Diagrammatic Algebra of First Order Logic

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(First Order) Logic is strange, when you're a stranger

- The syntax has binders, with all of the added baggage
- The proof theory (sequent calculus, natural deduction) is syntax directed and not compositional — rules look only at the outer connective
    - although Guglielmi et al have been doing very interesting work on deep inference
- There are some more subtle technical issues caused by **bad syntax**
    - e.g. Gödel completeness (e.g. Henkin's proof) has as an assumption model non-emptiness, which seems strange — surely first order logic with empty model should be propositional logic?

"Logic in his adolescent phase was algebraic. There was Boole's algebra of classes and Peirce's algebra of relations. But in 1879 logic come of age, with Frege's quantification theory. Here the bound variables, so characteristic of analysis rather than of algebra, became central to logic."

W.V. Quine. 1971. Predicate-Functor Logics.

Some recent work

- **Relational calculus** with string diagrams, building on the concept of cartesian bicategories of relations of Carboni and Walters
    - logically, this corresponds to **regular logic**: the conjunctive existential fragment
    - Filippo Bonchi, Jens Seeber, PS. Graphical Conjunctive Queries. CSL 2018: 13:1-13:23
- **Peirce's existential graphs** seen as string diagrams
    - Nathan Haydon, PS. Compositional Diagrammatic First-Order Logic. Diagrams 2020: 402-418
- Adding **disjunction** to regular logic (= **coherent logic**)
    - Filippo Bonchi, Alessandro Di Giorgio, Alessio Santamaria. Deconstructing the Calculus of Relations with Tape Diagrams. PoPL 23: 1864-1894

The monoidal category of relations

- **Rel** = category with objects sets and arrows $X \to Y$ relations $R \subseteq X \times Y$
  - composition $x\ (R\ ;\ S)\ z$ iff $\exists y.\ xRy \wedge ySz$
  - monoidal product is cartesian product
  - identies are $x\ I\ y$ iff $x=y$

Results

- An algebraic calculus for full first order logic with equality
  - diagrammatic syntax with a sound and complete axiomatisation
  - an axiomatisation that is justified by some underlying categorical structure

- A functorial semantics for first order theories

$$M : \mathbf{S}_{\mathrm{Th}} \to \mathbf{Rel}$$

Two starting points

- **Aurelio Carboni and RFC Walters (1987) Cartesian Bicategories**
  - **an algebra of relations with the expressive power of regular logic**

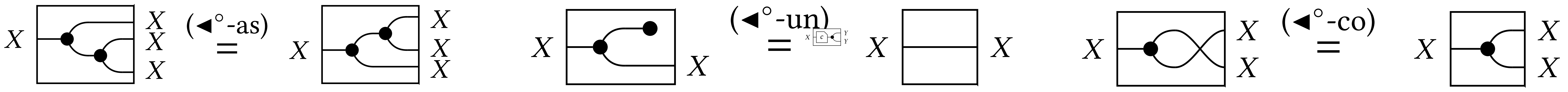
- Charles Peirce’s Calculus of Relations (1883)
  - featuring linear distributivity and linear adjoints

Towards cartesian bicategories i

- Lawvere in the 1960s realised the power of **cartesian categories**
  - **free cartesian categories** on a signature are the same as categories of terms and substitutions (**classical syntax**)
  - cartesian category induced by a (presentation of an) algebraic theory is a **presentation-independent** notion of algebraic theory in the universal algebraic sense
  - **functorial semantics**: models are cartesian functors to **Set**, homomorphisms are natural transformations

Aside - Fox's theorem

- A category is cartesian iff it is symmetric monoidal st every object is equipped with a cocommutative **comonoid structure**



- which is **natural**

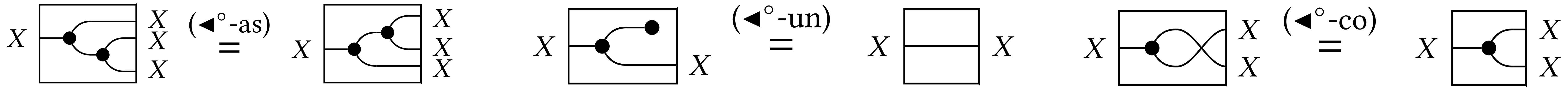
- and **coherent**

Towards cartesian bicategories ii

- But what if one wants to move to more expressive theories?
  - e.g. what if one wants models in **Rel**?
  - **Rel** = category with objects sets and arrows $X \to Y$ relations $R \subseteq X \times Y$
    - composition $x\ (R\ ;\ S)\ z$ iff $\exists y.\ xRy \wedge ySz$
    - identies are $x\ I\ y$ iff $x=y$
- Cartesian product is **still** important (n-ary relations can be seen as a relation of type $X^n \to 1$)
- But cartesian product is **not** the categorical product in **Rel**…
- Note though: it does make **Rel** a symmetric monoidal category and every homset is a poset

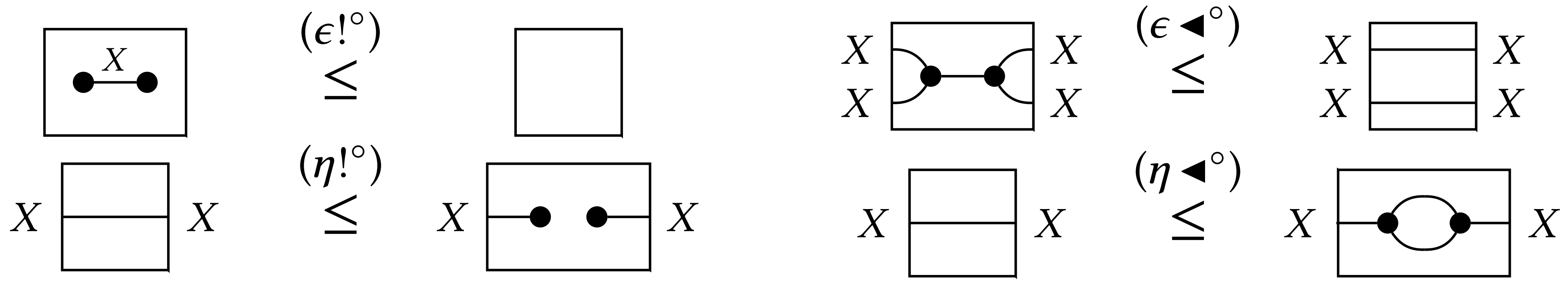
Cartesian bicategories

- every homset is a poset
- every object X is equipped with a cocommutative comonoid structure

