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Diagrammatic Algebra of First Order Logic joint work with Filippo Bonchi and Alessandro Di Giorgio, U. Pisa and Nathan Haydon, Tallinn University of Technology

(First Order) Logic is strange, when you're a stranger

- The syntax has binders, with all of the added baggage
- The proof theory (sequent calculus, natural deduction) is syntax directed and not compositional — rules look only at the outer connective
	- although Guglielmi et al have been doing very interesting work on deep inference
- There are some more subtle technical issues caused by **bad syntax**
	- e.g. Gödel completeness (e.g. Henkin's proof) has as an assumption model non-emptiness, which seems strange — surely first order logic with empty model should be propositional logic?

W.V. Quine. 1971. Predicate-Functor Logics.

"Logic in his adolescent phase was algebraic. There was Boole's algebra of classes and Peirce's algebra of relations. But in 1879 logic come of age, with Frege's quantification theory. Here the bound variables, so characteristic of analysis rather than of algebra, became central to logic."

Some recent work

• **Relational calculus** with string diagrams, building on the concept of cartesian bicategories of

• Filippo Bonchi, Jens Seeber, PS. Graphical Conjunctive Queries. CSL 2018: 13:1-13:23

- relations of Carboni and Walters
	- logically, this corresponds to **regular logic**: the conjunctive existential fragment
	-
- **Peirce's existential graphs** seen as string diagrams
	-
- Adding **disjunction** to regular logic (= **coherent logic**)
	- Relations with Tape Diagrams. PoPL 23: 1864-1894

• Nathan Haydon, PS. Compositional Diagrammatic First-Order Logic. Diagrams 2020: 402-418

• Filippo Bonchi, Alessandro Di Giorgio, Alessio Santamaria. Deconstructing the Calculus of

The monoidal category of relations

- - **•** composition x (R ; S) z iff ∃y. xRy ∧ ySz
	- **•** monoidal product is cartesian product
	- identies are x I y iff x=y

• Rel = category with objects sets and arrows X→Y relations R ⊆ X×Y

Results

- An algebraic calculus for full first order logic with equality
	- diagrammatic syntax with a sound and complete axiomatisation
	- an axiomatisation that is justified by some underlying categorical structure

- A functorial semantics for first order theories
	-

$M : S_{Th} \rightarrow Rel$

Two starting points

- **• Aurelio Carboni and RFC Walters (1987) Cartesian Bicategories**
	-

• an algebra of relations with the expressive power of regular logic

- Charles Peirce's Calculus of Relations (1883)
	- featuring linear distributivity and linear adjoints

Towards cartesian bicategories i

- Lawvere in the 1960s realised the power of **cartesian categories**
	- **free cartesian categories** on a signature are the same as categories of terms and substitutions (**classical syntax**)
	- cartesian category induced by a (presentation of an) algebraic theory is a **presentation-independent** notion of algebraic theory in the universal algebraic sense
	- **functorial semantics**: models are cartesian functors to **Set**, homomorphisms are natural transformations

Aside - Fox's theorem

• A category is cartesian iff it is symmetric monoidal st every object is equipped **stru** $\overline{\mathbf{r}}$ \overline{a} $\frac{1}{2}$ \bigcup $niect$ || -
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• which is **natural** VVIIIUI I is nat . which is natural • which is natural \mathbf{I}

> • and **coherent** and coherent · and con $- 1$

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X & \xleftarrow{\circ} & X & \xleftarrow{\circ} & X\n\end{array}
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$$
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$$

Towards cartesian bicategories ii

- But what if one wants to move to more expressive theories?
	- e.g. what if one wants models in **Rel**?
	- **• Rel** = category with objects sets and arrows X→Y relations R ⊆ X×Y
		- **•** composition x (R ; S) z iff ∃y. xRy ∧ ySz
		- identies are x I y iff x=y
-
- **•** But cartesian product is **not** the categorical product in **Rel**…
-

Cartesian product is still important (n-ary relations can be seen as a relation of type $Xⁿ \rightarrow 1$)

• Note though: it does make **Rel** a symmetric monoidal category and every homset is a poset

Cartesian bicategories

- every homset is a poset $\ddot{ }$ S $\overline{\mathbf{a}}$
- every object X is equipped with a cocommutative comonoid structure ith a co $\overline{\mathsf{L}}$ - · every object $\overline{}$ $\overline{\Theta}$ cdaihhea $h₂$ n a cocommu · every object X is equipped $\frac{1}{2}$ - $\frac{1}{2}$ - $\frac{1}{2}$

