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Quantum isometries of noncommutative polygonal spheres

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Abstract. The real sphere $S^{N-1}_{\mathbb{R}}$ appears as increasing union, over $d \in \{1, \ldots, N\}$, of its "polygonal" versions $S^{N-1,d-1}_{\mathbb{R}} = \{x \in S^{N-1}_{\mathbb{R}} \mid x_{i_0} \ldots x_{i_d} = 0 \text{ for all } i_0, \ldots, i_d \text{ distinct}\}$. Motivated by general classification questions for the undeformed noncommutative spheres, smooth or not, we study here the quantum isometries of $S^{N-1,d-1}_{\mathbb{R}}$ and of its various noncommutative analogs, obtained via liberation and twisting. We also discuss a complex version of these results, with $S^{N-1}_{\mathbb{R}}$ replaced by the complex sphere $S^{N-1}_{\mathbb{C}}$.

Introduction

Goswami has shown in [14] that any noncommutative compact Riemannian manifold X has a quantum isometry group $G^+(X)$. While the classical, connected manifolds cannot have genuine quantum isometries [16], the situation changes when looking at manifolds which are (1) disconnected, or (2) not smooth, or (3) not classical.

The fact that a disconnected manifold can have indeed quantum isometries is well known and goes back to Wang's paper [24], where a free analog S^+_N of the symmetric group S_N, acting on the N-point space $X_N = \{1, \ldots, N\}$, was constructed. For non-smooth (connected) manifolds this is a relatively new discovery, due to Huang [17], the simplest example here being the action of S^+_N on the union $Y_N = \bigcup_{i=1}^N [0,1]^{(i)}$ of the N copies of the $[0,1]$-segment on the coordinate axes of $\mathbb{R}^N$. Finally, for the non-classical manifolds this is once again well known (see [14]); a basic example here being the action of the free quantum group O^+_N on the free real sphere $S^{N-1}_{\mathbb{R},+}$, as discussed in [4].

Generally speaking, it is an open question which exact geometric features of X allow the existence of genuine quantum group actions. In view of the above results and examples, the answer probably involves a subtle mixture of non-connectedness, non-smoothness, and non-commutativity, which remains yet to be determined.

The present paper is a continuation of [1, 4], where we proposed the framework of "undeformed noncommutative spheres" and their submanifolds, as a reasonably general setting for investigating various quantum isometry phenomena. We will study here certain non-smooth versions of $S^{N-1}_{\mathbb{R}}$ and their various noncommutative analogs.

More precisely, we will be interested in the "polygonal spheres" and their noncommutative analogs appearing via liberation and twisting. The polygonal spheres are real algebraic manifolds, depending on integers $1 \leq d \leq N$, defined as follows:

$$S^{N-1,d-1}_{\mathbb{R}} = \big\{x \in S^{N-1}_{\mathbb{R}} \mid x_{i_0} \ldots x_{i_d} = 0 \text{ for all } i_0, \ldots, i_d \text{ distinct}\big\}.$$

This type of construction also applies to the noncommutative versions of $S^{N-1}_{\mathbb{R}}$ constructed in [1, 4]. The cases $d = 1, 2$ are of particular interest, because we can recover in this way some key examples from [1], originally dismissed there because of their non-smoothness. We have in fact nine basic polygonal spheres, as follows:

$$\begin{array}{ccccc}
S^{N-1}_{\mathbb{R}} & \longrightarrow & S^{N-1}_{\mathbb{R},*} & \longrightarrow & S^{N-1}_{\mathbb{R},+} \\
\uparrow & & \uparrow & & \uparrow \\
S^{N-1,1}_{\mathbb{R}} & \longrightarrow & S^{N-1,1}_{\mathbb{R},*} & \longrightarrow & \bar{S}^{N-1}_{\mathbb{R},*} \\
\uparrow & & \uparrow & & \uparrow \\
S^{N-1,0}_{\mathbb{R}} & \longrightarrow & \bar{S}^{N-1,1}_{\mathbb{R}} & \longrightarrow & \bar{S}^{N-1}_{\mathbb{R}}.
\end{array}$$

Here all the maps are inclusions. The three spheres on top are those in [4], the three spheres on the right are their twists, introduced in [1], with the free sphere $S^{N-1}_{\mathbb{R},+}$ being equal to its own twist, and the four spheres at bottom left appear as intersections.

We will first perform an axiomatic study of these nine spheres, with some noncommutative algebraic geometry results, of diagrammatic type, extending those in [1, 4]. We will prove then that the corresponding quantum isometry groups are as follows:

$$\begin{array}{ccccc}
O_N & \longrightarrow & O_N^* & \longrightarrow & O_N^+ \\
\uparrow & & \uparrow & & \uparrow \\
H_N & \longrightarrow & H_N^{[\infty]} & \longrightarrow & \bar{O}_N^* \\
\uparrow & & \uparrow & & \uparrow \\
H_N^+ & \longrightarrow & H_N & \longrightarrow & \bar{O}_N.
\end{array}$$

Here the five results on top and at right are known from [1, 4]. The four new results, at bottom left, concern the hyperoctahedral group H_N and its versions

$H_N^+, H_N^{[\infty]}$ from [2, 3]. The proof uses methods from [1, 4, 9, 11, 20] and some ad-hoc tricks.

We also have a complex version of these results, concerning the nine complex analogs of the above spheres and quantum groups, which once again extends some previous findings from [1]. We refer to the body of the paper for the precise statements of our results and to the final section below for a list of questions raised by the present work.

The paper is organized as follows: In Sections 1 and 2 we introduce the real polygonal spheres. In Sections 3 and 4 we study their quantum isometries. In Sections 5 and 6 we state and prove our main results, we discuss the complex extensions, and we end with a few concluding remarks.

1. Noncommutative spheres

According to [4], the free analog $S_{\mathbb{R},+}^{N-1}$ of the real sphere $S_{\mathbb{R}}^{N-1}$ is the noncommutative real manifold whose coordinates $x_1, \ldots, x_N$ are subject to the condition $\sum_i x_i^2 = 1$. To be more precise, $S_{\mathbb{R},+}^{N-1}$ is the abstract spectrum of the following universal C^*-algebra:

$$C(S_{\mathbb{R},+}^{N-1}) = C^*\big(x_1, \ldots, x_N \mid x_i = x_i^*,\ x_1^2 + \cdots + x_N^2 = 1\big).$$

In what follows we are interested in various "subspheres" of $S_{\mathbb{R},+}^{N-1}$. As explained in [4], besides $S_{\mathbb{R}}^{N-1}$, another fundamental example is the half-liberated sphere $S_{\mathbb{R},*}^{N-1}$, which appears as an intermediate object,

$$S_{\mathbb{R}}^{N-1} \subset S_{\mathbb{R},*}^{N-1} \subset S_{\mathbb{R},+}^{N-1}.$$

Moreover, as explained in [1], we have two more basic spheres obtained by twisting,

$$\bar{S}_{\mathbb{R}}^{N-1} \subset \bar{S}_{\mathbb{R},*}^{N-1} \subset S_{\mathbb{R},+}^{N-1}.$$

Here is the precise definition of the three extra spheres.

Definition 1.1. The subspheres $\bar{S}_{\mathbb{R}}^{N-1}, S_{\mathbb{R},*}^{N-1}, \bar{S}_{\mathbb{R},*}^{N-1} \subset S_{\mathbb{R},+}^{N-1}$ are constructed by imposing the following conditions on the standard coordinates $x_1, \ldots, x_N$:

$$\bar{S}_{\mathbb{R}}^{N-1}: \quad x_i x_j = -x_j x_i \quad \text{for any } i \neq j,$$

$$S_{\mathbb{R},*}^{N-1}: \quad x_i x_j x_k = x_k x_j x_i \quad \text{for any } i, j, k,$$

$$\bar{S}_{\mathbb{R},*}^{N-1}: \quad x_i x_j x_k = \begin{cases} -x_k x_j x_i & \text{for any } i, j, k \text{ distinct}, \\ x_k x_j x_i & \text{otherwise}. \end{cases}$$

The fact that we have indeed $\bar{S}_{\mathbb{R}}^{N-1} \subset \bar{S}_{\mathbb{R},*}^{N-1}$ comes from $abc = -bac = bca = -cba$ for $a, b, c \in \{x_i\}$ distinct and $aab = -aba = baa$ for $a, b \in \{x_i\}$ distinct, where $x_1, \ldots, x_N$ are the standard coordinates on $\bar{S}_{\mathbb{R}}^{N-1}$. In addition, it is known that the inclusions

$$S_{\mathbb{R}}^{N-1} \subset S_{\mathbb{R},*}^{N-1} \subset S_{\mathbb{R},+}^{N-1} \quad \text{and} \quad \bar{S}_{\mathbb{R}}^{N-1} \subset \bar{S}_{\mathbb{R},*}^{N-1} \subset S_{\mathbb{R},+}^{N-1}$$

are all proper at $N \geq 3$; see [1].

As pointed out in [1], when intersecting twisted and untwisted spheres, non-smooth manifolds can appear. More precisely, $S_{\mathbb{R}}^{N-1}\cap\bar{S}_{\mathbb{R},*}^{N-1}$ consists by definition of the points $x\in S_{\mathbb{R}}^{N-1}$ having the property $x_ix_jx_k=0$ for any i,j,k distinct and is therefore a union of $\binom{N}{2}$ copies of the unit circle $\mathbb{T}$, which is not smooth; see [1].

In what follows we will enlarge the formalism in [1], as to cover these intersections as well, originally dismissed there, but which are quite interesting.

Proposition 1.2. *The five main spheres and the intersections between them are*

$$\begin{array}{ccccc}
S_{\mathbb{R}}^{N-1} & \longrightarrow & S_{\mathbb{R},*}^{N-1} & \longrightarrow & S_{\mathbb{R},+}^{N-1}\\
\uparrow & & \uparrow & & \uparrow\\
S_{\mathbb{R}}^{N-1,1} & \longrightarrow & S_{\mathbb{R},*}^{N-1,1} & \longrightarrow & \bar{S}_{\mathbb{R},*}^{N-1}\\
\uparrow & & \uparrow & & \uparrow\\
S_{\mathbb{R}}^{N-1,0} & \longrightarrow & \bar{S}_{\mathbb{R}}^{N-1,1} & \longrightarrow & \bar{S}_{\mathbb{R}}^{N-1},
\end{array}$$

where $\dot{S}_{\mathbb{R},\times}^{N-1,d-1}\subset\dot{S}_{\mathbb{R},\times}^{N-1}$ *is obtained by assuming* $x_{i_0}\ldots x_{i_d}=0$ *for* $i_0,\ldots,i_d$ *distinct.*

Proof. We must prove that the 4-diagram obtained by intersecting the five main spheres coincides with the 4-diagram appearing at bottom left in the statement:

$$\begin{array}{ccc}
S_{\mathbb{R}}^{N-1}\cap\bar{S}_{\mathbb{R},*}^{N-1} & \longrightarrow & S_{\mathbb{R},*}^{N-1}\cap\bar{S}_{\mathbb{R},*}^{N-1}\\
\uparrow & & \uparrow\\
S_{\mathbb{R}}^{N-1}\cap\bar{S}_{\mathbb{R}}^{N-1} & \longrightarrow & S_{\mathbb{R},*}^{N-1}\cap\bar{S}_{\mathbb{R}}^{N-1}
\end{array}
\quad=\quad
\begin{array}{ccc}
S_{\mathbb{R}}^{N-1,1} & \longrightarrow & S_{\mathbb{R},*}^{N-1,1}\\
\uparrow & & \uparrow\\
S_{\mathbb{R}}^{N-1,0} & \longrightarrow & \bar{S}_{\mathbb{R}}^{N-1,1}.
\end{array}$$

But this is clear, because combining the commutation and anticommutation relations leads to the vanishing relations defining spheres of type $\dot{S}_{\mathbb{R},\times}^{N-1,d-1}$. More precisely:

(1) $S_{\mathbb{R}}^{N-1}\cap\bar{S}_{\mathbb{R}}^{N-1}$ consists of the points $x\in S_{\mathbb{R}}^{N-1}$ satisfying $x_ix_j=-x_jx_i$ for $i\neq j$. Since $x_ix_j=x_jx_i$, this latter relation reads $x_ix_j=0$ for $i\neq j$, which means $x\in S_{\mathbb{R}}^{N-1,0}$.

(2) $S_{\mathbb{R}}^{N-1}\cap\bar{S}_{\mathbb{R},*}^{N-1}$ consists of the points $x\in S_{\mathbb{R}}^{N-1}$ satisfying $x_ix_jx_k=-x_kx_jx_i$ for i,j,k distinct. Once again by commutativity, this relation is equivalent to $x\in S_{\mathbb{R}}^{N-1,1}$.

