equiv\ \subseteq\ \sim$
CBN	✓	×
CBV	✓	×

[Ong1993, Lassen1998]

The Probabilistic Case

- $\Lambda_{\oplus}$ with probabilistic semantics.

	$\sim \subseteq \equiv$	$\equiv \subseteq \sim$
CBN	✓	✗
CBV	✓	✓

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- Counterexample for CBN: $(\lambda x.I) \oplus (\lambda x.\Omega) \not\sim \lambda x.I \oplus \Omega$
- **Where** these discrepancies come from?

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- From **testing**!

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CBN	✓	✗
CBV	✓	✓

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- **Where** these discrepancies come from?
- From **testing**!
- Bisimulation can be characterized by testing equivalence as follows:

Calculus	**Testing**
Λ	$T ::= \omega \mid a \cdot T$
$P\Lambda_{\oplus}$	$T ::= \omega \mid a \cdot T \mid \langle T, T \rangle$
$N\Lambda_{\oplus}$	$T ::= \omega \mid a \cdot T \mid \wedge_{i \in I} T_i \mid \ldots$

The Probabilistic Case

- $\Lambda_{\oplus}$ with probabilistic semantics.

	$\precsim \subseteq \leq$	$\leq \subseteq \precsim$
CBN	✓	×
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- Full abstraction can be recovered if endowing $\Lambda_{\oplus}$ with parallel disjunction [CDLSV2015].

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$$\frac{}{\Gamma, x \vdash x} \qquad \frac{x, \Gamma \vdash M}{\Gamma \vdash \lambda x.M} \qquad \frac{\Gamma \vdash M \qquad \Delta \vdash N}{\Gamma, \Delta \vdash MN} \qquad \frac{\Gamma \vdash M \qquad \Gamma \vdash N}{\Gamma \vdash M \oplus N}$$

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- **Example**: $\delta^t(I, I \oplus \Omega) = \delta^t(I \oplus \Omega, \Omega) = \frac{1}{2}$.

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  u...
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- **Soundness and Comple**

What would a *sensible* notion of distance look like?

$\delta^t \leq \delta^c$	$\delta^c \leq \delta^t$
✓	✓

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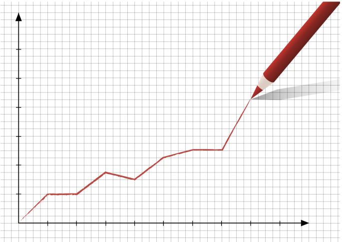
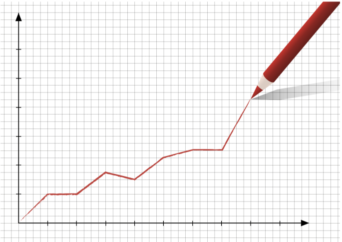
Part II

Bayesian Functional Programming

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    observe 4 from $poisson(r)$;

    return(x))

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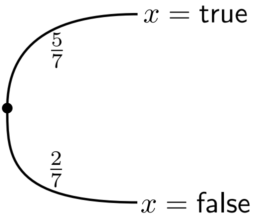

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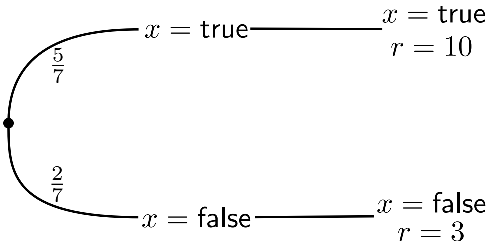

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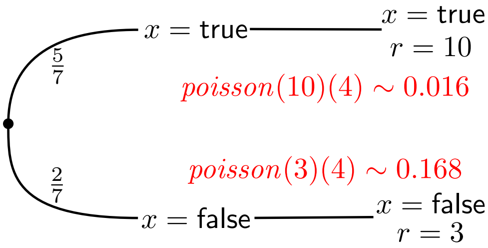