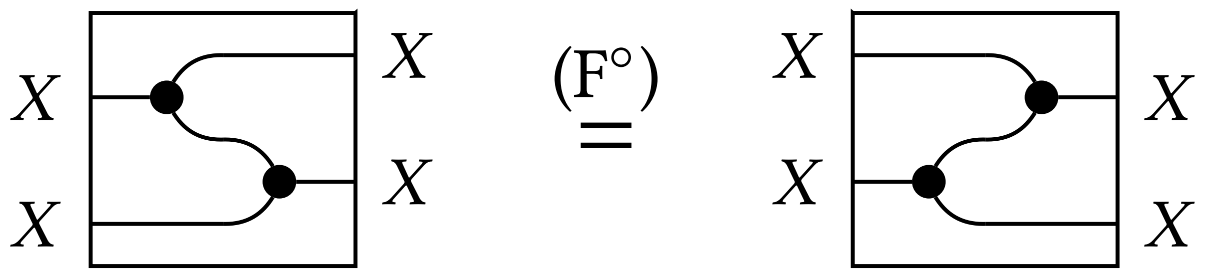


- but now the naturality is only weak

- and there is new structure!
  - the comonoid structure has **right adjoints**



  - and together they satisfy the **Frobenius equation**



Functorial semantics for relational theories

- A la Lawvere, once you know that the notion of cartesian bicategory replaces cartesian category
    - term syntax is given by string diagrams
    - models are functors of cartesian bicategories to **Rel**
    - homomorphisms are the canonical notion of natural transformation
- completeness (CSL 2018)
- This same general functorial semantics recipe is repeated for partial algebraic theories (PoPL 21) and coherent theories (PoPL 23)

Two starting points

- Aurelio Carboni and RFC Walters (1987) Cartesian Bicategories
  - an algebra of relations with the expressive power of regular logic

- **Charles Peirce's Calculus of Relations (1883)**
  - **featuring linear distributivity and linear adjoints**

Aside: **Rel**'s weird cousin

- From now on let us call the usual category of relations **Rel**$^{\circ}$
- Lets meet its strange cousin, **Rel**$^{\bullet}$
  - objects are still sets and arrows are still relations
  - composition is $x\ (R\ ;\ S)\ z$ iff $\forall\ y.\ xRy \vee ySz$
  - identities are $x\ I\ y$ iff $x \neq y$
  - cartesian product on objects still makes it a symmetric monoidal category, and homsets are posets
- But it is a **cocartesian bicategory** (the inequalities go the other way!)

Peirce's calculus of relations (1883)