(3) $S_{\mathbb{R},*}^{N-1}\cap\bar{S}_{\mathbb{R}}^{N-1}$ is obtained from $\bar{S}_{\mathbb{R}}^{N-1}$ by imposing to the standard coordinates the half-commutation relations $abc=cba$. On the other hand, we know from $\bar{S}_{\mathbb{R}}^{N-1}\subset\bar{S}_{\mathbb{R},*}^{N-1}$ that the standard coordinates on $\bar{S}_{\mathbb{R}}^{N-1}$ satisfy $abc=-cba$ for a,b,c distinct and $abc=cba$ otherwise. Thus, the relations brought by intersecting with $S_{\mathbb{R},*}^{N-1}$ reduce to the relations $abc=0$ for a,b,c distinct, and so we are led to the sphere $\bar{S}_{\mathbb{R}}^{N-1,1}$.

(4) $S^{N-1}_{\mathbb{R},*} \cap \bar{S}^{N-1}_{\mathbb{R},*}$ is obtained from $\bar{S}^{N-1}_{\mathbb{R},*}$ by imposing the relations $abc = -cba$ for a, b, c distinct and $abc = cba$ otherwise. Since we know that $abc = cba$ for any a, b, c, the extra relations reduce to $abc = 0$ for a, b, c distinct, and so we are led to $S^{N-1,1}_{\mathbb{R},*}$. □

Let us find now a suitable axiomatic framework for the nine spheres in Proposition 1.2. We denote by $P(k, l)$ the set of partitions between an upper row of k points and a lower row of l points. We set $P = \bigcup_{kl} P(k, l)$ and denote by $P_{\text{even}} \subset P$ the subset of partitions having all the blocks of even size. Observe that $P_{\text{even}}(k, l) = \varnothing$ for $k + l$ odd.

We use the fact, from [1], that there is a signature map $\varepsilon : P_{\text{even}} \to \{-1, 1\}$, extending the usual signature of permutations, $\varepsilon : S_\infty \to \{-1, 1\}$. This map is obtained by setting $\varepsilon(\pi) = (-1)^c$, where $c \in \mathbb{N}$ is the number of switches between neighbors required for making π noncrossing, and which can be shown to be well-defined modulo 2.

Definition 1.3 ([1])**.** Given variables $x_1, \ldots, x_N$, any permutation $\sigma \in S_k$ produces two collections of relations between these variables, as follows:

(1) Untwisted relations: $x_{i_1} \ldots x_{i_k} = x_{i_{\sigma(1)}} \ldots x_{i_{\sigma(k)}}$ for any $i_1, \ldots, i_k$.
(2) Twisted relations:

$$x_{i_1} \ldots x_{i_k} = \varepsilon\big(\ker\big(\begin{smallmatrix} i_1 & \cdots & i_k \\ i_{\sigma(1)} & \cdots & i_{\sigma(k)} \end{smallmatrix}\big)\big) x_{i_{\sigma(1)}} \ldots x_{i_{\sigma(k)}}$$

for any $i_1, \ldots, i_k$.

The untwisted relations are denoted by $\mathcal{R}_\sigma$ and the twisted ones by $\bar{\mathcal{R}}_\sigma$.

Observe that the relations $\mathcal{R}_\sigma$ are trivially satisfied for the standard coordinates on $S^{N-1}_{\mathbb{R}}$ for any $\sigma \in S_k$. A twisted analog of this fact holds, in the sense that the standard coordinates on $\bar{S}^{N-1}_{\mathbb{R}}$ satisfy the relations $\bar{\mathcal{R}}_\sigma$ for any $\sigma \in S_k$. Indeed, by anticommutation we must have a formula of type $x_{i_1} \ldots x_{i_k} = \pm x_{i_{\sigma(1)}} \ldots x_{i_{\sigma(k)}}$, and the sign $\pm$ obtained in this way is precisely the one given above (see [1]),

$$\pm = \varepsilon\big(\ker\big(\begin{smallmatrix} i_1 & \cdots & i_k \\ i_{\sigma(1)} & \cdots & i_{\sigma(k)} \end{smallmatrix}\big)\big).$$

Finally, we agree as in [1] to distinguish the untwisted and twisted cases by using a dot symbol, which is null in the untwisted case and a bar in the twisted case.

We have now all the needed ingredients for axiomatizing the various spheres.

Definition 1.4. We have three types of noncommutative spheres $S \subset S^{N-1}_{\mathbb{R},+}$, as follows:

(1) Monomial: $\dot{S}^{N-1}_{\mathbb{R},E}$, with $E \subset S_\infty$, obtained via the relations $\{\dot{\mathcal{R}}_\sigma \mid \sigma \in E\}$.
(2) Mixed monomial: $S^{N-1}_{\mathbb{R},E,F} = S^{N-1}_{\mathbb{R},E} \cap \bar{S}^{N-1}_{\mathbb{R},F}$, with $E, F \subset S_\infty$.
(3) Polygonal: $S^{N-1,d-1}_{\mathbb{R},E,F} = S^{N-1}_{\mathbb{R},E,F} \cap S^{N-1,d-1}_{\mathbb{R},+}$, with $E, F \subset S_\infty$ and $d \in \{1, \ldots, N\}$.

Here the subsphere $S^{N-1,d-1}_{\mathbb{R},+} \subset S^{N-1}_{\mathbb{R},+}$ appearing in (3) is constructed as in Proposition 1.2, by imposing the relations $x_{i_0} \ldots x_{i_d} = 0$ for $i_0, \ldots, i_d$ distinct.

With the above notions, we cover all spheres appearing so far. More precisely, the five spheres in [1] are monomial, the nine spheres in Proposition 1.2 are mixed monomial, and the polygonal sphere formalism covers all the examples given so far in this paper.

Observe that the set of mixed monomial spheres is closed under intersections. The same holds for the set of polygonal spheres, because we have the following formula:

$$S^{N-1,d-1}_{\mathbb{R},E,F} \cap S^{N-1,d'-1}_{\mathbb{R},E',F'} = S^{N-1,\min(d,d')-1}_{\mathbb{R},E\cup E',F\cup F'}.$$

Let us try now to understand the structure of the various types of noncommutative spheres. We call a group of permutations $G \subset S_\infty$ filtered if, with $G_k = G \cap S_k$, we have $G_k \times G_l \subset G_{k+l}$ for any k, l. We use the following simple fact, coming from [1].

Proposition 1.5. *The various spheres can be parametrized by groups, as follows:*

(1) *Monomial case:* $\dot{S}^{N-1}_{\mathbb{R},G}$*, with filtered group* $G \subset S_\infty$*.*
(2) *Mixed monomial case:* $S^{N-1}_{\mathbb{R},G,H}$*, with filtered groups* $G, H \subset S_\infty$*.*
(3) *Polygonal case:* $S^{N-1,d-1}_{\mathbb{R},G,H}$*, with filtered groups* $G, H \subset S_\infty$ *and* $d \in \{1, \ldots, N\}$*.*

Proof. As explained in [1], in order to prove (1) for a monomial sphere $S = \dot{S}_{\mathbb{R},E}$, we can take $G \subset S_\infty$ to be the set of permutations $\sigma \in S_\infty$ having the property that the relations $\dot{\mathcal{R}}_\sigma$ hold for the standard coordinates of S. We have then $E \subset G$ and also $S = \dot{S}^{N-1}_{\mathbb{R},G}$. The fact that G is a filtered group is clear as well; see [1]. Cases (2) and (3) follow from (1) by taking intersections. □

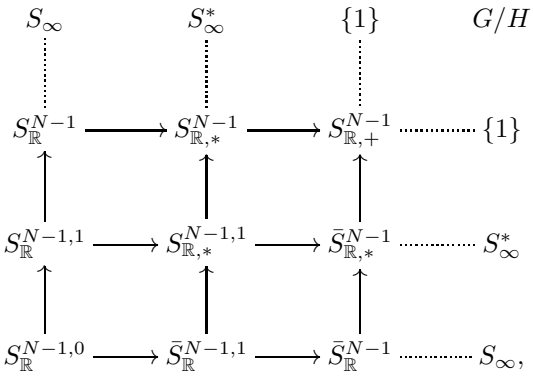
Let us write now the nine main polygonal spheres as in Proposition 1.5 (2). We recall from [1] that the permutations $\sigma \in S_\infty$ having the property that when labelling clockwise their legs $\circ \bullet \circ \bullet \ldots$ and string joins a white leg to a black leg, form a filtered group, denoted $S^*_\infty \subset S_\infty$. This group comes from the half-liberation considerations in [6] and its structure is very simple (see [1]),

$$S^*_{2n} \simeq S_n \times S_n, \quad S^*_{2n+1} \simeq S_n \times S_{n+1}.$$

We call a mixed monomial sphere parametrization $S = S^{N-1}_{\mathbb{R},G,H}$ standard when both filtered groups $G, H \subset S_\infty$ are chosen to be maximal. In this case, Proposition 1.5 and its proof tell us that G, H encode all the monomial relations which hold in S.

We have the following result, extending some previous findings from [1].

Theorem 1.6. *The standard parametrization of the nine main spheres is*



where $S_\infty^* \subset S_\infty$ *is given by* $S_{2n}^* \simeq S_n \times S_n$, $S_{2n+1}^* \simeq S_n \times S_{n+1}$.

Proof. The fact that we have parametrizations as in the statement is known to hold for the five main spheres from [1], as explained there. For the remaining four spheres the result follows by intersecting and using the following formula, valid for any $E, F \subset S_\infty$:

$$S_{\mathbb{R},E,F}^{N-1} \cap S_{\mathbb{R},E',F'}^{N-1} = S_{\mathbb{R},E\cup E',F\cup F'}^{N-1}.$$

In order to prove that the parametrizations are standard, we must compute the following two filtered groups and show that we get the groups in the statement:

$$G = \{\sigma \in S_\infty \mid \text{the relations } \mathcal{R}_\sigma \text{ hold over } X\},$$
$$H = \{\sigma \in S_\infty \mid \text{the relations } \bar{\mathcal{R}}_\sigma \text{ hold over } X\}.$$

As a first observation, using the various inclusions between spheres, we just have to compute G for the spheres on the bottom and H for the spheres on the left:

$$X = S_{\mathbb{R}}^{N-1,0}, \bar{S}_{\mathbb{R}}^{N-1,1}, \bar{S}_{\mathbb{R}}^{N-1} \implies G = S_\infty, S_\infty^*, \{1\},$$
$$X = S_{\mathbb{R}}^{N-1,0}, S_{\mathbb{R}}^{N-1,1}, S_{\mathbb{R}}^{N-1} \implies H = S_\infty, S_\infty^*, \{1\}.$$

The results for $S_{\mathbb{R}}^{N-1,0}$ being clear, we are left with computing the remaining four groups, for the spheres $S_{\mathbb{R}}^{N-1}, \bar{S}_{\mathbb{R}}^{N-1}, S_{\mathbb{R}}^{N-1,1}, \bar{S}_{\mathbb{R}}^{N-1,1}$. The proof here goes as follows:

(1) $S_{\mathbb{R}}^{N-1}$. According to the definition of $H = (H_k)$, we have

$$\begin{aligned} H_k &= \Big\{\sigma \in S_k \mid x_{i_1} \ldots x_{i_k} = \varepsilon\Big(\ker\Big(\begin{smallmatrix} i_1 & \cdots & i_k \\ i_{\sigma(1)} & \cdots & i_{\sigma(k)} \end{smallmatrix}\Big)\Big) x_{i_{\sigma(1)}} \ldots x_{i_{\sigma(k)}} \\ &\qquad\qquad \text{for all } i_1, \ldots, i_k\Big\} \\ &= \Big\{\sigma \in S_k \mid \varepsilon\Big(\ker\Big(\begin{smallmatrix} i_1 & \cdots & i_k \\ i_{\sigma(1)} & \cdots & i_{\sigma(k)} \end{smallmatrix}\Big)\Big) = 1 \text{ for all } i_1, \ldots, i_k\Big\} \\ &= \big\{\sigma \in S_k \mid \varepsilon(\tau) = 1 \text{ for all } \tau \leq \sigma\big\}. \end{aligned}$$

Now since for any $\sigma \in S_k$, $\sigma \neq 1_k$, we can always find a partition $\tau \leq \sigma$ satisfying $\varepsilon(\tau) = -1$, we deduce that we have $H_k = \{1_k\}$, and so $H = \{1\}$, as desired.

(2) $\bar{S}_{\mathbb{R}}^{N-1}$. The proof of $G=\{1\}$ here is similar to the proof of $H=\{1\}$ in case (1), using the same combinatorial ingredient at the end.

(3) $S_{\mathbb{R}}^{N-1,1}$. By definition of $H=(H_k)$, a permutation $\sigma\in S_k$ belongs to H_k when the following condition is satisfied for any choice of the indices $i_1,\ldots,i_k$:

$$x_{i_1}\ldots x_{i_k}=\varepsilon\big(\ker\begin{pmatrix}i_1&\cdots&i_k\\ i_{\sigma(1)}&\cdots&i_{\sigma(k)}\end{pmatrix}\big)x_{i_{\sigma(1)}}\ldots x_{i_{\sigma(k)}}.$$

When $|\ker i|=1$, this formula reads $x_r^k=x_r^k$, which is true. When $|\ker i|\geq 3$, this formula is automatically satisfied as well, because by using the relations $ab=ba$ and $abc=0$ for a,b,c distinct, which both hold over $S_{\mathbb{R}}^{N-1,1}$, the formula reduces to $0=0$. Thus, we are left with studying the case $|\ker i|=2$. Here the quantities on the left $x_{i_1}\ldots x_{i_k}$ will not vanish, so the sign on the right must be 1, and we therefore have

$$H_k=\big\{\sigma\in S_k\mid\varepsilon(\tau)=1\text{ for all }\tau\leq\sigma,\ |\tau|=2\big\}.$$

Now by coloring the legs of σ clockwise $\circ\bullet\circ\bullet\ldots$, the above condition is satisfied when each string of σ joins a white leg to a black leg. Thus $H_k=S_k^*$, as desired.

