The Ukraine Crisis
A Game Theoretic Approach

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Abstract

At the beginning of February 2022 the whole world wondered if Putin would attack Ukraine. In the following we present a simple game theoretic model for this conflict and calculate the Nash equilibrium in order to determine the probability of a Russian attack.

1 Setup

Simplifying one can say that there are two players in the Ukrainian conflict. Putin on one side threatens the western world by a possible military attack to Ukraine. He justifies such an action with Russia's legitimate need of safety caused by the improper Nato extension to East Europe. His opponent are the western democracies represented by the US, EU and Nato. They try to prevent a Russian attack by threatening Russia with economic backlash. For a game theoretic analysis one can give a model where both Putin and Nato have two possible strategies they can play. These are

Attack: Putin will overtake Ukraine by a military invasion.

Fallback: Putin will retreat his army from the Ukrainian frontier.

and for the Nato

Weak: The Nato will acknowledge in some sense Russia's safety needs and give Putin some concessions concerning the neutrality of Ukraine. For example they could offer that de facto Ukraine could not become a Nato member.

Hard: The Nato insists on their view that every country has the right of a Nato membership and this could not be denied in the Ukrainian case.

2 A Game Theoretic Analysis

The above specified two person game will be analysed in the following. To understand the analysis one should be aware that theoretically the game is played in the following way: at an initially known future time point both parties hand over simultaneously an envelope in which they declare their strategy they would act on. This is not that unrealistic as it seems to be. After Putin has finished to position his forces at the Ukrainian frontier he has only a short time window of some days to make his final decision, attack or fallback. Likewise the Nato has only that time window to place finally a negotiation offer.

2.1 Putin as Agressor

From the view of Putin he might have the following payoff matrix in mind that comes from the four principle combinations of actions.

	Attack	*Fallback*
Weak	ca	da
Hard	a	-ba

with $a, b, c, d > 0$ and $c > d$.

The number $a > 0$ quantifies the gain that Putin would receive by attacking the Ukraine while simultaneously the Nato reacts "Hard". This should be positive since otherwise an attack possibility would not make any sense. A value $c > d$ indicates that Putin will act as an aggressor. Even if the Nato is willing to accept some safety needs of Russia Putin values an attack payoff higher than that of a fallback. In this case Putin will attack however the Nato acts. The attack strategy is dominant and yields for Putin in both cases a better payoff than the Fallback action.

2.2 Putin as Doubter

To avoid Putin acting as aggressor the Nato could try the following:

1. Enlarge the attack's price in order to convince Putin that a should become negative. Then Putin would fall back since this strategy would become dominant. This means that the hard strategy would enhanced.

2. Support stronger the weak strategy by openly suggesting what Putin could win if he falls back. Then Putin could get doubts if an attack is reasonable any more and the c would become negative.

It seems to be that the Nato has tried the first alternative and failed. That they did not give the second alternative a strong attempt might be seen as fault. In order to prevent a war the Nato could have offered substantial concessions to Putin openly to give him an incentive for a fallback of his troops. This could have been an offer that Putin should not decline. Such a strategy would have led to a better chance of success although the Nato could be seen as susceptible to extortion.

However, from a game-theoretic view a stronger support of the weak strategy is interesting and analysed in the following.

The payoff matrix would change to

	Attack	*Fallback*
Weak	-ca	da
Hard	a	-ba

with $a, b, c, d > 0$.

The number $a > 0$ quantifies the gain that Putin would receive by attacking the Ukraine while simultaneously the Nato reacts “Hard”.

If Putin attacks on a “Weak” Nato action he would unmask himself as that aggressor that he always negates to be. Due to an international isolation and also a loosing backing by the Russian pupil Putin himself could evaluate this scenario as a loss and the number c denotes the factor the magnitude of loss would exceed that of the gain.

If the Russian army falls back due to a weak Nato reaction Putin can claim this as success and would evaluate this by a gain da where d can be again seen as factor that determines the fallback gain in comparison to that of an attack.

If a fallback is caused due to a hard Nato reaction Putin would be seen as loser in the conflict and this would provide him a severe loss of ba.

To analyse the game one has to clarify the objections both sides would like to achieve. Putin would try to take a strategy with maximal payoff to him. This means that he would attack if $d < 1$ and this is the case if he values the attack's over a fallback's gain. But he has to keep in mind that a weak Nato reaction would convert his supposed gain into a loss and a hard Nato action would convert a supposed fallback gain into a loss either. From a game theoretic point of view we may conclude that there is neither a best action for Putin nor for the Nato. So what strategy should Putin play? A solution to his dilemma could be to randomise his action. This means that he flips a coin with some probability p and writes attack in the envelope if head turns up. The same way the Nato might flip a coin with probability q independent of Putin and writes “weak” in the envelope if head turns up.

The set of all possible randomised actions for both players can be identified with all possible probabilities p for Putin and q for the Nato. The question remains which strategy is the most favourable?

2.2.1 Null-Zero Game

In the null-zero game it is assumed that the gain of one player is the other's loss and vice versa. Since Putin's payoff matrix is fixed, the payoff of the Nato is either.

One approach to find an optimal solution is to propose to play as in the Nash-Equilibrium which will be determined in the following.

