Second-Order Conditions in Optimal Control

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In this lecture, we shall present second order optimality conditions in the following optimal control *problem A*, considered on a variable time interval $[t_0, t_1]$, with initial-final time-state equality and inequality constraints and mixed state-control constraints such that the gradients w.r.t. control of active constraints are *linearly independent*:

$$\mathcal{J} = J(t_0, x(t_0), t_1, x(t_1)) \longrightarrow \min, \quad F(t_0, x(t_0), t_1, x(t_1)) \leq 0, \quad K(t_0, x(t_0), t_1, x(t_1)) = 0,$$
$$\dot{x}(t) = f(t, x(t), u(t)), \quad g(t, x(t), u(t)) = 0, \quad \varphi(t, x(t), u(t)) \leq 0 \text{ for a.a. } t \in (t_0, t_1),$$
$$(t_0, x(t_0), t_1, x(t_1)) \in \mathcal{P}, \ (t, x(t), u(t)) \in \mathcal{Q} \text{ for a.a. } t \in (t_0, t_1).$$

Here $\mathcal{P}$ and $\mathcal{Q}$ are open sets; the functions J, F, and K are twice continuously differentiable on $\mathcal{P}$; the functions f, g, and φ are twice continuously differentiable on $\mathcal{Q}$; the gradients with respect to control $g_{iu}(t, x, u)$, $i = 1, \dots, d(g)$ and $\varphi_{ju}(t, x, u)$, $j \in I_\varphi(t, x, u)$ are jointly linearly independent at each point $(t, x, u) \in \mathcal{Q}$ such that $g(t, x, u) = 0$, $\varphi(t, x, u) \leq 0$, where $I_\varphi(t, x, u) = \{j \in \{1, \dots, d(\varphi)\} \mid \varphi_j(t, x, u) = 0\}$ is the set of active indices; $d(\varphi)$ denotes the dimension of the vector φ. Minimum is sought among trajectories $\mathcal{T} = (x(t), u(t) \mid t \in [t_0, t_1])$ such that the state $x(t)$ is absolutely continuous and the control $u(t)$ is measurable and essentially bounded on $[t_0, t_1]$. We shall formulate no-gap second order conditions of weak, strong and Pontryagin minima in this problem in the cases of continuous and discontinuous optimal control. Sufficient conditions guarantee the growth of the cost function of the definite "order", which is quadratic in the case of continuous optimal control and has more complicated, non-homogeneous structure in the case of discontinuous optimal control. We shall also discuss recent results (obtained jointly with F. Bonnans) on the quadratic growth of the objective function in a special case of problem A under the assumption of *positive linear independence* of gradients w.r.t. control.