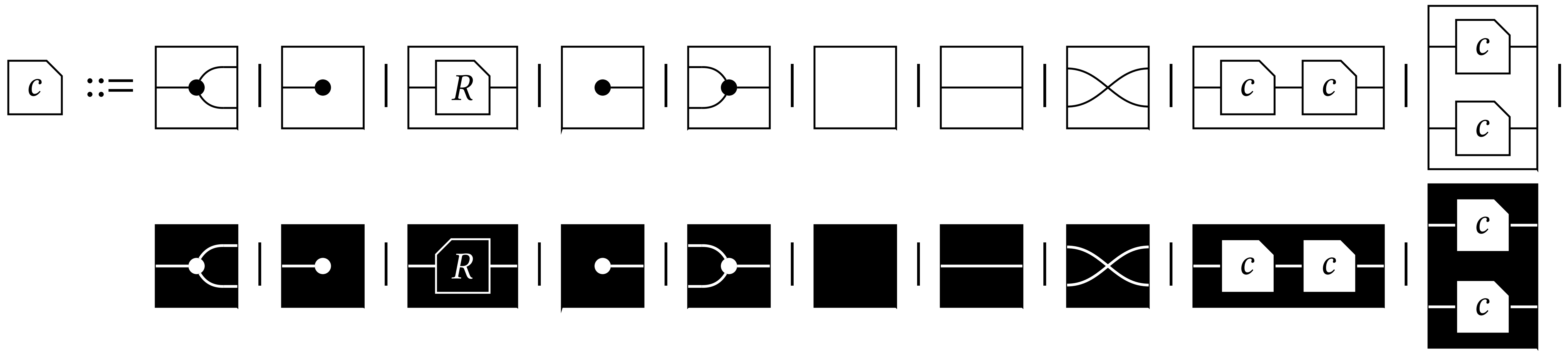
- Peirce liked the weird cousin

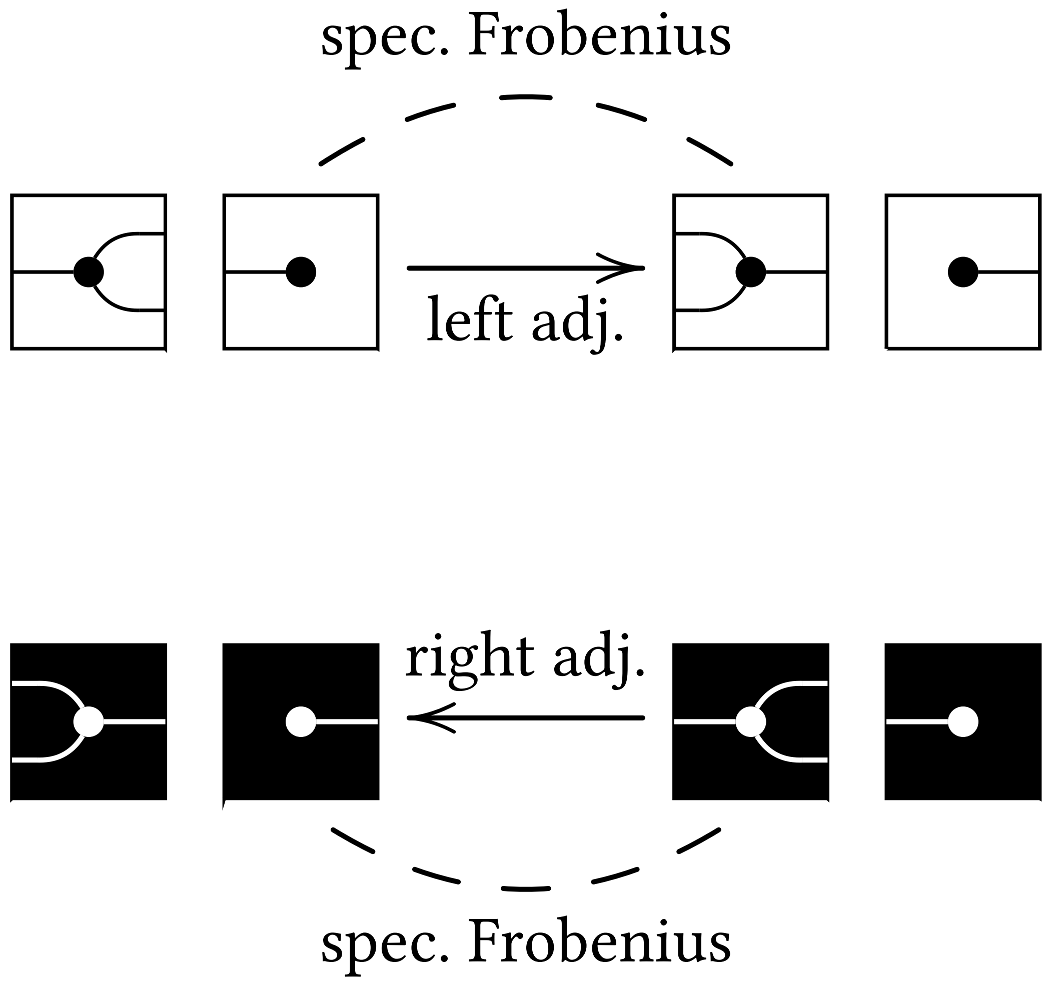
$$E \quad ::= \quad R \mid id^{\circ} \mid E \fatsemi E \mid id^{\bullet} \mid E \,\bullet\!;\, E \mid \bot \mid E \cup E \mid \top \mid E \cap E \mid E^{\dagger} \mid \overline{E}$$

- The calculus only deals with binary relations. Peirce did not like this and went on to work on **existential graphs** (19th century string diagrams)
- Later work on relational calculi (e.g. Tarski) discarded the "black" structure