• but now the naturality is only weak $10W$ the nature (J-un) y is only w: \mathbf{V} \bullet b • but now the naturality is on $-$) rality is or $\overline{\mathbf{v}}$ weak
"^{° ret)}

- and there is new structure! th J ing the control of t
The control of the c **CONTROLLET STREW SULLET** - -
	- the comonoid structure has **right adjoints** \overline{e} anoid structure has right adioints - - (I-as) - hoid structure has **right adjoints**
- and together they satisfy the **Frobenius equation** 455 454 - together they satisfy the Free \bullet and together they eatiefy the Erehanius equation \overline{X}

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Functorial semantics for relational theories

- A la Lawvere, once you know that the notion of cartesian bicategory replaces cartesian category
	- term syntax is given by string diagrams
	- models are functors of cartesian bicategories to **Rel**
	- homomorphisms are the canonical notion of natural transformation
- completeness (CSL 2018)
- This same general functorial semantics recipe is repeated for partial algebraic theories (PoPL 21) and coherent theories (PoPL 23)

Two starting points

- Aurelio Carboni and RFC Walters (1987) Cartesian Bicategories
	- an algebra of relations with the expressive power of regular logic

- **• Charles Peirce's Calculus of Relations (1883)**
	- **• featuring linear distributivity and linear adjoints**

Aside: Rel's weird cousin

- From now on let us call the usual category of relations **Rel◦**
- Lets meet its strange cousin, **Rel**
	- objects are still sets and arrows are still relations
	- composition is x (R; S) z iff ∀ y. xRy ∨ ySz
	- identities are $x \mid y$ iff $x \neq y$
	- homsets are posets
- But it is a **cocartesian bicategory** (the inequalities go the other way!)

• cartesian product on objects still makes it a symmetric monoidal category, and

Peirce's calculus of relations (1883) \overline{G} def **S GAIGAIUS OF FEIGHALD** def $\sqrt{1000}$

• Peirce liked the weird cousin unary operations and the property of the parties. tha waird cousing the set of the complement: th ' def ${\bf G}$, choc into a the world obdom

- on to work on *existential graphs* (19th century string diagrams) = ú h⇢¹ [⇢2i^I \overline{a} = 1
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iII am an Islam \bullet I at ater work on relati def bhal calculi (e.g. Tarski) c

$E \nightharpoonup E \nightharpoonup R \mid id^{\circ} \mid E \circ f E \mid id^{\bullet} \mid E \circ f E \mid \bot \mid E \cup E \mid \top \mid E \cap E \mid E^{\dagger} \mid \overline{E}$

• Later work on relational calculi (e.g. Tarski) discarded the "black" structure def scarded the "blacl $\overline{\mathbf{v}}$ $"$ structure

• The calculus only deals with binary relations. Peirce did not like this and went l L L = - ⇥ - h⇢¹ \ ⇢2i^I def = h⇢1i^I \ h⇢2i^I

• but how to understand two compositions and two tensors together? (The rightmost diagram does not only rightmost diagram does not only represent the term \mathbf{I}

• Use black background/white strings to emphasise the "De Morgan" aspects that for any relation ' ✓ - ⇥ ., the relation '? ✓ . ⇥ - def = {(~*,* G)|(G*,*~) 8 '} is its *linear adjoint*. ound/white strings to emphasise the "De Morgan" aspects

Diagrams in Rel• 3.1 Diagrams 2 152 \blacksquare categorical structure to deal with these situations is *linear bicategories* introduced in [18] as a

, and we we we we we we use the sumple of the we use distinguish them we use distinguish them we use distinguis 156 \overline{J}

> iderstand two compositions and two tensors together? !rst order bicategory C are morphisms M: LCB^T ! C. Taking C = Rel, the !rst order bicategory

- obvious extension of Cockett, Koslowski, Seely 2000
- linear distributivity

- and linear strengths for tensors
- + obvious laws for identities and symmetries

(symmetric monoidal) Linear bicategories Diagrammatic Algebra of First Order Logic Conference acronym 'XX, June 03–05, 2018, Woodstock, NY

• … and "linear" Frobenius \overline{M} par["] Frohanius ar · and "linear" Frobenius near" Frobenius

First order bicategories Conference acronym 'XX, June 03–05, 2018, Woodstock, NY Anon.