(4) $\bar{S}_{\mathbb{R}}^{N-1,1}$. The proof of $G=S_\infty^*$ here is similar to the proof of $H=S_\infty^*$ in case (3), using the same combinatorial ingredient at the end. □

As a conclusion, the $5+4=9$ spheres from Proposition 1.2 come from the 3×3 ways of selecting a pair of filtered groups (G,H), among the basic filtered groups $\{1\},S_\infty^*,S_\infty$. This result, improving some previous findings from [1], is the best one that we have.

2. Uniqueness results

In this section we discuss a number of conjectures, whose validity would improve the formalism in Theorem 1.6. These conjectures are all equivalent:

Proposition 2.1. *The following are equivalent:*

(1) *The three spheres in* [4] *are the only untwisted monomial ones.*
(2) *The five spheres in* [1] *are the only monomial ones.*
(3) *The nine spheres in Theorem 1.6 are the only mixed monomial ones.*

Proof. The implications (1) ⇒ (2) ⇒ (3) are obtained by intersecting, and (3) ⇒ (2) ⇒ (1) by restricting. □

These conjectures belong a priori to operator theory/algebras and more specifically to a branch that could be called "noncommutative algebraic geometry, with positivity", that we are trying to develop in this paper. Our claim here would be that there might be a purely combinatorial way of solving them. We have the following definition.

Definition 2.2. Consider a filtered group of permutations, that is, a group $G\subset S_\infty$, $G=(G_k)$, satisfying $G_k\times G_l\subset G_{k+l}$ for any k,l. We call this group

(1) *saturated*, if G consists of all permutations $\sigma\in S_k$ such that the relations $x_{i_1}\ldots x_{i_k}=x_{i_{\sigma(1)}}\ldots x_{i_{\sigma(k)}}$ hold over $S_{\mathbb{R},G}^{N-1}$ for any $i_1,\ldots,i_k$;

(2) *weakly saturated*, if whenever $\sigma \in G_k$ satisfies $\sigma(i+1) = \sigma(i) \pm 1$, the permutation $\sigma^{(i,i+1)} \in S_{k-2}$ obtained by deleting i, $i+1$ and their images belongs to G_{k-2}.

It follows from Proposition 1.5 that we have a saturation operation $G \to \widetilde{G}$ for the filtered groups, which can be obtained by setting $S^{N-1}_{\mathbb{R},G} = S^{N-1}_{\mathbb{R},\widetilde{G}}$, with $\widetilde{G} \subset S_\infty$ chosen maximal. With this remark in hand, the conjecture in Proposition 2.1 (1) simply states that there are exactly three saturated groups, namely $\{1\}, S^*_\infty, S_\infty$. Observe that these three groups are indeed saturated, as a consequence of Theorem 1.6.

Regarding now the weak saturation, once again this produces an operation $G \to \bar{G}$ for the filtered groups. Indeed, given $G \subset S_\infty$ we can add to it all the permutations $\sigma^{(i,i+1)}$ appearing in Definition 2.2 (2), then consider the filtered group generated by G and by these extra permutations, and then repeat the procedure, a finite or possibly countable number of times, until we obtain a weakly saturated group $\bar{G}$.

The interest in the above notions comes from the following result.

Proposition 2.3. *Any saturated group is weakly saturated. In particular, if the only weakly saturated groups are $\{1\}, S^*_\infty, S_\infty$, then the conjectures in Proposition 2.1 hold.*

Proof. Consider a saturated group $G \subset S_\infty$ and let $S = S^{N-1}_{\mathbb{R},G}$ be the corresponding sphere. We must show that if $\sigma \in G_k$ satisfies $\sigma(i+1) = \sigma(i) \pm 1$, then $\sigma^{(i,i+1)} \in G_{k-2}$.

We know from $\sigma \in G_k$ that the relations $x_{i_1} \ldots x_{i_k} = x_{i_{\sigma(1)}} \ldots x_{i_{\sigma(k)}}$ hold over S. In the case $\sigma(i+1) = \sigma(i)+1$ these relations are of type $XabY = ZabT$, and by setting $a = b^*$ and summing over a, we obtain $XY = ZT$. But these are exactly the relations associated to the permutation $\sigma^{(i,i+1)} \in S_{k-2}$. We deduce that we have $\sigma^{(i,i+1)} \in G_{k-2}$.

In the case $\sigma(i+1) = \sigma(i) - 1$ the proof is similar. Indeed, the relations associated to σ are now of type $XabY = ZbaT$. Once again setting $a = b^*$ and summing over a, we obtain $XY = ZT$ and conclude that $\sigma^{(i,i+1)} \in G_{k-2}$.

Finally, the last assertion is clear from the above considerations. □

We have the following result, of interest in connection with Proposition 2.1.

Proposition 2.4. *If a filtered group $G \subset S_\infty$, $G = (G_k)$ is weakly saturated and $|G_5| > 1$, then G must be one of the groups S_∞, S^*_∞.*

Proof. Our claim, which will basically prove the result, is that for $k \leq 5$, $\sigma \in S_k$ implies that there exist $\tau \in \langle 1 \otimes \sigma, \sigma \otimes 1 \rangle \subset S_{k+1}$ and i such that $\tau(i+1) = \tau(i) \pm 1$.

We have no conceptual proof for this claim, so we will first discuss the cases $k = 3, 4$, following some previous work in [1], and then the case $k = 5$.

Case $k = 3$. Here we just have to investigate the 3-cycles and by symmetry we can restrict attention to the cycle $\sigma = (231)$. As explained in [1], a standard C^*-algebra trick shows that the corresponding sphere collapses to $S^{N-1}_{\mathbb{R}}$. The

point now is that this trick can be converted into a proof of the above claim. More precisely, we have $(1\otimes\sigma)(\sigma\otimes 1)=(2143)$, which satisfies the requirements for τ in the above claim.

Case $k=4$. Here, as explained in [1], for 22 of the 24 permutations $\sigma\in S_4$, the above claim holds, with $\tau=\sigma$. The remaining two permutations are $\sigma_1=(3412)$ and $\sigma_2=(2413)$. The point now is that we have $(1\otimes\sigma_1)(\sigma_1\otimes 1)=(52143)$ and $\sigma_2^2=(4321)$, which both satisfy the requirements for τ in the above claim; see [1].

Case $k=5$. We have to study the 120 elements $\sigma\in S_5$. Best here is to consider the corresponding group $\langle\sigma\rangle\subset S_5$, which is $G=\mathbb{Z}_s$ with $s=1,2,3,4,5,6$. At $s=1$ the result is clear. At $s=3,4,5$ what happens is that we can always find $\tau\in G$ satisfying $\tau(i+1)=\tau(i)\pm 1$, for some i. At $s=6$ the result follows from the $s=3$ result. Thus we are left with the case $s=2$. Here the cycle structure of σ can be either (2111), where the result is clear, or (221), which is the case left. But here σ must appear from one of the elements $(2143),(4321),(3412)\in S_4$ by adding a "fixed point". When this fixed point is at right or at left, the result is clear, so by symmetry it remains to study the two cases where this fixed point is either in the middle or at left of the middle point. Thus we have $3\times 2=6$ cases to be investigated, five of these cases are trivial, in the sense that σ itself satisfies $\sigma(i+1)=\sigma(i)\pm 1$ for some i. The remaining case is $\sigma=(42513)$, and here $(1\otimes\sigma)(\sigma\otimes 1)=(435621)$, which satisfies the requirements for τ in the above claim.

Thus we are done with the proof of the above claim. The point now is that, given $G\subset S_\infty$ as in the statement, we can pick $\sigma\in G_5-\{1_5\}$ and apply to it the above claim, perhaps several times, until we obtain either the basic crossing $(12)\in S_2$ or the half-liberated partition $(321)\in S_3$. We deduce from this that G must be generated by one of these two partitions, and so we have $G=S_\infty$ or $G=S_\infty^*$, as desired. □

Combining now Propositions 2.3 and 2.4, we deduce the following result.

Theorem 2.5. *The conjectures in Proposition 2.1 hold, provided that the spheres in question are generated by relations coming from permutations* $\sigma\in S_k$ *with* $k\leq 5$.

Proof. Propositions 2.3 and 2.4 tell us that at $k\leq 5$ we have:

$$\sigma\in S_k,\ \sigma\neq 1_k\implies S^{N-1}_{\mathbb{R},\sigma}\in\{S^{N-1}_{\mathbb{R}},S^{N-1}_{\mathbb{R},*}\}.$$

Thus the conjecture in Proposition 2.1 (1) holds under the $k\leq 5$ assumption. The statements coming from Proposition 2.1 (2) and (3) follow as well, by intersecting. □

We believe that Proposition 2.4 should hold under the assumption $G\neq\{1\}$, therefore proving the conjectures in Proposition 2.1, but we were unable so far to extract something conceptual from the above proof which would extend from $k\leq 5$ to $k\in\mathbb{N}$.

As a second piece of evidence for the conjectures in Proposition 2.1, we can try to intersect an arbitrary untwisted monomial sphere $S = S^{N-1}_{\mathbb{R},E}$ with the three untwisted monomial spheres, or with the five monomial spheres, or with the nine mixed monomial spheres, and see if we get indeed the results predicted by $S \in \{S^{N-1}_{\mathbb{R}}, S^{N-1}_{\mathbb{R},*}, S^{N-1}_{\mathbb{R},+}\}$.

There are many interesting statements here and as an example we have:

Proposition 2.6. *For any $F \subset S_\infty$ we have*

$$S^{N-1}_{\mathbb{R}} \cap \bar{S}^{N-1}_{\mathbb{R},F} = S^{N-1,d-1}_{\mathbb{R}}$$

for a certain number $d \in \{1, \ldots, N\}$. In addition, we have $d \in \{1, 2, N\}$.

Proof. We can assume $F = \{\sigma\}$, with $\sigma \in S_k$, $\sigma \neq 1_k$.

(1) The anticommutation relations, when compared to the corresponding commutation relations, translate into relations of type $a_{i_1} \ldots a_{i_k} = 0$ for certain indices i. Since we can permute the terms and also replace $x_i^r \to x_i$ for any $r \geq 2$, we are led to relations of type $x_{i_0} \ldots x_{i_r} = 0$ for any $i_0, \ldots, i_r$ distinct. Now since the spheres $S^{N-1,r-1}_{\mathbb{R}}$ form an increasing sequence, by setting $d = \min(r)$ we obtain the formula in the statement.

(2) We use the defining formulae for $\bar{S}^{N-1}_{\mathbb{R},\sigma}$, which are

$$x_{i_1} \ldots x_{i_k} = \varepsilon\big(\ker\big(\begin{smallmatrix} i_1 & \cdots & i_k \\ i_{\sigma(1)} & \cdots & i_{\sigma(k)} \end{smallmatrix}\big)\big) x_{i_{\sigma(1)}} \ldots x_{i_{\sigma(k)}}.$$

Comparing with the commutation relation $x_{i_1} \ldots x_{i_k} = x_{i_{\sigma(1)}} \ldots x_{i_{\sigma(k)}}$, valid for the classical sphere $S^{N-1}_{\mathbb{R}}$, we conclude that the intersection $S^{N-1}_{\mathbb{R}} \cap \bar{S}^{N-1}_{\mathbb{R},\sigma}$ consists of the points $x \in S^{N-1}_{\mathbb{R}}$ which are subject to the following relations:

$$\varepsilon\big(\ker\big(\begin{smallmatrix} i_1 & \cdots & i_k \\ i_{\sigma(1)} & \cdots & i_{\sigma(k)} \end{smallmatrix}\big)\big) = -1 \implies x_{i_1} \ldots x_{i_k} = 0.$$

In other words, given our permutation $\sigma \in S_k$, we can consider all the partitions $\pi \leq \sigma$, obtained by collapsing blocks. The partitions satisfying $\varepsilon(\pi) = 1$ do not produce new relations and the partitions satisfying $\varepsilon(\pi) = -1$ produce the following relations, where $r = |\pi|$ comes from the compression procedure explained in part (1):

$$\ker\big(\begin{smallmatrix} i_1 & \cdots & i_k \\ i_{\sigma(1)} & \cdots & i_{\sigma(k)} \end{smallmatrix}\big) = \pi \implies x_{i_1} \ldots x_{i_r} = 0.$$

We use now the fact that $\sigma \in S_\infty$ with $\sigma \neq 1_1, 1_2, 1_3, \ldots$ implies that there exists $\pi \leq \sigma$, $\varepsilon(\pi) = -1$, $|\pi| \leq 3$, which comes by selecting a suitable crossing for σ and then by collapsing all the other strings to a single block. Thus $d + 1 = \min(r)$ satisfies $d \in \{2, 3\}$ and we are done. □

As a last comment, a useful ingredient for dealing with the conjectures in Proposition 2.1 would be a good diagrammatic framework for the polygonal spheres. Observe that all the relations that we need are of the following type, with $\alpha, \beta \in \{-, 1, 0, 1\}$:

$$x_{i_1} \ldots x_{i_k} = \begin{cases} \alpha \cdot x_{i_{\sigma(1)}} \ldots x_{i_{\sigma(k)}} & \text{if } \varepsilon = 1, \\ \beta \cdot x_{i_{\sigma(1)}} \ldots x_{i_{\sigma(k)}} & \text{if } \varepsilon = -1. \end{cases}$$

Here the number $\varepsilon = \pm 1$ on the right is by definition given by

$$\varepsilon = \varepsilon\big(\ker\big(\begin{smallmatrix} i_1 & \cdots & i_k \\ i_{\sigma(1)} & \cdots & i_{\sigma(k)} \end{smallmatrix}\big)\big).$$

Thus the diagrams that we need are a priori the usual permutations, colored in $3 \times 3 = 9$ ways, according to the values of (α, β). It is quite unclear, however, on how to turn this idea into an efficient computational tool, that can solve our conjectures.