Diagrams in Rel•

- Use black background/white strings to emphasise the “De Morgan” aspects

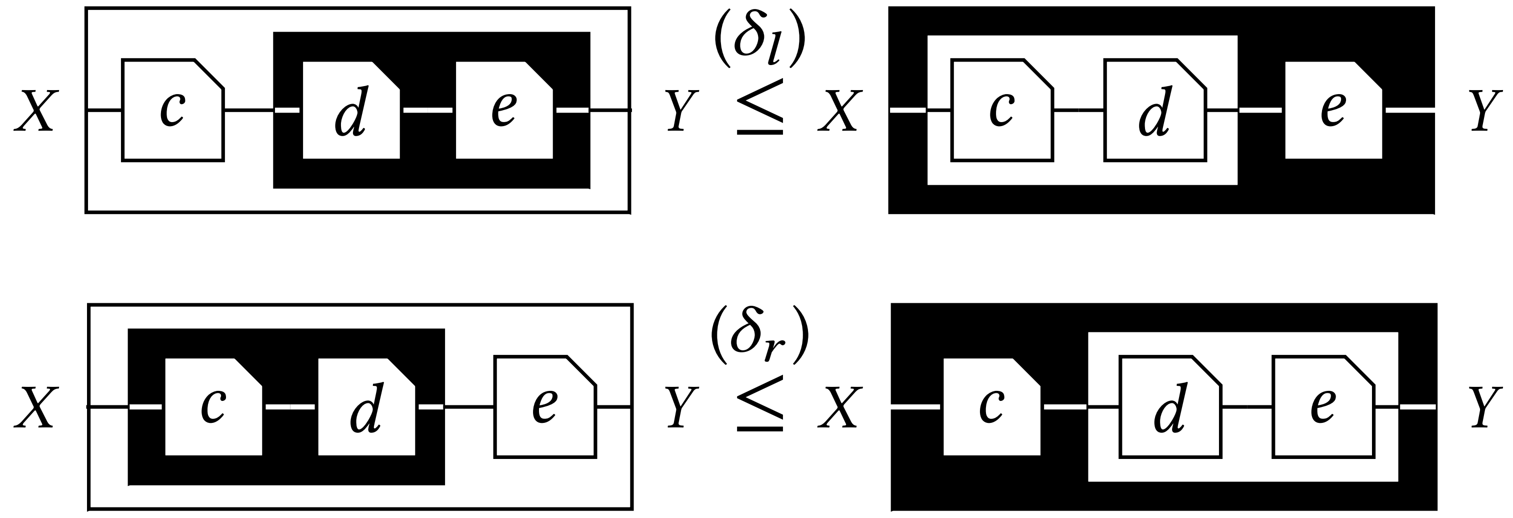




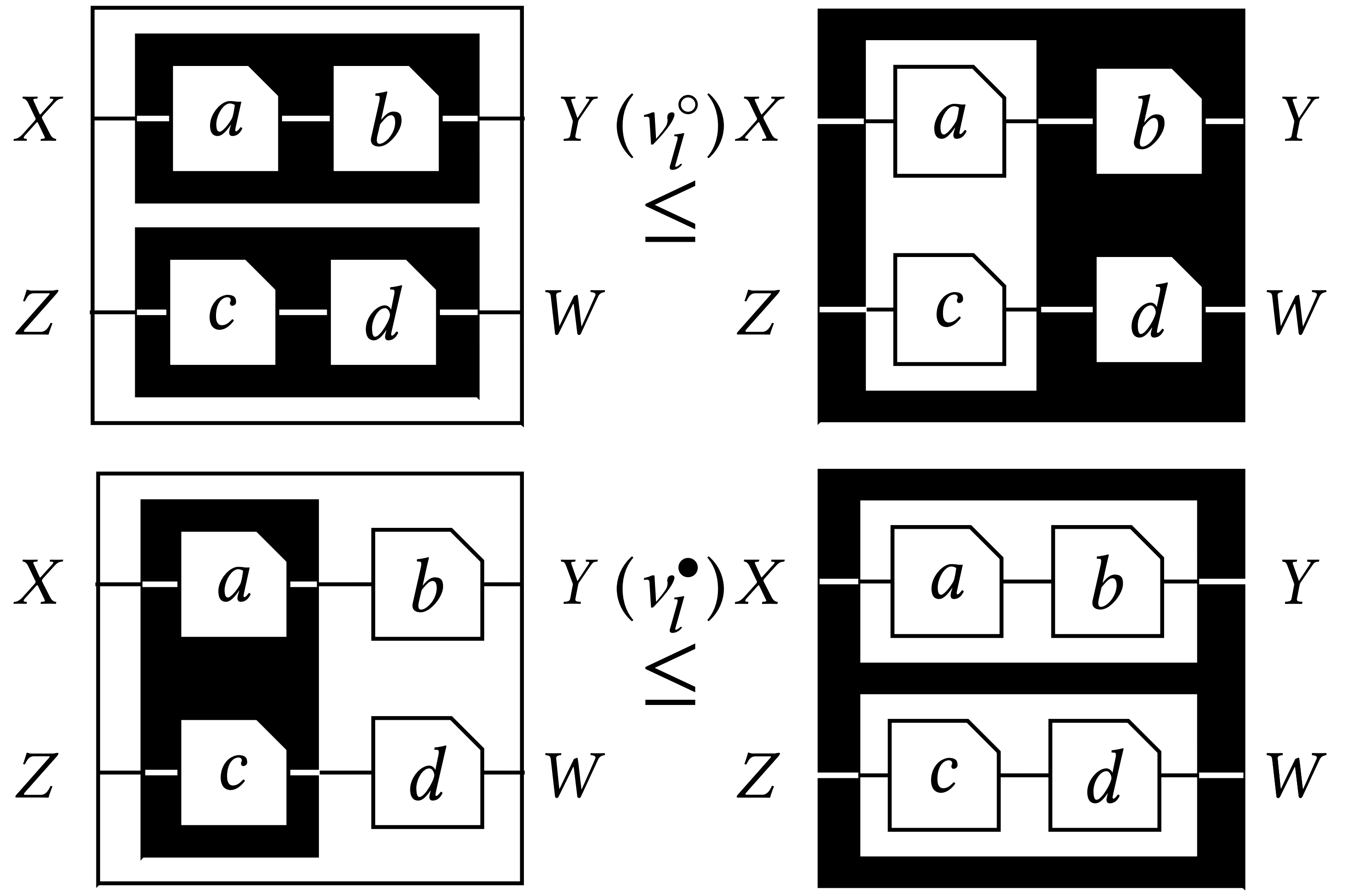
- but how to understand two compositions and two tensors together?

(symmetric monoidal) Linear bicategories

- obvious extension of Cockett, Koslowski, Seely 2000
- linear distributivity



- and linear strengths for tensors
- + obvious laws for identities and symmetries