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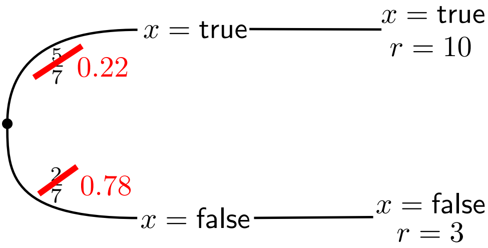
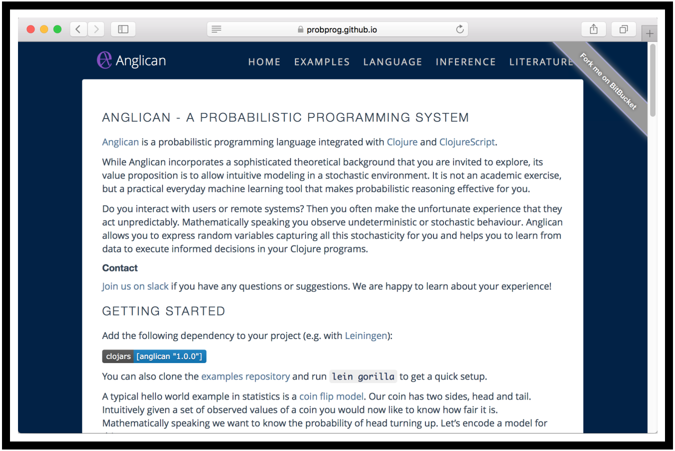
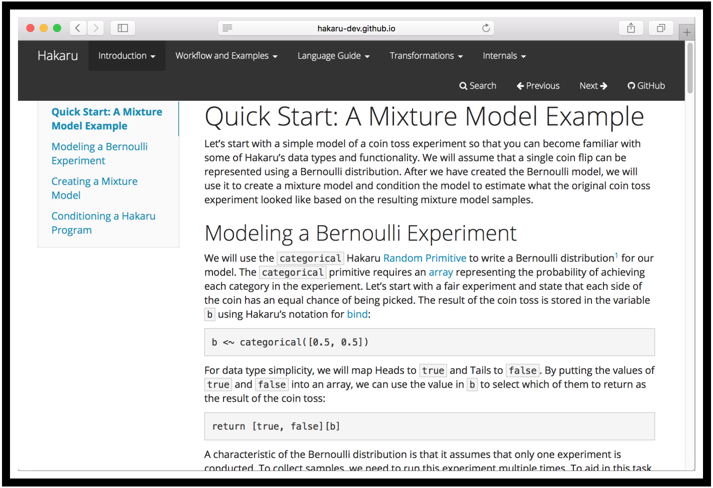


Bayesian Functional Programming



ANGLICAN

Bayesian Functional Programming



HAKARU

normalize(
    let $x =$ sample($gauss\,(0,1)$) in
        observe d from $exp(1/f(x))$;
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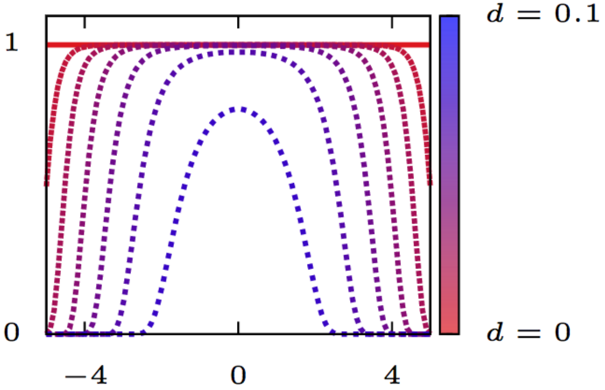

Bayesian Programming: Semantics

- Giving semantics to programming languages like Anglican or Hakaru is nontrivial:
  - Real numbers;
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  - **Conditioning**.