3. Affine actions

We discuss now the computation of the quantum isometry groups of our spheres. We use the quantum group formalism developed by Woronowicz in [25, 26].

There are several ways of defining quantum isometries, depending on the type of manifold involved, see [5, 10, 12, 13, 14, 15, 18, 21, 23, 24]. In what follows we use an algebraic geometry approach, inspired from Goswami's paper [15].

Assume that we are given an algebraic manifold $X \subset S^{N-1}_{\mathbb{R},+}$, in the sense that X appears via a presentation result as follows, for certain noncommutative polynomials P_α:

$$C(X) = C(S^{N-1}_{\mathbb{R},+})/\langle P_\alpha(x_1, \ldots, x_N) = 0\rangle.$$

We say that a closed subgroup $G \subset O_N^+$ acts affinely on X when we have a morphism of C^*-algebras $\Phi : C(X) \to C(G) \otimes C(X)$, given by $x_i \to \sum_j u_{ij} \otimes x_i$. Observe that such a morphism is automatically coassociative and counital and unique.

Proposition 3.1. *Given an algebraic submanifold $X \subset S^{N-1}_{\mathbb{R},+}$, the category of closed quantum subgroups $G \subset O_N^+$ acting affinely on X has a universal object, $G^+(X)$.*

Proof. Assume indeed that X is defined by polynomials P_α as above. Our claim is that $G = G^+(X)$ appears as

$$C(G) = C(O_N^+)/\langle P_\alpha(X_1, \ldots, X_N) = 0\rangle,$$

where $X_i = \sum_j u_{ij} \otimes x_j \in C(O_N^+) \otimes C(X)$.

In order to prove this claim, we have to clarify how the above relations $P_\alpha(X_1, \ldots, X_N) = 0$ are interpreted inside $C(O_N^+)$ and then show that G is indeed a quantum group.

So, pick one of the defining polynomials, $P = P_\alpha$, and write it as follows:

$$P(x_1, \ldots, x_N) = \sum_r \alpha_r \cdot x_{i_1^r} \ldots x_{i_{s(r)}^r}.$$

With $X_i = \sum_j u_{ij} \otimes x_j$ as above, we have the following formula:

$$P(X_1, \ldots, X_N) = \sum_r \alpha_r \sum_{j_1^r \ldots j_{s(r)}^r} u_{i_1^r j_1^r} \ldots u_{i_{s(r)}^r j_{s(r)}^r} \otimes x_{j_1^r} \ldots x_{j_{s(r)}^r}.$$

Since the space spanned by the variables at right is finite-dimensional, the relations $P(X_1,\ldots,X_N)=0$ correspond indeed to certain relations between the variables u_{ij}.

It order to show that G is indeed a quantum group, consider the following elements:

$$u_{ij}^{\Delta}=\sum_k u_{ik}\otimes u_{kj},\quad u_{ij}^{\varepsilon}=\delta_{ij},\quad u_{ij}^{S}=u_{ji}.$$

Now if we consider the associated elements $X_i^\gamma=\sum_j u_{ij}^\gamma\otimes x_j$, with $\gamma\in\{\Delta,\varepsilon,S\}$, then from the relations $P(X_1,\ldots,X_N)=0$ we deduce

$$P(X_1^\gamma,\ldots,X_N^\gamma)=(\gamma\otimes\mathrm{id})P(X_1,\ldots,X_N)=0.$$

Thus, using the universal property of G, we can construct morphisms of algebras mapping $u_{ij}\to u_{ij}^\gamma$ for any $\gamma\in\{\Delta,\varepsilon,S\}$. This finishes the proof. □

As an illustration, we have the following statement, coming from [9, 15].

Proposition 3.2. *Assume that $X\subset S_{\mathbb{R}}^{N-1}$ is invariant under $x_i\to -x_i$ for any i.*

(1) *If the coordinates $x_1,\ldots,x_N$ are linearly independent inside $C(X)$, then the group $G(X)=G^+(X)\cap O_N$ consists of the usual isometries of X.*
(2) *In addition, in the case where the products of coordinates $\{x_ix_j\mid i\leq j\}$ are linearly independent inside $C(X)$, we have $G^+(X)=G(X)$.*

Proof. This follows from [9, 15], the idea being as follows:

(1) The assertion here is well known, $G(X)=G^+(X)\cap O_N$ being by definition the biggest subgroup $G\subset O_N$ acting affinely on X. We refer to [15] for details and for a number of noncommutative extensions of this fact, with $G(X)$ replaced by $G^+(X)$.

(2) Here we must prove that, whenever we have a coaction $\Phi:C(X)\to C(G)\otimes C(X)$, given by $\Phi(x_i)=\sum_j u_{ij}\otimes x_j$, the variables u_{ij} commute. But this follows by using a standard trick from [9] that we will briefly recall now. We can write:

$$\Phi([x_i,x_j])=\sum_{k\leq l}\big([u_{ik},u_{jl}]-[u_{jk},u_{il}]\big)\otimes\Big(1-\frac{\delta_{kl}}{2}\Big)x_kx_l.$$

Now since the variables $\{x_kx_l\mid k\leq l\}$ are linearly independent, we obtain from this $[u_{ik},u_{jl}]=[u_{jk},u_{il}]$ for any i,j,k,l. Moreover, if we apply now the antipode, we further obtain $[u_{lj},u_{ki}]=[u_{li},u_{kj}]$ and, by relabelling, $[u_{ik},u_{jl}]=[u_{il},u_{jk}]$. We therefore conclude that we have $[u_{ik},u_{jl}]=0$ for any i,j,k,l, and this finishes the proof; see [9]. □

With the above notion in hand, let us investigate the polygonal spheres. We recall from [1, 4] that the quantum isometry groups of the five main spheres

are as follows:

$$\begin{array}{ccccccccc}
S_{\mathbb{R}}^{N-1} & \longrightarrow & S_{\mathbb{R},*}^{N-1} & \longrightarrow & S_{\mathbb{R},+}^{N-1} & \longleftarrow & \bar{S}_{\mathbb{R},*}^{N-1} & \longleftarrow & \bar{S}_{\mathbb{R}}^{N-1} \\
\wr & & \wr & & \wr & & \wr & & \wr \\
O_N & \longrightarrow & O_N^* & \longrightarrow & O_N^+ & \longleftarrow & \bar{O}_N^* & \longleftarrow & \bar{O}_N.
\end{array}$$

Here O_N is the orthogonal group, O_N^+ is its free version constructed in [23], $\bar{O}_N$ is its twist constructed in [2], O_N^* is its half-liberated version studied in [6], and $\bar{O}_N^*$ is its twisted half-liberated version constructed in [1]. We refer to [1] for full details.

In the polygonal case now, we begin with the classical case. We use the hyperoctahedral group H_N and its free version H_N^+ constructed in [2].

Proposition 3.3. *The quantum isometry group of $S_{\mathbb{R}}^{N-1,d-1}$ is as follows:*
(1) *At $d=1$ we obtain the free hyperoctahedral group H_N^+.*
(2) *At $d=2,\ldots,N-1$ we obtain the hyperoctahedral group H_N.*
(3) *At $d=N$ we obtain the orthogonal group O_N.*

Proof. Observe first that the sphere $S_{\mathbb{R}}^{N-1,d-1}$ appears by definition as a union on $\binom{N}{d}$ copies of the sphere $S_{\mathbb{R}}^{d-1}$, one for each choice of d coordinate axes, among the coordinate axes of $\mathbb{R}^N$. We can write this decomposition as follows, with $I_N=\{1,\ldots,N\}$:

$$S_{\mathbb{R}}^{N-1,d-1}=\bigcup_{I\subset I_N,|I|=d}(S_{\mathbb{R}}^{d-1})^I.$$

(1) At $d=1$ our sphere is $S_{\mathbb{R}}^{N-1,0}=\mathbb{Z}_2^{\oplus N}$, formed by the endpoints of the N copies of $[-1,1]$ on the coordinate axes of $\mathbb{R}^N$. Thus by [2] the quantum isometry group is H_N^+.

(2) Our first claim is that at $d\geq 2$, the elements $\{x_ix_j\mid i\leq j\}$ are linearly independent. Since $S_{\mathbb{R}}^{N-1,1}\subset S_{\mathbb{R}}^{N-1,d}$, we can restrict attention to the case $d=2$. Here the above decomposition is as follows, where $\mathbb{T}^{\{i,j\}}$ denote the various copies of $\mathbb{T}$:

$$S_{\mathbb{R}}^{N-1,d-1}=\bigcup_{i<j}\mathbb{T}^{\{i,j\}}.$$

Now since $\{x^2,y^2,xy\}$ are linearly independent over $\mathbb{T}\subset\mathbb{R}^2$, we deduce from this that $\{x_ix_j\mid i\leq j\}$ are linearly independent over $S_{\mathbb{R}}^{N-1,d-1}$ and we are done. Thus, our claim is proved and so Proposition 3.2 (2) above applies and gives $G^+(X)=G(X)$.

We are therefore left with proving $G(X)=H_N$ for any $d\in\{2,\ldots,N-1\}$.

Let us first discuss the case $d=2$. Here any affine isometric action $U\curvearrowright S_{\mathbb{R}}^{N-1,1}$ must permute the $\binom{N}{2}$ circles $\mathbb{T}^I$. So we can write $U(\mathbb{T}^I)=\mathbb{T}^{I'}$ for a certain permutation of the indices $I\to I'$. Now since U is bijective, we deduce that for any I,J we have

$$U(\mathbb{T}^I\cap\mathbb{T}^J)=\mathbb{T}^{I'}\cap\mathbb{T}^{J'}.$$

Since for $|I\cap J|=0,1,2$ we have $\mathbb{T}^I\cap\mathbb{T}^J\simeq\varnothing,\{-1,1\},\mathbb{T}$, by taking the union over I,J with $|I\cap J|=1$, we deduce that $U(\mathbb{Z}_2^{\oplus N})=\mathbb{Z}_2^{\oplus N}$. Thus $U\in H_N$ and we are done.

In the general case now, $d\in\{2,\ldots,N-1\}$, we can proceed similarly, by recurrence. Indeed, for any subsets $I,J\subset I_N$ with $|I|=|J|=d$ we have

$$(S_{\mathbb{R}}^{d-1})^I\cap(S_{\mathbb{R}}^{d-1})^J=(S_{\mathbb{R}}^{|I\cap J|-1})^{I\cap J}.$$

Using $d\leq N-1$, we deduce that we have the following formula:

$$S_{\mathbb{R}}^{N-1,d-2}=\bigcup_{|I|=|J|=d,|I\cap J|=d-1}(S_{\mathbb{R}}^{|I\cap J|-1})^{I\cap J}.$$

On the other hand, using the same argument as in the $d=2$ case, we deduce that the space on the right is invariant, under any affine isometric action on $S_{\mathbb{R}}^{N-1,d-1}$. Thus by recurrence we obtain

$$G(S_{\mathbb{R}}^{N-1,d-1})=G(S_{\mathbb{R}}^{N-1,d-2})=H_N$$

and we are done.

(3) At $d=N$ the result is known from [4], with the proof coming from the equality $G^+(X)=G(X)$, deduced from Proposition 3.2 (2), as explained above. □

In order to discuss the twisted case, we recall the following definition from [2].

Definition 3.4. $\bar{O}_N$ is the quantum group obtained by imposing the relations

$$ab=\begin{cases}-ba & \text{for } a\neq b \text{ on the same row or column of } u,\\ ba & \text{otherwise}\end{cases}$$

to the standard coordinates u_{ij} of the quantum group O_N^+.