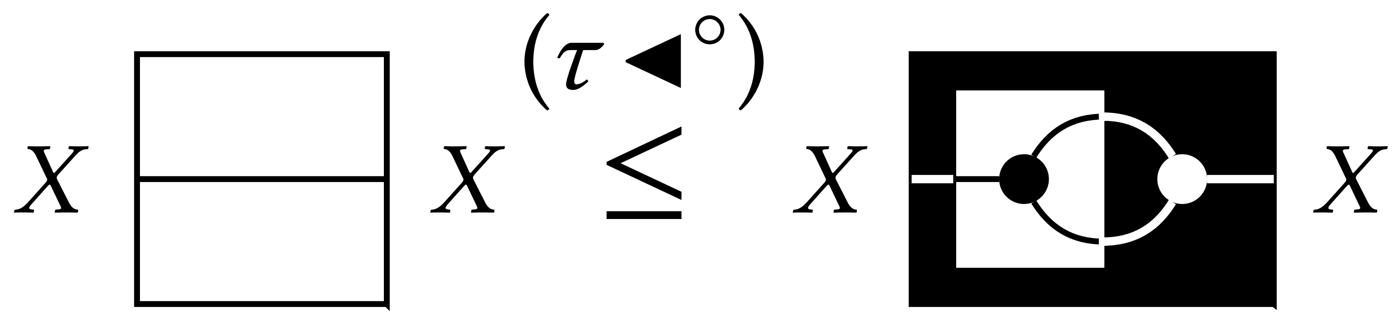


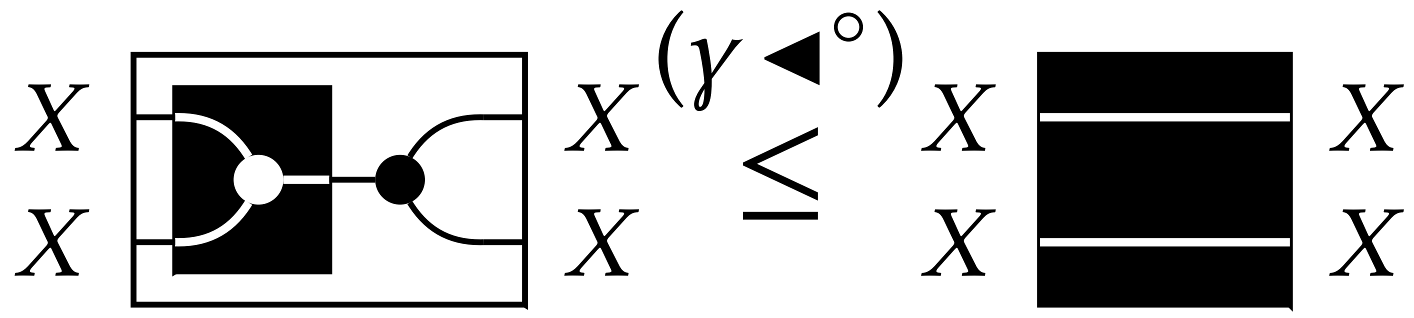
First order bicategories

- The missing thing is to characterise how the two (co)cartesian structures interact:

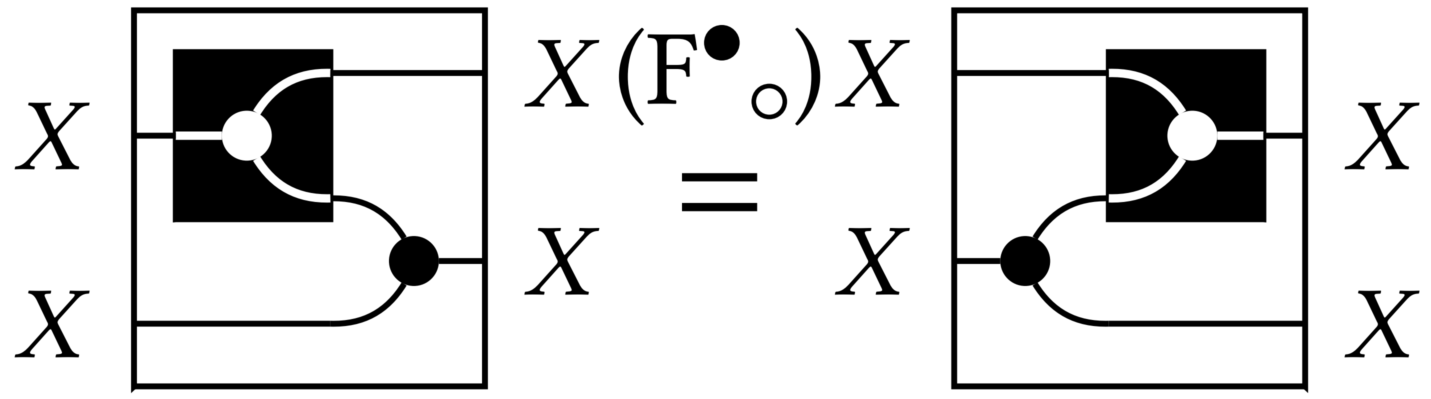
- there are **linear adjunctions**

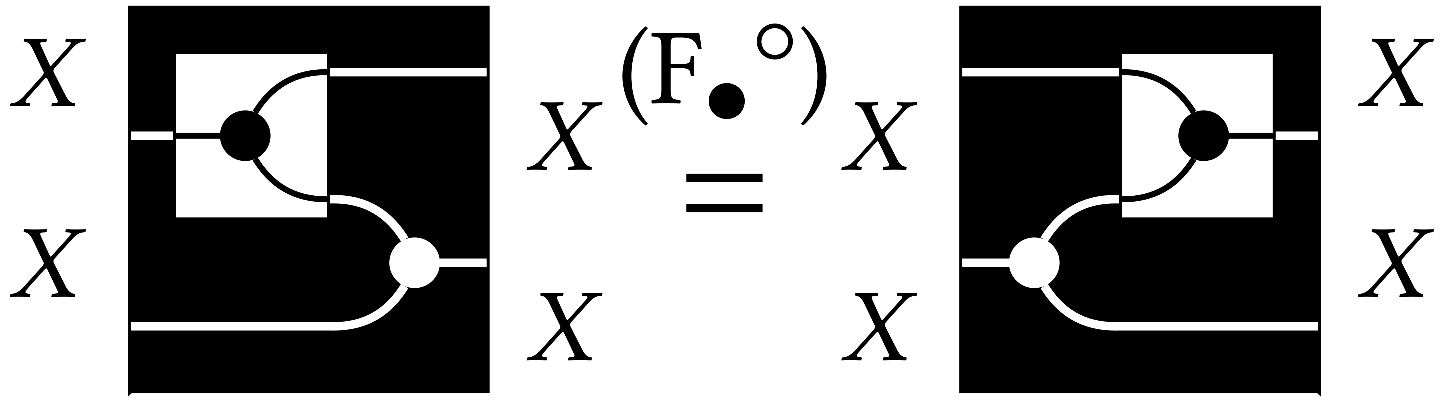
  - e.g.