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  - From

$$\frac{M \Downarrow \mathscr{K} \qquad \{P[N/x] \Downarrow \mathscr{E}_P\}_{\lambda x.P \in \mathsf{S}\mathscr{K}}}{MN \Downarrow \sum_{\lambda x.P \in \mathsf{S}\mathscr{K}} \mathscr{K}(\lambda x.P) \cdot \mathscr{E}_P}$$

  we go to

$$\frac{M \Downarrow \mathscr{K} \qquad \{P[N/x] \Downarrow \mathscr{E}_P\}_{\lambda x.P \in \mathsf{S}\mathscr{K}}}{MN \Downarrow \int \mathscr{E}_P \cdot d\mathscr{K}(\lambda x.P)}$$

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$$\frac{\qquad \in \mathsf{S}\mathscr{K}}{\qquad P}$$

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- This Lebesgue integral does not necessarily exist.
- We must ensure that $\Downarrow$ gives rise to a *stochastic kernel*.
- In presence of conditioning, we need even more.

Part III

Termination

The Landscape: *Type* Theory

Simple Types

$$\tau ::= \iota \quad \Big| \quad \tau \to \tau$$

The Landscape: *Type* Theory

Simple Types

$$\tau ::= \iota \quad | \quad \tau \rightarrow \tau$$

- Sound for termination, in absence of recursion.
- Poor expressive power.
- Intuitionistic Logic.

The Landscape: *Type* Theory

Simple Types

$$\tau ::= \iota \;\big|\; \tau \to \tau$$

Polymorphic Types

$$\tau ::= \cdots \;\big|\; \alpha \;\big|\; \forall \alpha . \tau$$

The Landscape: *Type* Theory

- Second-order Logic.
- Very expressive, extensionally.
- Still poor, intensionally.

ypes

$\tau \to \tau$

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The Landscape: *Type* Theory

- Motivated by Semantics.
- *Complete* for termination.
- Type inference is undecidable.

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Sized Types

$\tau ::= \cdots \mid \iota[\xi]$

The Landscape: *Type* Theory

- Reasonably expressive, intensionally.
- Type inference remains decidable

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The Landscape: *Recursion* Theory

Determinism

$M\overline{s} \to^* N_s$

The Landscape: *Recursion* Theory

Determinism	**Probabilism**
$M\overline{s} \rightarrow^* N_s$	$[\![M\overline{s}]\!] = \mathcal{D}_s$

The Landscape: *Recursion* Theory

$\sum \mathcal{D}_s$ can be smaller than 1.

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Undecidable;
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The Landscape: *Recursion* Theory

Almost-Sure Termination
Π^0_2-complete.

	De	
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- For every type τ, define a set of reducible terms Red_τ.
- Prove that all reducible terms are normalizing...
- ...and that all typable terms are reducible.

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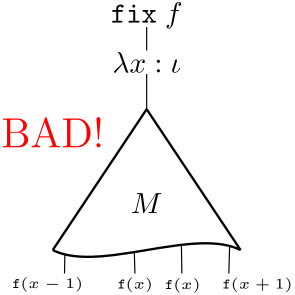
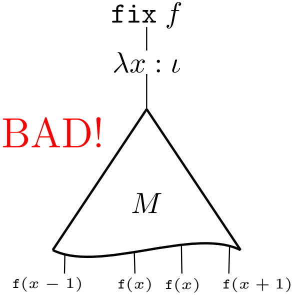


Deterministic Sized Types

- Pure λ-calculus with simple types is terminating.
  - This can be proved in many ways, including by **reducibility**.
  - But useless as a programming language.
- What if we endow it with **full recursion** as a `fix` binder?
- All the termination properties are **lost**, for very good reasons.
- Is **everything** lost?
- **NO!**



Deterministic Sized Types, Technically

- **Types**.

$$\xi ::= a \;\Big|\; \omega \;\Big|\; \xi + 1; \qquad\qquad \tau ::= \iota[\xi] \;\Big|\; \tau \to \tau.$$

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Index Terms

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- **Typing Fixpoints.**

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  - Proved by **Reducibility**.
  - ...but of an indexed form.

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- Reducibility sets are of the form Red_{τ}^{θ}.
- θ is an environment for index variables.
- Proof of reducibility for $\texttt{fix}\, x.M$ is rather delicate.

- **Q**
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  - . . . but of an indexed form.
- **Type Inference**.
  - It is indeed *decidable*.
  - But *nontrivial*.