As explained in [2], this quantum group has an interesting noncommutative geometric meaning, because it is the quantum isometry group of the hypercube $\mathbb{Z}_2^N\subset\mathbb{R}^N$. Thus, $\bar{O}_N$ is a natural analog of the hyperoctahedral group H_N. However, quite surprisingly, $\bar{O}_N$ is not the free version of H_N. The correct free version of H_N is the quantum isometry group H_N^+ of the space $\mathbb{Z}_2^{\oplus N}\subset\mathbb{R}^N$ formed by the N copies of $\mathbb{Z}_2\subset\mathbb{R}$ on the coordinate axes of $\mathbb{R}^N$, that we already met in Proposition 3.3 (1); see [2].

Now back to the polygonal spheres, the study in the twisted case is considerably more difficult than in the classical case, and we have complete results only at $d=1,2,N$. Our next statement will be enhanced later on only with a few minor results.

Theorem 3.5. *The quantum isometry group of $\bar{S}_{\mathbb{R}}^{N-1,d-1}$ is as follows:*

(1) *At $d=1$ we obtain the free hyperoctahedral group H_N^+.*
(2) *At $d=2$ we obtain the hyperoctahedral group H_N.*
(3) *At $d=N$ we obtain the twisted orthogonal group $\bar{O}_N$.*

Proof. The idea is to adapt the proof of Proposition 3.3:

(1) At $d=1$ we have $\bar{S}_{\mathbb{R}}^{N-1,0}=S_{\mathbb{R}}^{N-1,0}=\mathbb{Z}_2^{\oplus N}$ and by Proposition 3.3 (1), coming from [2], the corresponding quantum isometry group is indeed H_N^+.

(2) As a first ingredient, we will need the twisted analog of the trick from [9], explained in the proof of Proposition 3.2 (2). This twisted trick was already worked out in [1] for the sphere $\bar{S}_{\mathbb{R}}^{N-1}$ itself, and the situation is similar for any closed subset $X\subset\bar{S}_{\mathbb{R}}^{N-1}$, having the property that the variables $\{x_ix_j\mid i\leq j\}$ are linearly independent. More precisely, our claim is that if $G\subset O_N^+$ acts on X, then we must have $G\subset\bar{O}_N$.

Indeed, given a coaction $\Phi(x_i)=\sum_j u_{ij}\otimes x_j$, we can write, as in [1]:

$$\Phi(x_ix_j)=\sum_k u_{ik}u_{jk}\otimes x_k^2+\sum_{k<l}(u_{ik}u_{jl}-u_{il}u_{jk})\otimes x_kx_l.$$

We deduce that with $[[a,b]]=ab+ba$ we have the following formula:

$$\Phi([[x_i,x_j]])=\sum_k[[u_{ik},u_{jk}]]\otimes x_k^2+\sum_{k<l}\big([u_{ik},u_{jl}]-[u_{il},u_{jk}]\big)\otimes x_kx_l.$$

Now assuming $i\neq j$, we have $[[x_i,x_j]]=0$ and therefore obtain $[[u_{ik},u_{jk}]]=0$ for any k and $[u_{ik},u_{jl}]=[u_{il},u_{jk}]$ for any $k<l$. By applying the antipode and then relabelling, the latter relation gives $[u_{ik},u_{jl}]=0$. Thus we have reached to the defining relations for the quantum group $\bar{O}_N$ and so we have $G\subset\bar{O}_N$, as claimed.

Our second claim is that the above trick applies to any $\bar{S}_{\mathbb{R}}^{N-1,d-1}$ with $d\geq 2$. Indeed, using the maps $\pi_{ij}:C(\bar{S}_{\mathbb{R}}^{N-1,d-1})\to C(\bar{S}_{\mathbb{R}}^1)$ obtained by setting $x_k=0$ for $k\neq i,j$, we conclude that the variables $\{x_ix_j\mid i\leq j\}$ are indeed linearly independent over $\bar{S}_{\mathbb{R}}^{N-1,d-1}$.

Summarizing, we have proved so far that if a compact quantum group $G\subset O_N^+$ acts on a polygonal sphere $\bar{S}_{\mathbb{R}}^{N-1,d-1}$ with $d\geq 2$, then we must have $G\subset\bar{O}_N$. We must now adapt the second part of the proof of Proposition 3.3. Since this is quite unobvious at $d\geq 3$, we will restrict our attention to the case $d=2$, as in the statement.

So, consider a compact quantum group $G\subset\bar{O}_N$. In order to have a coaction map

$$\Phi:C(\bar{S}_{\mathbb{R}}^{N-1,1})\to C(G)\otimes C(\bar{S}_{\mathbb{R}}^{N-1,1}),$$

given as usual by

$$\Phi(x_i)=\sum_j u_{ij}\otimes x_j,$$

the elements $X_i=\sum_j u_{ij}\otimes x_j$ must satisfy the relations $X_iX_jX_k=0$ for any i,j,k distinct.

For i,j,k distinct, we have

$$\begin{aligned}X_iX_jX_k&=\sum_{a,b,c}u_{ia}u_{jb}u_{kc}\otimes x_ax_bx_c\\&=\sum_{a,b,c\text{ not distinct}}u_{ia}u_{jb}u_{kc}\otimes x_ax_bx_c\\&=\sum_{a\neq b}u_{ia}u_{ja}u_{kb}\otimes x_a^2x_b+\sum_{a\neq b}u_{ia}u_{jb}u_{ka}\otimes x_ax_bx_a\\&\quad+\sum_{a\neq b}u_{ib}u_{ja}u_{ka}\otimes x_bx_a^2+\sum_a u_{ia}u_{ja}u_{ka}\otimes x_a^3.\end{aligned}$$

Using $x_ax_bx_a=-x_a^2x_b$ and $x_bx_a^2=x_a^2x_b$, we deduce

$$\begin{aligned}X_iX_jX_k&=\sum_{a\neq b}(u_{ia}u_{ja}u_{kb}-u_{ia}u_{jb}u_{ka}+u_{ib}u_{ja}u_{ka})\otimes x_a^2x_b\\&\quad+\sum_a u_{ia}u_{ja}u_{ka}\otimes x_a^3\\&=\sum_{a,b}(u_{ia}u_{ja}u_{kb}-u_{ia}u_{jb}u_{ka}+u_{ib}u_{ja}u_{ka})\otimes x_a^2x_b.\end{aligned}$$

Using now the defining relations for $\bar{O}_N$, which apply to the variables u_{ij}, we can write this formula in a cyclic way, as follows:

$$X_iX_jX_k=\sum_{a,b}(u_{ia}u_{ja}u_{kb}+u_{ja}u_{ka}u_{ib}+u_{ka}u_{ia}u_{jb})\otimes x_a^2x_b.$$

We use now the fact, coming from [1], that the variables $x_a^2x_b$ on the right are linearly independent. We conclude that, in order for our quantum group $G\subset\bar{O}_N$ to act on $\bar{S}_{\mathbb{R}}^{N-1,1}$, its coordinates must satisfy the following relations for any i,j,k distinct:

$$u_{ia}u_{ja}u_{kb}+u_{ja}u_{ka}u_{ib}+u_{ka}u_{ia}u_{jb}=0.$$

Multiplying the left-hand side by u_{kb} and then summing over b, we deduce that we have $u_{ia}u_{ja}=0$ for any i,j. Now since the quotient of $C(\bar{O}_N)$ by these latter relations is $C(H_N)$, we conclude that we have $G^+(\bar{S}_{\mathbb{R}}^{N-1,1})=H_N$, as claimed.

(3) At $d=N$ the result is already known from [1] and its proof follows in fact from the "twisted trick" explained in the proof of (2) above, applied to $\bar{S}_{\mathbb{R}}^{N-1}$. □

Observe that the results that we have so far, namely those in [1] and in Proposition 3.3 and Theorem 3.5, give us the quantum isometry groups of eight of the nine spheres in Theorem 1.6. The sphere left, $S_{\mathbb{R},*}^{N-1,1}$, will be investigated in the next two sections.

In the context of Theorem 3.5, we do not know what happens at $d = 3, \ldots, N-1$. It is easy to see that the hyperoctahedral group H_N acts on any polygonal sphere $\bar{S}_{\mathbb{R}}^{N-1,d-1}$ and our conjecture would be that this action is the universal one.

4. Hyperoctahedral groups

As explained above, our main objective now will be that of computing the quantum isometry group of $S_{\mathbb{R},*}^{N-1,1}$. The computation is quite nontrivial and requires a number of quantum group preliminaries, that we will develop in this section.

We recall from [3, 6] that the quantum group $O_N^* \subset O_N^+$ is obtained by imposing the half-commutation relations $abc = cba$ to the standard coordinates u_{ij}. This quantum group has a twist $\bar{O}_N^*$, constructed in [1], whose definition is as follows.

Definition 4.1. $\bar{O}_N^* \subset O_N^+$ is the quantum group obtained by imposing the relations

$$abc = \begin{cases} -cba & \text{for } r \leq 2,\ s = 3 \text{ or } r = 3,\ s \leq 2, \\ cba & \text{for } r \leq 2,\ s \leq 2 \text{ or } r = s = 3, \end{cases}$$

where $r, s \in \{1, 2, 3\}$ are the numbers of rows and columns, respectively, of u spanned by $a, b, c \in \{u_{ij}\}$.

In order to deal with $\bar{O}_N^*$, it is useful to keep in mind the following table, encoding the choice of the above half-commutation/half-anticommutation signs:

$r \backslash s$	1	2	3
1	+	+	−
2	+	+	−
3	−	−	+

We have intersected twisted and untwisted spheres in Section 2 and we will do the same now for the corresponding orthogonal groups.

Proposition 4.2. *The main five quantum groups and the intersections between them, are*

$$\begin{array}{ccccc} O_N & \longrightarrow & O_N^* & \longrightarrow & O_N^+ \\ \uparrow & & \uparrow & & \uparrow \\ H_N & \longrightarrow & H_N^* & \longrightarrow & \bar{O}_N^* \\ \uparrow & & \uparrow & & \uparrow \\ H_N & \longrightarrow & H_N & \longrightarrow & \bar{O}_N \end{array}$$

at $N \geq 3$. At $N = 2$ the same holds, with the lower left square being $\left[\begin{smallmatrix} O_2 & O_2^+ \\ H_2 & \bar{O}_2 \end{smallmatrix}\right]$.

Proof. We study the following four quantum group intersections:

(1) $O_N \cap \bar{O}_N$. Here an element $U \in O_N$ belongs to the intersection when its entries satisfy $ab = 0$ for any $a \neq b$ on the same row or column of U. But this means that our matrix $U \in O_N$ must be monomial and so we get $U \in H_N$, as claimed.

(2) $O_N \cap \bar{O}_N^*$. At $N = 2$ the defining relations for $\bar{O}_N^*$ disappear and so we have $O_2 \cap \bar{O}_2^* = O_2 \cap O_2^+ = O_2$, as claimed. At $N \geq 3$ now, the inclusion $H_N \subset O_N \cap \bar{O}_N^*$ is clear. In order to prove the converse inclusion, pick $U \in O_N$ in the intersection and assume that U is not monomial. Permuting the entries, we can further assume $U_{11} \neq 0$ and $U_{12} \neq 0$. From $U_{11}U_{12}U_{i3} = 0$ for any i we deduce that the third column of U is filled with 0 entries, a contradiction. Thus we must have $U \in H_N$, as claimed.

(3) $O_N^* \cap \bar{O}_N$. At $N = 2$ we have $O_2^* \cap \bar{O}_2 = O_2^+ \cap \bar{O}_2 = \bar{O}_2$, as claimed. At $N \geq 3$ now, best is to use the result in (4) below. Indeed, knowing $O_N^* \cap \bar{O}_N^* = H_N^*$, our intersection is then $G = H_N^* \cap \bar{O}_N$. Now since the standard coordinates on H_N^* satisfy $ab = 0$ for $a \neq b$ on the same row or column of u, the commutation/anticommutation relations defining $\bar{O}_N$ reduce to plain commutation relations. Thus G follows to be classical, $G \subset O_N$. Using (1), we obtain $G = H_N^* \cap \bar{O}_N \cap O_N = H_N^* \cap H_N = H_N$, as claimed.

(4) $O_N^* \cap \bar{O}_N^*$. The result here is nontrivial and we must use technology from [11]. The quantum group $H_N^\times = O_N^* \cap \bar{O}_N^*$ is indeed half-classical in the sense of [11], and since we have $H_N^* \subset H_N^\times$, this quantum group is not classical. Thus the main result in [11] applies and shows that $H_N^\times \subset O_N^*$ must come, via the crossed product construction there, from an intermediate compact group $\mathbb{T} \subset G \subset U_N$. Now observe that the standard coordinates on $H_N^\times$ are by definition subject to the conditions $abc = 0$ when $(r, s) = (\leq 2, 3), (3, \leq 2)$, with the conventions in Definition 4.1. It follows that the standard coordinates on G are subject to the conditions $\alpha\beta\gamma = 0$ when $(r, s) = (\leq 2, 3), (3, \leq 2)$, where $r, s = \text{span}(a, b, c)$ as in Definition 4.1 and $\alpha = a, a^*$, $\beta = b, b^*$, $\gamma = c, c^*$. Thus we have $G \subset \bar{U}_N^{**}$, where $\bar{U}_N^{**}$ is the twisted half-liberated version of U_N constructed in [1].