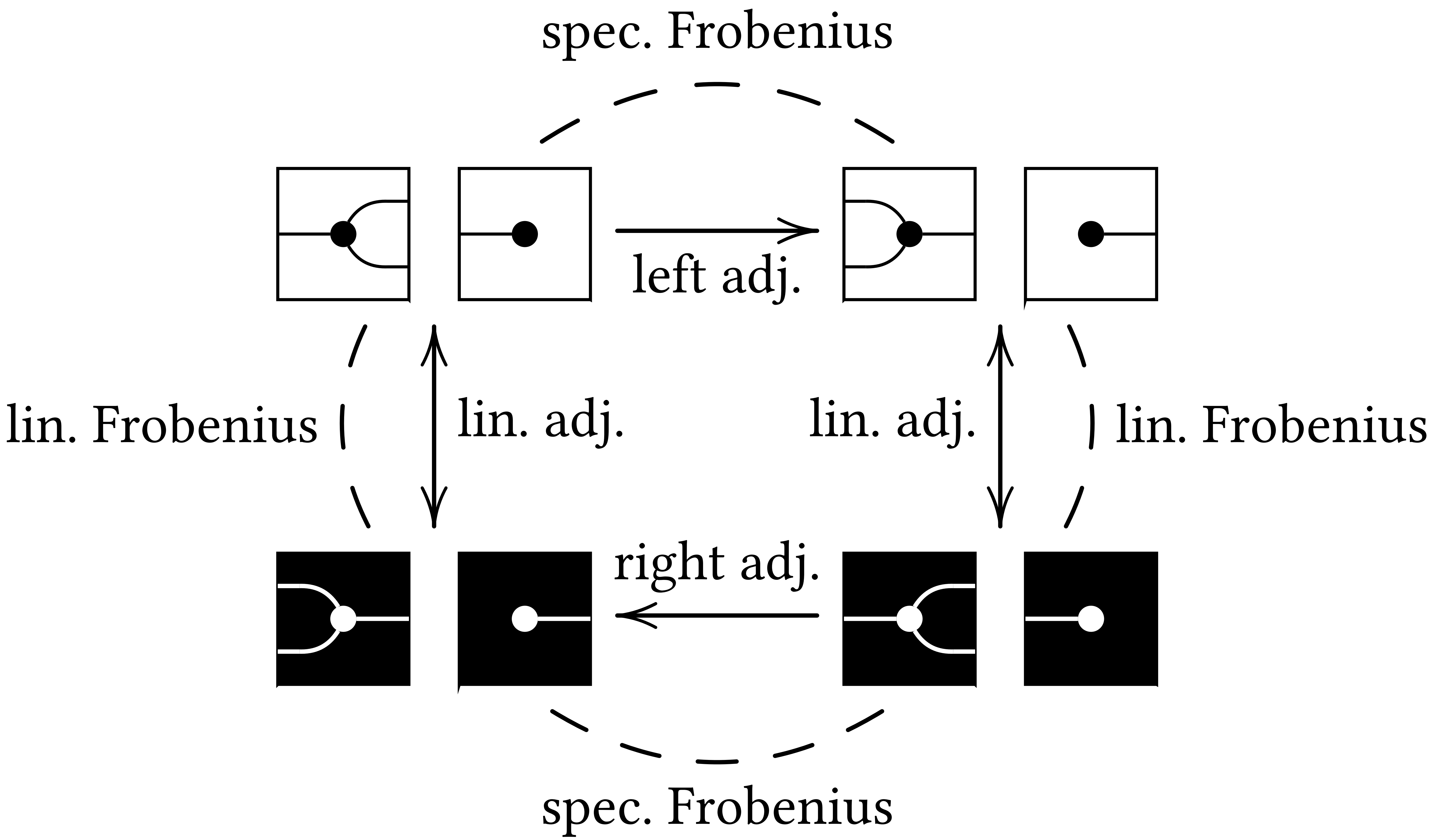


- … and "linear" Frobenius

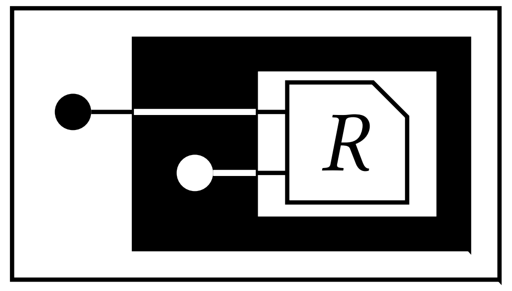




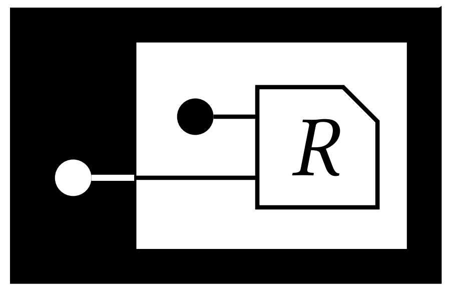
Summarising



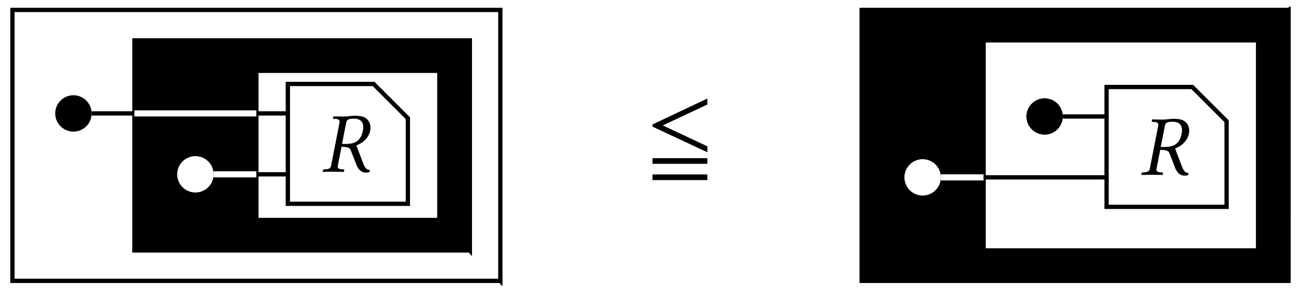
Worked example

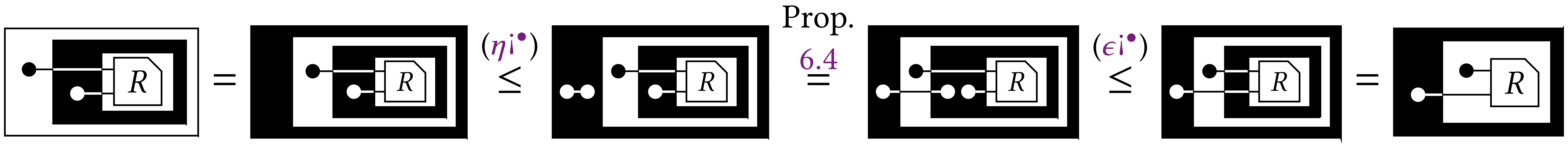


$\exists x.\, \forall y.\, R(x, y)$



$\forall y.\, \exists x.\, R(x, y)$





Highlights

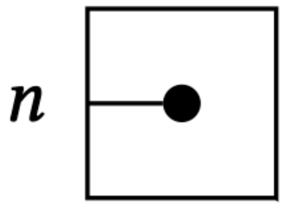
- Gödel completeness by adapting Henkin's proof to the string diagrammatic language (more on this on the next slide)
- Functorial semantics for first order theories following the usual recipe
- No variables, no quantifiers
    - Easy and natural encodings of other variable free approaches (e.g. Quine predicate functor logic)

What’s new, different?

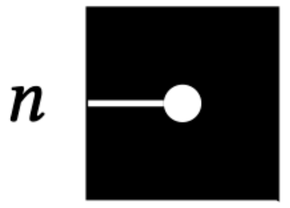
- Diagrammatic syntax is closely related to Peirce’s existential graphs
  - Although negation **is not** a primitive
    - it is a derived operation that operates on syntax