- The missing thing is to characterise how the two (co)cartesian structures interact: g is ing thing is to ch) Criaracteris
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- there are **linear adjunctions** e ilileal \mathbf{r} . There are linear adjunctions adiuncti $\overline{}$

e how the two (\sim (co/cared) res wo (co)cartesian structures - -

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$$

$$
\begin{array}{c}\nX \\
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X\n\end{array} =\n\begin{array}{c}\nX \\
X\n\end{array} \begin{array}{c}\nX\n\end{array}
$$

2 and then its photographic negative 2 . For instance, the linear adjoint of ' is ' . **Summarising**

Worked example *A taste of diagrammatic logic.* Before we introduce the calculus of neo-Peircean relations, we start $$ result of !rst order logic. Doing so lets us illustrate the methodology of proof within the calculus,

 $\exists x. \forall y. R(x, y)$ $\forall y. \exists x. R(x, y)$

Highlights

- Gödel completeness by adapting Henkin's proof to the string diagrammatic language (more on this on the next slide)
- Functorial semantics for first order theories following the usual recipe
- No variables, no quantifiers
	- Easy and natural encodings of other variable free approaches (e.g. Quine predicate functor logic)

What's new, different?

- Diagrammatic syntax is closely related to Peirce's existential graphs
	- Although negation **is not** a primitive
		- it is a derived operation that operates on syntax
		- e.g. ¬¬¬φ is **syntactically equal** as a diagram to ¬φ
- string diagrams let one to discover places where the traditional syntax has caused problems
	- **trivial** vs **contradictory** theories is a meaningful distinction
- trivial theories are propositional logic
	-
- our axiomatisation becomes Guglielmi's deep inference Calculus of Structures (SKSg) • completeness theorem extends Gödel's to all theories

From traditional syntax to string diagrams

$$
\mathcal{E}(\mathbf{x}_n + \tau) \stackrel{\text{def}}{=} n \underbrace{\left(\frac{\mathcal{E}(\mathbf{x}_n + t_1)}{\mathcal{E}(\mathbf{x}_n + t_2)} \right)}_{\text{for } n \in \mathbb{Z}}.
$$
\n
$$
\mathcal{E}(\mathbf{x}_n + t_1 = t_2) \stackrel{\text{def}}{=} n \underbrace{\left(\frac{\mathcal{E}(\mathbf{x}_n + t_1)}{\mathcal{E}(\mathbf{x}_n + t_2)} \right)}_{\text{for } n \in \mathbb{Z}}.
$$
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\mathcal{E}(\mathbf{x}_n + \varphi_1 \wedge \varphi_2) \stackrel{\text{def}}{=} n \underbrace{\left(\frac{\mathcal{E}(\mathbf{x}_n + t_1)}{\mathcal{E}(\mathbf{x}_n + \varphi_1)} \right)}_{\text{for } n \in \mathbb{Z}}.
$$
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$$
\mathcal{E}(\mathbf{x}_{n+1} + \exists x_n, \varphi) \stackrel{\text{def}}{=} n \underbrace{\left(\frac{\mathcal{E}(\mathbf{x}_n + t_1)}{\mathcal{E}(\mathbf{x}_n + \varphi_2)} \right)}_{\text{for } n \in \mathbb{Z}}.
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$$
\mathcal{E}(\mathbf{x}_n + \perp) \stackrel{\text{def}}{=} n
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\mathcal{E}(\mathbf{x}_n + R(t_1, ..., t_m)) \stackrel{\text{def}}{=} n \underbrace{\left(\frac{\mathcal{E}(\mathbf{x}_n + t_1)}{\mathcal{E}(\mathbf{x}_n + t_m)}\right)}_{\text{max}}
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\mathcal{E}(\mathbf{x}_n + \varphi_1 \vee \varphi_2) \stackrel{\text{def}}{=} n \underbrace{\left(\frac{\mathcal{E}(\mathbf{x}_n + t_n)}{\mathcal{E}(\mathbf{x}_n + \varphi_1)}\right)}_{\text{max}}
$$
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$$
\mathcal{E}(\mathbf{x}_{n-1} + \forall x_n, \varphi) \stackrel{\text{def}}{=} n - 1 \underbrace{\left(\frac{\mathcal{E}(\mathbf{x}_n + t_n)}{\mathcal{E}(\mathbf{x}_n + \varphi_2)}\right)}_{\text{max}}
$$

Relationship with Peirce's existential graphs 21 nsnip with Peirce's exi

Trivial vs contradictory

• A theory is trivial if $\vert \rightarrow \vert \leq_{\mathbb{T}} \vert$

• A theory is contradictory $|$ \leq T