We deduce from this that we have $G \subset K_N^\circ$, where $K_N^\circ = U_N \cap \bar{U}_N^{**}$. But this intersection can be computed exactly as in the real case, in the proof of (2), and we obtain $K_2^\circ = U_2$ and $K_N^\circ = \mathbb{T} \wr S_N$ at $N \geq 3$. But the half-liberated quantum groups obtained from U_2 and $\mathbb{T} \wr S_N$ via the construction in [11] are well known, these being $O_2^* = O_2^+$ and H_N^*. Thus by functoriality we have $H_2^\times \subset O_2^+$ and $H_N^\times \subset H_N^*$ at $N \geq 3$, and since the reverse inclusions are clear, we obtain $H_2^\times = O_2^+$ and $H_N^\times = H_N^*$ at $N \geq 3$, as claimed. □

Observe that the diagram in Proposition 4.2 is not exactly the quantum isometry group diagram from the introduction. In order to evolve now towards that diagram, we must first introduce a new quantum group, $H_N^{[\infty]}$. This quantum group was constructed in [3] and its main properties, worked out in [3, 19, 20], can be summarized as follows.

Proposition 4.3. *Let $H_N^{[\infty]} \subset O_N^+$ be the compact quantum group obtained via the relations $abc = 0$, whenever $a \neq c$ are on the same row or column of u.*

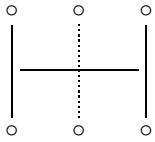
(1) *We have inclusions $H_N^* \subset H_N^{[\infty]} \subset H_N^+$.*

(2) *We have $ab_1 \ldots b_r c = 0$, whenever $a \neq c$ are on the same row or column of u.*

(3) *We have $ab^2 = b^2 a$ for any two entries a, b of u.*

Proof. We briefly recall the proof from [3, 19, 20]. Our first claim is that $H_N^{[\infty]}$ comes, as an easy quantum group, from the following diagram:

π =



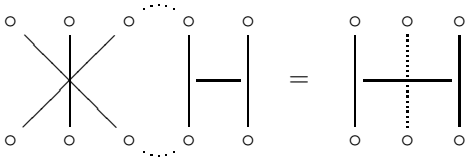
Indeed, since this diagram acts via the map $T_\pi(e_{ijk}) = \delta_{ik} e_{ijk}$, we obtain

$$T_\pi u^{\otimes 3} e_{abc} = T_\pi \sum_{i,j,k} e_{ijk} \otimes u_{ia} u_{jb} u_{kc} = \sum_{i,j,k} e_{ijk} \otimes \delta_{ik} u_{ia} u_{jb} u_{kc},$$

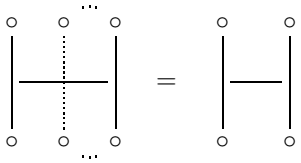
$$u^{\otimes 3} T_\pi e_{abc} = u^{\otimes 3} \delta_{ac} e_{abc} = \sum_{i,j,k} e_{ijk} \otimes \delta_{ac} u_{ia} u_{jb} u_{kc}.$$

Thus $T_\pi \in \mathrm{End}(u^{\otimes 3})$ is equivalent to $(\delta_{ik} - \delta_{ac}) u_{ia} u_{jb} u_{kc} = 0$. The nontrivial cases are $i = k$, $a \neq c$ and $i \neq k$, $a = c$. They produce the relations $u_{ia} u_{jb} u_{ic} = 0$ for any $a \neq c$ and $u_{ia} u_{jb} u_{ka} = 0$ for any $i \neq k$. Thus, we have reached to the relations for $H_N^{[\infty]}$.

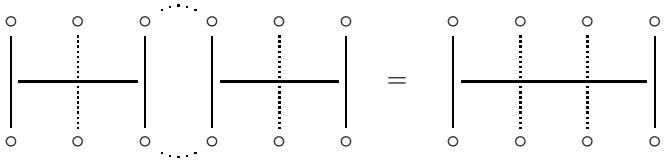
(1) The fact that we have inclusions $H_N^* \subset H_N^{[\infty]} \subset H_N^+$ comes from



and

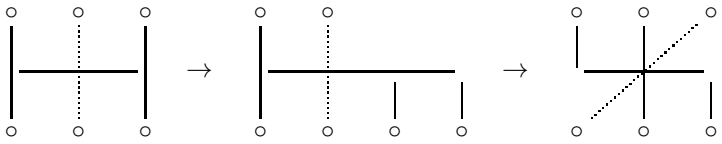


(2) At $r = 2$, the relations $ab_1 b_2 c = 0$ come indeed from the following diagram:



In the general case $r \geq 2$ the proof is similar, see [3] for details.

(3) We use here an idea from [19, 20]. Rotating π, we obtain



Let us denote by σ the partition on the right. Since $T_\sigma(e_{ijk}) = \delta_{ij}e_{kji}$, we obtain

$$T_\sigma u^{\otimes 3}e_{abc} = T_\sigma \sum_{i,j,k} e_{ijk} \otimes u_{ia}u_{jb}u_{kc} = \sum_{i,j,k} e_{kji} \otimes \delta_{ij}u_{ia}u_{jb}u_{kc},$$

$$u^{\otimes 3}T_\sigma e_{abc} = u^{\otimes 3}\delta_{ab}e_{cba} = \sum_{i,j,k} e_{kji} \otimes \delta_{ab}u_{kc}u_{jb}u_{ia}.$$

Thus $T_\sigma \in \mathrm{End}(u^{\otimes 3})$ is equivalent to $\delta_{ij}u_{ia}u_{jb}u_{kc} = \delta_{ab}u_{kc}u_{jb}u_{ia}$. Setting $j = i$ and $b = a$, we obtain the commutation relation $u_{ia}^2u_{kc} = u_{kc}u_{ia}^2$ in the statement. □

The relation of $H_N^{[\infty]}$ with the polygonal spheres comes from the following fact.

Proposition 4.4. *Let $X \subset S_{\mathbb{R},+}^{N-1}$ be closed, $d \geq 2$, and $X^{d-1} = X \cap S_{\mathbb{R},+}^{N-1,d-1}$. Then for a quantum group $G \subset H_N^{[\infty]}$ the following statements are equivalent:*

(1) *$x_i \to \sum_j u_{ij} \otimes x_j$ defines a coaction $\Phi : C(X^{d-1}) \to C(G) \otimes C(X^{d-1})$.*

(2) *$x_i \to \sum_j u_{ij} \otimes x_j$ defines a morphism $\widetilde{\Phi} : C(X) \to C(G) \otimes C(X^{d-1})$.*

In particular, $G^+(X) \cap H_N^{[\infty]}$ acts on X^{d-1} for any $d \geq 2$.

Proof. The idea here is to use the relations in Proposition 4.3 (2):

(1) $\Rightarrow$ (2) This implication follows by composing Φ with the projection map $C(X) \to C(X^{d-1})$.

(2) $\Rightarrow$ (1) In order for a coaction $C(X^{d-1}) \to C(G) \otimes C(X^{d-1})$ to exist, the variables $X_i = \sum_j u_{ij} \otimes x_j$ must satisfy the relations defining X, which hold indeed by (2), and also the relations $X_{i_0} \ldots X_{i_d} = 0$ for $i_0, \ldots, i_d$ distinct, which define $S_{\mathbb{R},+}^{N-1,d-1}$.

The point now is that, under the assumption $G \subset H_N^{[\infty]}$, these latter relations are automatic. Indeed, using Proposition 4.3 (2), we obtain, for $i_0, \ldots, i_d$ distinct,

$$\begin{aligned} X_{i_0} \ldots X_{i_d} &= \sum_{j_0 \ldots j_d} u_{i_0 j_0} \ldots u_{i_d j_d} \otimes x_{j_0} \ldots x_{j_d} \\ &= \sum_{j_0 \ldots j_d \text{ distinct}} u_{i_0 j_0} \ldots u_{i_d j_d} \otimes 0 + \sum_{j_0 \ldots j_d \text{ not distinct}} 0 \otimes x_{j_0} \ldots x_{j_d} \\ &= 0 + 0 = 0. \end{aligned}$$

Thus the coaction in (1) exists precisely when (2) is satisfied, and we are done.

Finally, the last assertion is clear from (2) $\Rightarrow$ (1), because the universal coaction of $G = G^+(X)$ gives rise to a map as in (2):

$$\widetilde{\Phi} : C(X) \to C(G) \otimes C(X^{d-1}). \qquad \square$$

As an illustration, we have the following result.

Theorem 4.5. *$H_N, H_N, H_N^*, H_N^*, H_N^{[\infty]}$ act respectively on the spheres*

$$S_{\mathbb{R}}^{N-1,d-1}, \quad \bar{S}_{\mathbb{R}}^{N-1,d-1}, \quad S_{\mathbb{R},*}^{N-1,d-1}, \quad \bar{S}_{\mathbb{R},*}^{N-1,d-1}, \quad S_{\mathbb{R},+}^{N-1,d-1}$$

at any $d \geq 2$.

Proof. We use Proposition 4.4. We know from [1] that the quantum isometry groups at $d = N$ are respectively equal to $O_N, \bar{O}_N, O_N^*, \bar{O}_N^*, O_N^+$ and our claim is that, by intersecting with $H_N^{[\infty]}$, we obtain the quantum groups in the statement. Indeed:

(1) $O_N \cap H_N^{[\infty]} = H_N$ is clear from definitions.

(2) $\bar{O}_N \cap H_N^{[\infty]} = H_N$ follows from $\bar{O}_N \cap H_N^+ \subset O_N$, which in turn follows from the computation (3) in the proof of Proposition 4.2, with H_N^* replaced by H_N^+.

(3) $O_N^* \cap H_N^{[\infty]} = H_N^*$ follows from $O_N^* \cap H_N^+ = H_N^*$.

(4) $\bar{O}_N^* \cap H_N^{[\infty]} \supset H_N^*$ is clear and the reverse inclusion can be proved by a direct computation, similar to the computation (3) in the proof of Proposition 4.2.

(5) $O_N^+ \cap H_N^{[\infty]} = H_N^{[\infty]}$ is clear from definitions. $\square$

Observe that Theorem 4.5 is sharp, in the sense that the actions there are the universal ones, in the classical case at any $d \in \{2, \ldots, N-1\}$, as well as in the twisted case at $d = 2$. Indeed, this follows from Proposition 3.3 and Theorem 3.5.

5. Quantum isometries

In this section we complete the computation of the quantum isometry groups of the nine main spheres, as to prove our main result, announced in the introduction. As already pointed out, we already have results for eight spheres, the sphere left being $S_{\mathbb{R},*}^{N-1,1}$.

We already know from Theorem 4.5 that the quantum group H_N^* from [3] acts on $S_{\mathbb{R},*}^{N-1,1}$. This action, however, is not universal, because of the following result.

Proposition 5.1. *$\widehat{\mathbb{Z}_2^{*N}}$ acts on $S_{\mathbb{R},*}^{N-1,1}$.*

Proof. The standard coordinates on $S_{\mathbb{R},*}^{N-1,1}$ are subject to the relations

$$x_i x_j x_k = \begin{cases} 0 & \text{for } i,j,k \text{ distinct}, \\ x_k x_j x_i & \text{otherwise}. \end{cases}$$

Thus, in order to have a coaction map $\Phi : C(S^{N-1,1}_{\mathbb{R},*}) \to C(G) \otimes C(S^{N-1,1}_{\mathbb{R},*})$, given by $\Phi(x_i) = \sum_j u_{ij} \otimes x_j$, the variables $X_i = \sum_j u_{ij} \otimes x_j$ must satisfy the above relations.

For the group dual $G = \widehat{\mathbb{Z}_2^{*N}}$ we have by definition $u_{ij} = \delta_{ij} g_i$, where $g_1, \ldots, g_N$ are the standard generators of $\mathbb{Z}_2^{*N}$. We therefore have

$$X_i X_j X_k = g_i g_j g_k \otimes x_i x_j x_k, \quad X_k X_j X_i = g_k g_j g_i \otimes x_k x_j x_i.$$

Thus the formula $X_i X_k X_k = 0$ for i, j, k distinct is clear and the formula $X_i X_j X_k = X_k X_j X_i$ for i, j, k not distinct requires $g_i g_j g_k = g_k g_j g_i$ for i, j, k not distinct, which is clear as well. Indeed, at $i = j$ this latter relation reduces to $g_k = g_k$, at $i = k$ this relation is trivial, $g_i g_j g_i = g_i g_j g_i$, and at $j = k$ this relation reduces to $g_i = g_i$. □

More generally, we have the following result.