Probabilistic Termination

- **Examples**:

  $\texttt{fix}\ f.\lambda x.\texttt{if}\ x > 0\ \texttt{then if}\ FairCoin\ \texttt{then}\ f(x-1)\ \texttt{else}\ f(x+1);$

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Unbiased Random Walk

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Biased Random Walk, the "wrong" way.

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- Probabilistic termination **is** thus:
  - Sensitive to *the actual distribution* from which we sample.
  - Sensitive to *how many recursive calls* we perform.

One-Counter Blind Markov Chains

- They are automata of the form (Q, δ) where
  - Q is a finite set of *states*.
  - $\delta : Q \to \mathsf{Dist}(Q \times \{-1, 0, 1\})$.
- They are a very special form of One-Counter Markov Decision Processeses [BBEK2011].
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  - The model is fully probabilistic, there is no nondeterminism.
  - The counter value is ignored.
- The probability of reaching a configuration where the counter is 0 can be approximated arbitrarily well *in polynomial time*.

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Every higher-order variable occurs **at most once**.

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This is sufficient for typing:

- Unbiased random walks;
- Biased random walks.

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- Reducibility sets are now on the form $Red_{\tau}^{\theta,p}$
- p stands for the *probability* of being reducible.
- Reducibility sets are continuous:

$$Red_{\tau}^{\theta,p} = \bigcup_{q<p} Red_{\tau}^{\theta,q}$$

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- **Completeness**
  - By *subject expansion*, the dual of subject reduction.

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This is **unavoidable**, due to recursion theory.

$\Gamma \vdash N : q \cdot A$

$\Gamma \vdash M \oplus N : 0s \cdot A$ $\qquad$ $\Gamma \vdash MN : (rqs) \cdot B$

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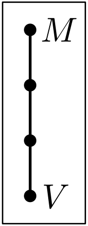
Intersection Types and Computations



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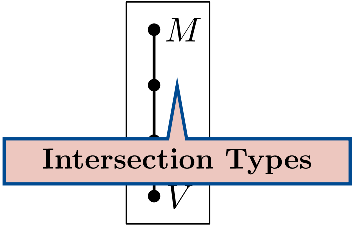

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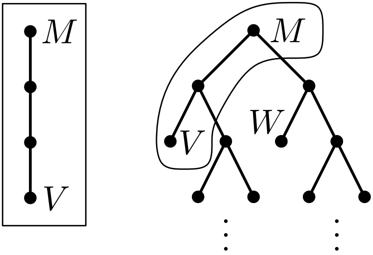

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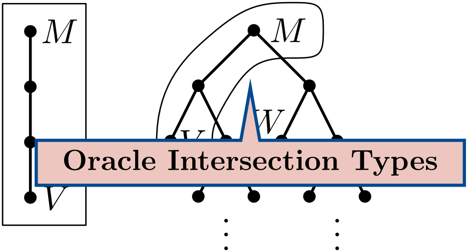

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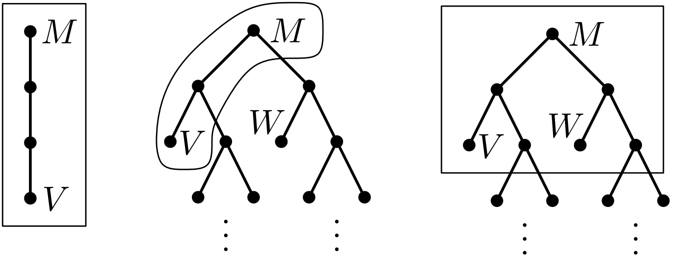
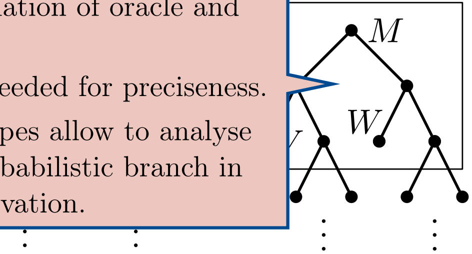


Intersection Types and Computations

Monadic Intersection Types [BDL2017]

- They are a combination of oracle and sized types.
- Intersections are needed for preciseness.
- Distributions of types allow to analyse more than one probabilistic branch in the same type derivation.



These Slides, and More...

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Questions?