    - e.g. $\neg\neg\neg\phi$ is **syntactically equal** as a diagram to $\neg\phi$
- string diagrams let one to discover places where the traditional syntax has caused problems
  - **trivial** vs **contradictory** theories is a meaningful distinction
- trivial theories are propositional logic
  - our axiomatisation becomes Guglielmi’s deep inference Calculus of Structures (SKSg)
- completeness theorem extends Gödel’s to all theories

From traditional syntax to string diagrams

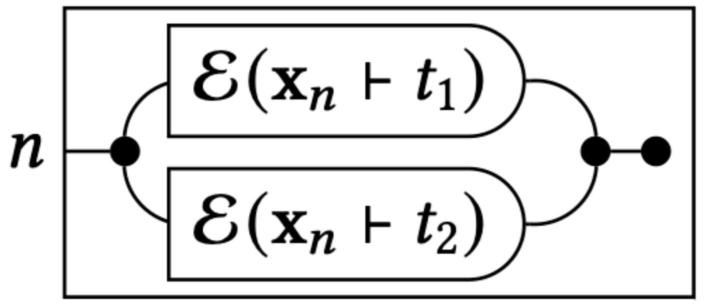
$\mathcal{E}(\mathbf{x}_n \vdash \top) \overset{\text{def}}{=}$



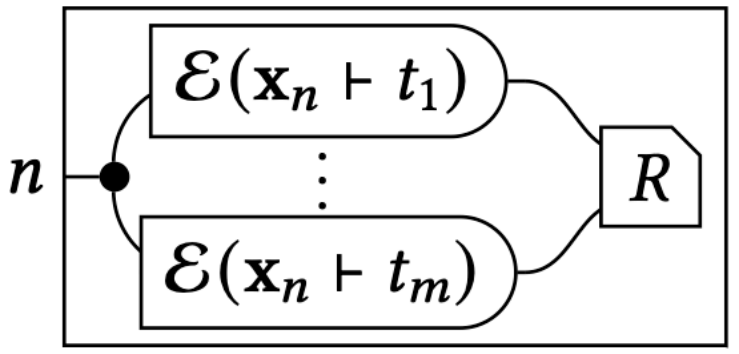
$\mathcal{E}(\mathbf{x}_n \vdash \bot) \overset{\text{def}}{=}$



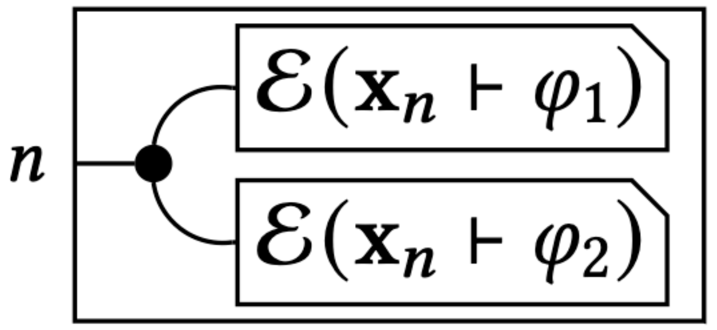
$\mathcal{E}(\mathbf{x}_n \vdash t_1 = t_2) \overset{\text{def}}{=}$



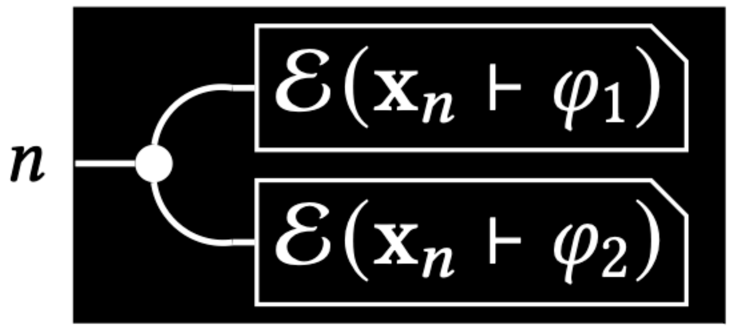
$\mathcal{E}(\mathbf{x}_n \vdash R(t_1, \ldots, t_m)) \overset{\text{def}}{=}$