Proposition 5.2. *$H_N^{[\infty]}$ acts on $S^{N-1,1}_{\mathbb{R},*}$.*

Proof. We proceed as in the proof of Theorem 3.5. Expanding the formula of $X_i X_j X_k$ and using the relations for the sphere $S^{N-1,1}_{\mathbb{R},*}$, we have

$$\begin{aligned} X_i X_j X_k &= \sum_{a,b,c} u_{ia} u_{jb} u_{kc} \otimes x_a x_b x_c \\ &= \sum_{a,b,c \text{ not distinct}} u_{ia} u_{jb} u_{kc} \otimes x_a x_b x_c \\ &= \sum_{a \neq b} (u_{ia} u_{ja} u_{kb} + u_{ib} u_{ja} u_{ka}) \otimes x_a^2 x_b \\ &\quad + \sum_{a \neq b} u_{ia} u_{jb} u_{ka} \otimes x_a x_b x_a + \sum_a u_{ia} u_{ja} u_{ka} \otimes x_a^3. \end{aligned}$$

Now assuming $G = H_N^{[\infty]}$ and using the various formulae in Proposition 4.3, we obtain, for any i, j, k distinct,

$$\begin{aligned} X_i X_j X_k &= \sum_{a \neq b} (0 \cdot u_{kb} + u_{ib} \cdot 0) \otimes x_a^2 x_b \\ &\quad + \sum_{a \neq b} 0 \otimes x_a x_b x_a + \sum_a (0 \cdot u_{ka}) \otimes x_a^3 = 0. \end{aligned}$$

It remains to prove that we have $X_i X_j X_k = X_k X_j X_i$ for i, j, k not distinct. Replacing $i \leftrightarrow k$ in the above formula of $X_i X_j X_k$, we obtain

$$\begin{aligned} X_k X_j X_i &= \sum_{a \neq b} (u_{ka} u_{ja} u_{ib} + u_{kb} u_{ja} u_{ia}) \otimes x_a^2 x_b \\ &\quad + \sum_{a \neq b} u_{ka} u_{jb} u_{ia} \otimes x_a x_b x_a + \sum_a u_{ka} u_{ja} u_{ia} \otimes x_a^3. \end{aligned}$$

Let us compare this formula with the above formula of $X_i X_j X_k$. The last sum being 0 in both cases, we must prove that for any i, j, k not distinct and any

$a \neq b$ we have

$$u_{ia}u_{ja}u_{kb} + u_{ib}u_{ja}u_{ka} = u_{ka}u_{ja}u_{ib} + u_{kb}u_{ja}u_{ia},$$
$$u_{ia}u_{jb}u_{ka} = u_{ka}u_{jb}u_{ia}.$$

By symmetry the three cases $i = j$, $i = k$, $j = k$ reduce to two cases, $i = j$ and $i = k$. The case $i = k$ being clear, we are left with the case $i = j$, where we must prove

$$u_{ia}u_{ia}u_{kb} + u_{ib}u_{ia}u_{ka} = u_{ka}u_{ia}u_{ib} + u_{kb}u_{ia}u_{ia},$$
$$u_{ia}u_{ib}u_{ka} = u_{ka}u_{ib}u_{ia}.$$

By using $a \neq b$, the first equality reads

$$u_{ia}^2u_{kb} + 0 \cdot u_{ka} = u_{ka} \cdot 0 + u_{kb}u_{ia}^2.$$

Since by Proposition 4.3 (3) we have $u_{ia}^2u_{kb} = u_{kb}u_{ia}^2$, we are done. As for the second equality, this reads $0 \cdot u_{ka} = u_{ka} \cdot 0$, which is true as well. This ends the proof. □

We will prove now that the action in Proposition 5.2 is universal. In order to do so, we need to convert the formulae of type $X_iX_jX_k = 0$ and $X_iX_jX_k = X_kX_jX_i$ into relations between the quantum group coordinates u_{ij} and this requires a good knowledge of the linear relations between the variables $x_a^2x_b$, $x_ax_bx_a$, x_a^3 over the sphere $S_{\mathbb{R},*}^{N-1,1}$.

We must first study these variables. The answer is given by the next lemma.

Lemma 5.3. *The variables* $\{x_a^2x_b, x_ax_bx_a, x_a^3 \mid a \neq b\}$ *are linearly independent over the sphere* $S_{\mathbb{R},*}^{N-1,1}$.

Proof. We use a trick from [11]. Consider the 1-dimensional polygonal version of the complex sphere $S_{\mathbb{C}}^{N-1}$, which is by definition given by

$$S_{\mathbb{C}}^{N-1,1} = \big\{z \in S_{\mathbb{C}}^{N-1} \mid z_iz_jz_k = 0 \text{ for all } i, j, k \text{ distinct}\big\}.$$

We have then a 2×2 matrix model for the coordinates of $S_{\mathbb{R},*}^{N-1,1}$, as follows:

$$x_i \to \gamma_i = \left(\begin{smallmatrix} 0 & z_i \\ \bar{z}_i & 0 \end{smallmatrix}\right).$$

Indeed, the matrices γ_i on the right are all selfadjoint, their squares sum up to 1, they half-commute and they satisfy $\gamma_i\gamma_j\gamma_k = 0$ for i, j, k distinct. Thus we have indeed a morphism $C(S_{\mathbb{R},*}^{N-1,1}) \to M_2(C(S_{\mathbb{C}}^{N-1,1}))$ mapping $x_i \to \gamma_i$, as claimed.

We can use this model in order to prove the linear independence. Indeed, the variables $x_a^2x_b$, $x_ax_bx_a$, x_a^3 that we are interested in are mapped to the following variables:

$$\gamma_a^2\gamma_b = \begin{pmatrix} 0 & |z_a|^2z_b \\ |z_a|^2\bar{z}_b & 0 \end{pmatrix}, \quad \gamma_a\gamma_b\gamma_a = \begin{pmatrix} 0 & z_a^2\bar{z}_b \\ \bar{z}_a^2z_b & 0 \end{pmatrix}, \quad \gamma_a^3 = \begin{pmatrix} 0 & |z_a|^2z_a \\ |z_a|^2\bar{z}_a & 0 \end{pmatrix}.$$

Now since $|z_1|^2z_2$, $|z_2|^2z_1$, $z_1^2\bar{z}_2$, $z_2^2\bar{z}_1$, $|z_1|^2z_1$, $|z_2|^2z_2$ are linearly independent over $S_{\mathbb{C}}^1$, the upper right entries of the above matrices are linearly independent over $S_{\mathbb{C}}^{N-1,1}$. Thus the matrices themselves are linearly independent, which proves our result. □

With the above lemma in hand, we can now reformulate the coaction problem into a purely quantum group-theoretical problem, as follows:

Lemma 5.4. *A quantum group $G \subset O_N^+$ acts on $S^{N-1,1}_{\mathbb{R},*}$ precisely when its standard coordinates u_{ij} satisfy the following relations:*

(1) $u_{ia}u_{ja}u_{kb} + u_{ib}u_{ja}u_{ka} = 0$ *for any i, j, k distinct.*
(2) $u_{ia}u_{jb}u_{ka} = 0$ *for any i, j, k distinct.*
(3) $u_{ia}^2 u_{kb} = u_{kb}u_{ia}^2$.
(4) $u_{ka}u_{ia}u_{ib} = u_{ib}u_{ia}u_{ka}$.
(5) $u_{ia}u_{ib}u_{ka} = u_{kb}u_{ib}u_{ia}$.

Proof. We use notations from the beginning of the proof of Proposition 5.2, along with the following formula, also established there:

$$X_iX_jX_k = \sum_{a\neq b}(u_{ia}u_{ja}u_{kb} + u_{ib}u_{ja}u_{ka}) \otimes x_a^2x_b + \sum_{a\neq b} u_{ia}u_{jb}u_{ka} \otimes x_ax_bx_a + \sum_a u_{ia}u_{ja}u_{ka} \otimes x_a^3.$$

In order to have an action as in the statement, these quantities must satisfy $X_iX_kX_k = 0$ for i, j, k distinct and $X_iX_kX_k = X_kX_jX_i$ for i, j, k not distinct. Now using Lemma 5.3, we conclude that the relations to be satisfied are as follows:

(A) For i, j, k distinct, the following must hold:

$$\begin{aligned} u_{ia}u_{ja}u_{kb} + u_{ib}u_{ja}u_{ka} &= 0 \quad \text{for all } a \neq b,\\ u_{ia}u_{jb}u_{ka} &= 0 \quad \text{for all } a \neq b,\\ u_{ia}u_{ja}u_{ka} &= 0 \quad \text{for all } a. \end{aligned}$$

(B) For i, j, k not distinct, the following must hold:

$$\begin{aligned} u_{ia}u_{ja}u_{kb} + u_{ib}u_{ja}u_{ka} &= u_{ka}u_{ja}u_{ib} + u_{kb}u_{ja}u_{ia} \quad \text{for all } a \neq b,\\ u_{ia}u_{jb}u_{ka} &= u_{ka}u_{jb}u_{ia} \quad \text{for all } a \neq b,\\ u_{ia}u_{ja}u_{ka} &= u_{ka}u_{ja}u_{ia} \quad \text{for all } a. \end{aligned}$$

In order to simplify this set of relations, the first observation is that the last relations in both (A) and (B) can be merged with the other ones, and we are led to:

(A′) For i, j, k distinct, the following must hold:

$$\begin{aligned} u_{ia}u_{ja}u_{kb} + u_{ib}u_{ja}u_{ka} &= 0 \quad \text{for all } a, b,\\ u_{ia}u_{jb}u_{ka} &= 0 \quad \text{for all } a, b. \end{aligned}$$

(B′) For i, j, k not distinct, the following must hold:

$$\begin{aligned} u_{ia}u_{ja}u_{kb} + u_{ib}u_{ja}u_{ka} &= u_{ka}u_{ja}u_{ib} + u_{kb}u_{ja}u_{ia} \quad \text{for all } a, b,\\ u_{ia}u_{jb}u_{ka} &= u_{ka}u_{jb}u_{ia} \quad \text{for all } a, b. \end{aligned}$$

Observe that the relations (A′) are exactly the relations (1) and (2) in the statement.

Let us further process the relations (B′). In the case $i=k$ the relations are automatic, and in the cases $j=i$, $j=k$ the relations that we obtain coincide, via $i\leftrightarrow k$. Thus (B′) reduces to the set of relations obtained by setting $j=i$, which are as follows:

$$u_{ia}u_{ia}u_{kb}+u_{ib}u_{ia}u_{ka}=u_{ka}u_{ia}u_{ib}+u_{kb}u_{ia}u_{ia},$$
$$u_{ia}u_{ib}u_{ka}=u_{ka}u_{ib}u_{ia}.$$

Observe that the second relation is relation (5) in the statement. Regarding now the first relation, with the notation $[x,y,z]=xyz-zyx$, this is as follows:

$$[u_{ia},u_{ia},u_{kb}]=[u_{ka},u_{ia},u_{ib}].$$

Applying the antipode, we obtain $[u_{bk},u_{ai},u_{ai}]=[u_{bi},u_{ai},u_{ak}]$. Then relabelling $a\leftrightarrow i$ and $b\leftrightarrow k$, this relation becomes $[u_{kb},u_{ia},u_{ia}]=[u_{ka},u_{ia},u_{ib}]$. Now since we have $[x,y,z]=-[z,y,x]$, by comparing this latter relation with the original one, a simplification occurs and the resulting relations are

$$[u_{ia},u_{ia},u_{kb}]=[u_{ka},u_{ia},u_{ib}]=0.$$

But these are exactly the relations (3) and (4) in the statement. □

Now solving the quantum group problem raised by Lemma 5.4, we obtain:

Proposition 5.5. *We have* $G^+(S^{N-1,1}_{\mathbb{R},*})=H_N^{[\infty]}$.

Proof. The inclusion $\supset$ is clear from Proposition 5.2. For the converse, we already have the result at $N=2$, so we assume $N\geq 3$. Using Lemma 5.4 (2), for $i\neq j$ we have

$$\begin{aligned}
&u_{ia}u_{jb}u_{ka}=0 \quad \text{for all } k\neq i,j\\
\implies\quad &u_{ia}u_{jb}u_{ka}^2=0 \quad \text{for all } k\neq i,j\\
\implies\quad &u_{ia}u_{jb}\Big(\sum_{k\neq i,j}u_{ka}^2\Big)=0 \quad \text{for all } i\neq j\\
\implies\quad &u_{ia}u_{jb}(1-u_{ia}^2-u_{ja}^2)=0 \quad \text{for all } i\neq j.
\end{aligned}$$

Now using Lemma 5.4 (3), we can move the variable u_{jb} to the right. Multiplying by u_{jb} from the right and then summing over b, we obtain

$$\begin{aligned}
&u_{ia}u_{jb}(1-u_{ia}^2-u_{ja}^2)=0 \quad \text{for all } i\neq j\\
\implies\quad &u_{ia}(1-u_{ia}^2-u_{ja}^2)u_{jb}=0 \quad \text{for all } i\neq j\\
\implies\quad &u_{ia}(1-u_{ia}^2-u_{ja}^2)u_{jb}^2=0 \quad \text{for all } i\neq j\\
\implies\quad &u_{ia}(1-u_{ia}^2-u_{ja}^2)=0 \quad \text{for all } i\neq j.
\end{aligned}$$

We can proceed now as follows, by summing over $j \neq i$:

$$\begin{aligned}
& u_{ia}(1-u_{ia}^2-u_{ja}^2)=0 && \text{for all } i\neq j\\
\implies\quad & u_{ia}u_{ja}^2=u_{ia}-u_{ia}^3 && \text{for all } i\neq j\\
\implies\quad & u_{ia}(1-u_{ia}^2)=(N-1)(u_{ia}-u_{ia}^3)\\
\implies\quad & u_{ia}=u_{ia}^3.
\end{aligned}$$

Thus the standard coordinates are partial isometries and so $G\subset H_N^+$. On the other hand, we know from the proof of Proposition 4.3 (3) that the quantum subgroup $G\subset H_N^+$ obtained via the relations $[a,b^2]=0$ is $H_N^{[\infty]}$. This finishes the proof. □

We have now complete results for the nine main spheres, as follows:

Theorem 5.6. *The quantum isometry groups of the nine main spheres are*

$$\begin{array}{ccccc}
O_N & \longrightarrow & O_N^* & \longrightarrow & O_N^+\\
\uparrow & & \uparrow & & \uparrow\\
H_N & \longrightarrow & H_N^{[\infty]} & \longrightarrow & \bar{O}_N^*\\
\uparrow & & \uparrow & & \uparrow\\
H_N^+ & \longrightarrow & H_N & \longrightarrow & \bar{O}_N,
\end{array}$$

where $H_N^+, H_N^{[\infty]}, \bar{O}_N, O_N^*, \bar{O}_N^*, O_N^*$ *are noncommutative versions of* H_N, O_N.

Proof. This follows indeed from [1, 4] and from the above results. □

As a first comment, in view of the conjectures in Section 2, Theorem 5.6 probably deals with the general mixed monomial case. We do not know if it is so.

In general, there are of course many questions left. Perhaps the very first question here regards $S_{\mathbb{R},+}^{N-1,1}$, whose quantum isometry group should be probably $H_N^{[\infty]}$. Technically speaking, the problem is that we have no good models for $S_{\mathbb{R},+}^{N-1,1}$ and hence no tools for dealing with independence questions for products of coordinates over it.

We should recall, however, that $S_{\mathbb{R},+}^{N-1,1}$ is a bit of a "pathological" sphere. Besides various issues with diagrams and axiomatization, coming from Sections 1 and 2, one problem is that the operation $S_{\mathbb{R}}^{N-1,1}\to S_{\mathbb{R},+}^{N-1,1}$ is not exactly a "liberation" in the sense of free probability theory [7, 22]. More precisely, as explained in [4], the operation $S_{\mathbb{R}}^{N-1}\to S_{\mathbb{R},+}^{N-1}$ is compatible with the Bercovici–Pata bijection [7], at the level of the corresponding hyperspherical laws, but this seems to fail for $S_{\mathbb{R}}^{N-1,1}\to S_{\mathbb{R},+}^{N-1,1}$.

Summarizing, if all our conjectures and guesses hold true, Theorem 5.6 above might be indeed the "final" statement regarding the quantum isometries of polygonal spheres. Note however that the Riemannian interpretation of our

various computations, in the smooth case, in the spirit of the constructions in [4], remains an open problem.

6. Complexification issues

In this section we discuss a straight-forward complex extension of the above results. Our starting point will be the following definition, from [1].

Definition 6.1. We consider the universal C^*-algebra

$$C(S^{N-1}_{\mathbb{C},+}) = C^*\Big(z_1, \ldots, z_N \mid \sum_i z_i z_i^* = \sum_i z_i^* z_i = 1\Big)$$

and call the underlying space $S^{N-1}_{\mathbb{C},+}$ free complex sphere.

As a first observation, the relation between the real and complex spheres is quite unobvious in the free case. Recall indeed that in the classical case we have an isomorphism $S^{N-1}_{\mathbb{C}} \simeq S^{2N-1}_{\mathbb{R}}$, obtained by setting $z_i = x_i + iy_i$. In the free case no such isomorphism is available and in fact both inclusions $S^{N-1}_{\mathbb{C},+} \subset S^{2N-1}_{\mathbb{R},+}$ and $S^{2N-1}_{\mathbb{R},+} \subset S^{N-1}_{\mathbb{C},+}$ fail to hold. This is due to the formula

$$(x + iy)(x - iy) = (x^2 + y^2) - i[x, y],$$

which makes appear the commutator $[x, y]$, which has no reasons to vanish for the free spherical coordinates.

We can define quantum isometry groups, in a complex sense, as follows.

Definition 6.2. Consider an algebraic manifold $Z \subset S^{N-1}_{\mathbb{C},+}$, assumed to be non-degenerate, in the sense that the variables $\mathrm{Re}(z_i), \mathrm{Im}(z_i) \in C(Z)$ are linearly independent.

(1) We let $G^+(Z) \subset U_N^+$ be the biggest quantum subgroup acting affinely on Z.
(2) We also set $G(Z) = G^+(Z) \cap U_N$, with the intersection taken inside U_N^+.

Here U_N^+ is the free analog of U_N, constructed by Wang in [23]. Existence and uniqueness of $G^+(Z)$ follow as in the proof of Proposition 3.1.

In the classical case, where $Z \subset S^{N-1}_{\mathbb{C}}$, we can use the isomorphism $S^{N-1}_{\mathbb{C}} \simeq S^{2N-1}_{\mathbb{R}}$ in order to view Z as a real manifold, $Z_r \subset S^{2N-1}_{\mathbb{R}}$. We can therefore construct a "real" quantum isometry group $G^+(Z_r) \subset O_{2N}^+$ and have $G^+(Z) = G^+(Z_r) \cap U_N^+$, where the intersection is taken inside O_{2N}^+, by using the embedding $U_N^+ \subset O_{2N}^+$ given by the fact that for $u = v + iw$ biunitary, the matrix $\left(\begin{smallmatrix} v & w \\ w & v \end{smallmatrix}\right)$ is orthogonal; see [11].

As an example here, consider the torus $\mathbb{T} \subset \mathbb{C}$. A straight-forward complex extension of the trick in Proposition 3.2 (2), explained in [1], shows that we have $G^+(\mathbb{T}) = G(\mathbb{T}) = U_1$. We should mention that it is true as well that we have $G^+(\mathbb{T}_r) = G(\mathbb{T}_r) = O_2$, therefore confirming the formula $G^+(\mathbb{T}) = G^+(\mathbb{T}_r) \cap U_1^+$, but this result holds due to much deeper reasons, explained by Bhowmick in [8]. For more on these issues, see also [16].

As explained in [1], the five real spheres have five complex analogs. We can extend this analogy to the level of polygonal spheres, as follows:

Definition 6.3. $S_{\mathbb{C}}^{N-1,d-1}$, $\bar{S}_{\mathbb{C}}^{N-1,d-1}$, $S_{\mathbb{C},**}^{N-1,d-1}$, $\bar{S}_{\mathbb{C},**}^{N-1,d-1}$, $S_{\mathbb{C},+}^{N-1,d-1}$ are constructed from $S_{\mathbb{C},+}^{N-1}$ in the same way as their real counterparts are constructed from $S_{\mathbb{R},+}^{N-1}$, by assuming that the corresponding relations hold between the variables $x_i = z_i, z_i^*$.

Here we use the convention that the subscript $**$ from the complex case corresponds to the subscript $*$ from the real case. For more on this issue, see [1, 11].

As an illustration, in the free case the polygonal spheres are as follows:

$$C(S_{\mathbb{C},+}^{N-1,d-1})$$
$$= C(S_{\mathbb{C},+}^{N-1})\Big/\Big\langle z_{i_0}^{\varepsilon_0}\ldots z_{i_d}^{\varepsilon_d} = 0 \text{ for all } i_0,\ldots,i_d \text{ distinct and all } \varepsilon_r \in \{1,*\}\Big\rangle.$$

As in the real case, we will restrict our attention to the five main spheres coming from [1] and their intersections. We have nine such spheres, which are given in the following proposition.

Proposition 6.4. *The five main spheres and the intersections between them are*

$$\begin{array}{ccccc}
S_{\mathbb{C}}^{N-1} & \longrightarrow & S_{\mathbb{C},**}^{N-1} & \longrightarrow & S_{\mathbb{C},+}^{N-1} \\
\uparrow & & \uparrow & & \uparrow \\
S_{\mathbb{C}}^{N-1,1} & \longrightarrow & S_{\mathbb{C},**}^{N-1,1} & \longrightarrow & \bar{S}_{\mathbb{C},**}^{N-1} \\
\uparrow & & \uparrow & & \uparrow \\
S_{\mathbb{C}}^{N-1,0} & \longrightarrow & \bar{S}_{\mathbb{C}}^{N-1,1} & \longrightarrow & \bar{S}_{\mathbb{C}}^{N-1}
\end{array}$$

with all the maps being inclusions.

Proof. This is similar to the proof of Proposition 1.2, by replacing in all the computations there the variables x_i by the variables $x_i = z_i, z_i^*$. □

As explained in [1], the axiomatization problems in the complex case are quite similar to those in the real case and the same happens in the present polygonal context. Thus, we will not review in detail the material from Sections 1 and 2. Let us mention, however, that there are a few subtleties appearing in the complex case. For instance the saturation notion in Definition 2.2 (1) has a straight-forward complex analog, but it is not clear whether the real and complex saturations of a filtered group $G \subset S_\infty$ coincide. In short, the "noncommutative algebraic geometry" questions discussed in Sections 1 and 2 are expected to be the same over $\mathbb{R}$ and $\mathbb{C}$, but we do not have a proof for this fact.

Let us discuss now the computation of the associated quantum isometry groups, following some previous results from [1] and the material from Sections 3–5.

We use the compact group $K_N = \mathbb{T} \wr S_N$ and its free version $K_N^+ = \mathbb{T} \wr_* S_N^+$, which appear as straight-forward complex analogs of the hyperoctahedral group

$H_N = \mathbb{Z}_2 \wr S_N$ and its free version $H_N^+ = \mathbb{Z}_2 \wr_* S_N^+$, constructed in [2]. Moreover, we define the complex version $K_N^{[\infty]} \subset U_N^+$ of the quantum group $H_N^{[\infty]} \subset O_N^+$ by using the relations $\alpha\beta\gamma = 0$ with $\alpha = a, a^*$, $\beta = b, b^*$, $\gamma = c, c^*$ for any $a \neq c$ on the same row of u.

With these conventions, we have the following result.

Theorem 6.5. *The quantum isometry groups of the nine main complex spheres are*

$$\begin{array}{ccccc} U_N & \longrightarrow & U_N^{**} & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow & & \uparrow \\ K_N & \longrightarrow & K_N^{[\infty]} & \longrightarrow & \bar{U}_N^{**} \\ \uparrow & & \uparrow & & \uparrow \\ K_N^+ & \longrightarrow & K_N & \longrightarrow & \bar{U}_N, \end{array}$$

where K_N and its versions are the complex analogs of H_N and its versions.

Proof. The idea is that the proof here is quite similar to the proof in the real case, by replacing H_N, O_N with their complex analogs K_N, U_N.

More precisely, the results for the five spheres on top and on the right are already known from [1]. Regarding the remaining four spheres, the proof here is as follows:

(1) We have $S_{\mathbb{C}}^{N-1,0} = \mathbb{T}^{\oplus N}$, whose quantum isometry group is indeed K_N^+. This follows as in [2], by adapting the proof from there of $G^+(\mathbb{Z}_2^{\oplus N}) = H_N^+$.

(2) We have a decomposition $S_{\mathbb{C}}^{N-1,1} = \bigcup_{i<j}(S_{\mathbb{C}}^1)^{\{i,j\}}$, which is similar to the one in the real case. The reduction method in the proof of Proposition 3.3 (2) applies and shows that the quantum isometry group is $K_N^+ \cap U_N = K_N$, as claimed.

(3) Regarding $\bar{S}_{\mathbb{C}}^{N-1,1}$, the first part of the proof of Theorem 3.5 extends to the complex case, by using as key ingredient the formula $G^+(\bar{S}_{\mathbb{C}}^{N-1}) = \bar{U}_N$ from [1]. The second part extends as well, by replacing everywhere the variables x_i by the variables $x_i = z_i, z_i^*$, and shows that the quantum isometry group is K_N, as claimed.

(4) Finally, regarding $\bar{S}_{\mathbb{C},**}^{N-1,1}$, all the computations in the proof of Lemma 5.4 and Proposition 5.5 extend to the complex case, by replacing everywhere the variables x_i by the variables $x_i = z_i, z_i^*$, and show that the quantum isometry group is $K_N^{[\infty]}$. □

Regarding the remaining complex polygonal spheres, the situation here is quite similar to the one in the real case. Technically speaking, the problem is that Proposition 3.3, whose complex analog can be shown to fully hold, is quite unobvious to extend.

As a conclusion, at the abstract classification level we have enlarged the set of ten spheres in [1] with eight more spheres, which should be generally

regarded as being not smooth. We should mention that the $10 \to 18$ extension announced in [1], via free complexification, is of course different from the one performed here. The extension via free complexification still remains to be done, but ideally under the present, upgraded formalism.

This adds to the various questions raised throughout the paper.

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