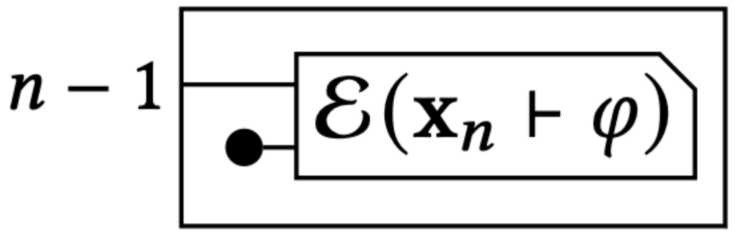
$\mathcal{E}(\mathbf{x}_n \vdash \varphi_1 \wedge \varphi_2) \overset{\text{def}}{=}$



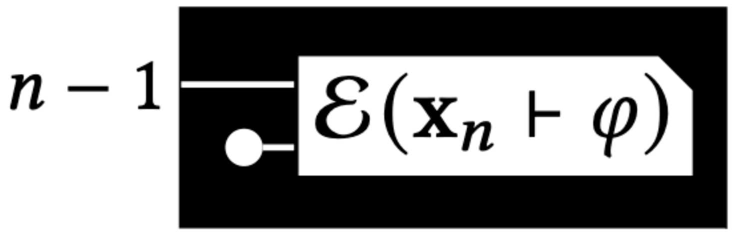
$\mathcal{E}(\mathbf{x}_n \vdash \varphi_1 \vee \varphi_2) \overset{\text{def}}{=}$



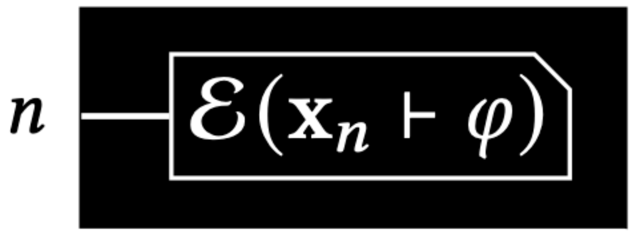
$\mathcal{E}(\mathbf{x}_{n-1} \vdash \exists x_n.\, \varphi) \overset{\text{def}}{=}$



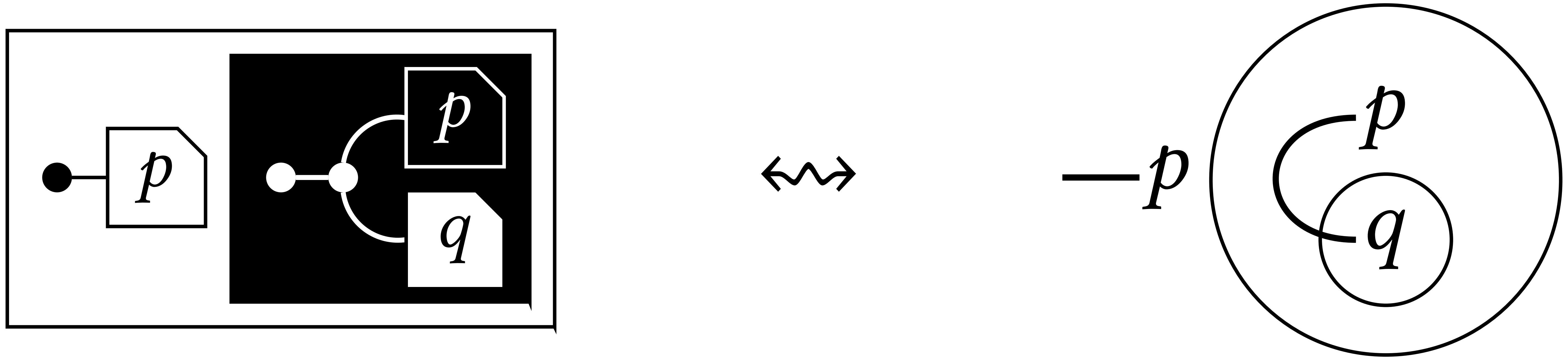
$\mathcal{E}(\mathbf{x}_{n-1} \vdash \forall x_n.\, \varphi) \overset{\text{def}}{=}$



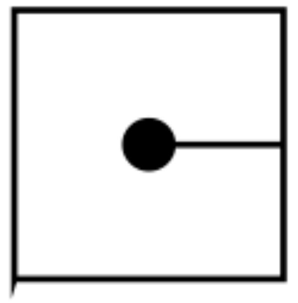
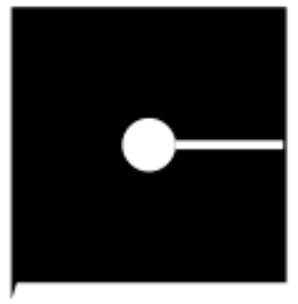
$\mathcal{E}(\mathbf{x}_n \vdash \neg\varphi) \overset{\text{def}}{=}$

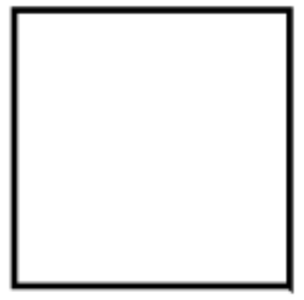


Relationship with Peirce's existential graphs



Trivial vs contradictory

- A theory is trivial if  $\lesssim_{\mathbb{T}}$ 

- A theory is contradictory  $\lesssim_{\mathbb{T}}$