If Putin attacks he would calculate the gain from a Nato response by

$$u_1(q) = -ca\,q + (1-q)a.$$

If he falls back he would calculate his expected payoff from the Nato's response by

$$u_2(q) = da\,q - (1-q)ab.$$

If we look at these two straight lines as function of q then there is one unique intersection point

$$q^* = \frac{1+b}{1+b+c+d}$$

and the best the Nato could do is play according to the random strategy q^* since each other strategy would result in a higher expected gain for Putin. In that sense q^* would be the best strategy for the Nato.

But what is the best strategy for Putin? Here we can turn the sides and make an analogous analysis

If the Nato acts weakly then a random strategy by Putin according to a probability p would lead to an expected payoff

$$w_1(p) = -ca\,p + da(1-p).$$

A hard Nato strategy would result in an expected payoff

$$w_2(p) = ap - ba(1-p).$$

Putin does not know which action the Nato performs. But he can be safe to receive the minimum of both responses, i.e.

$$\min\{w_1(p), w_2(p)\}$$

and he will take that action p such that the above minimum takes its maximal value as function of p. This is the case at

$$p^* = \frac{d+b}{1+b+c+d} \tag{1}$$

which is the unique intersection point of both straight lines.

These two values (p^*, q^*) determine the Nash-equilibrium of the game. This means that both players have no incentive to deviate from playing the game according to the strategies that are specified by p^* for Putin and q^* for the Nato. In this sense it is optimal for both players to play the game in the Nash Equilibrium and this means that a probability of an attack is given in this model by the above specified p^*.

Let us have a look at some examples.

	Attack	*Fallback*
Weak	-0.1	0.1
Hard	1	-2

Then

$$p^* = \frac{d+b}{1+b+c+d} = 0.65625$$

If we take a closer look at the formula (1) then we see that the two quantities

$$1+c \quad , \quad b+d$$

determine the attack probability. For the attack action Putin receives a payoff 1 or $-c$ according as the Nato reacts hard or weak. Hence

$$1 + c = 1 - (-c)$$

denotes the payoff's difference for Putin in the case of an attack.

For the fallback action there are the two possible payoffs $-b$ and d and

$$d + b = d - (-b)$$

denotes again the fallback's payoff difference for Putin.

We can conclude that an attack probability becomes small if the payoff difference $1 + c$ for an attack is large compared to the payoff difference for a fallback.

It is reasonable to look at the case $a = 1, b = 1$. Then the payoff matrix becomes to

	Attack	*Fallback*
Weak	-c	1
Hard	1	-d

and the attack probability is given by

$$p^* = \frac{1+d}{1+c}.$$

Hence, if $c = d$ then $p^* = \frac{1}{2}$. Hence if Putin values his both possible losses with the same magnitude an attack probability of $\frac{1}{2}$ follows. The Nato could try to place such a concession to Putin that he values the loss of an attack response to a weak Nato action substantial higher than that of a fallback response to a hard Nato action, i.e. $c >> d$. For example $c = 5$ and $d = 1$ would lead to an attack probability $p^* = \frac{2}{6} = \frac{1}{3}$ and $c = 7$ and $d = 1$ to $p^* = \frac{2}{8} = \frac{1}{4}$.

However, the only way to reduce the attack probability is to convince Putin that a decline of a diplomatic solution would lead to a much higher loss than a fallback from a hard Nato action.

With other words: in order to reduce the attack probability the Nato has to offer something substantially to Putin.

2.2.2 Bimatrix Game

Instead of a zero-sum game where the one player's gain is the others loss and vice versa one can also consider a bimatrix game. Here for both players individual payoff matrices are assumed. This takes the advantage that both players may declare their payoff independently. In the zero-sum game the payoff of one player determines the payoff of its opponent. In the bimatrix case as before we would propose the following payoff matrix for Putin

	Attack	*Fallback*
Weak	-ca	da
Hard	a	-ba

with $a, b, c, d > 0$ and for the Nato

	Attack	*Fallback*
Weak	$-\gamma a$	δa
Hard	$-\alpha a$	βa

with $\alpha, \beta, \delta, \gamma > 0$.

From the Nato's view an attack would cause a loss and this loss would be less if a weak action occurs than a hard one, i.e. $\alpha > \gamma$. A fallback would lead to a gain and this gain would be higher for a hard reaction than a weak reaction, hence $\beta > \delta$.

The expected outcome for Putin, when he chooses attack with probability p and the Nato decides for weak with probability q is

$$u(p, q) = -capq + ap(1-q) + da(1-p)q - ab(1-p)(1-q)$$

and the expected outcome for the Nato is

$$v(p, q) = -\gamma apq - \alpha ap(1-q) + \delta aq(1-p) + \beta a(1-p)(1-q)$$

respectively.

A pair (p^*, q^*) is called Nash-Equilibrium if both players have no incentive to deviate single-sided from their strategies. This means

$$\begin{array}{ll} u(p^*, q^*) \geq u(p, q^*) & \text{for every } \; p \in [0,1] \\ v(p^*, q^*) \geq v(p^*, q) & \text{for every } \; q \in [0,1] \end{array}$$

In the following we will determine the Nash-Equilibrium.

At first we fix a p that could correspond to a Putin action. Then we choose that q for the Nato that maximises its expected outcome. This is a $q(p)$ such that

$$v(p, q(p)) \geq v(p, q) \quad \text{for every } \; q \in [0,1].$$

Hence we have to

$$\max_{q \in [0,1]} q(\delta(1-p) - \gamma p) + (1-q)(\beta(1-p) - \alpha p)$$

and a maximiser is given by

$$\begin{array}{ll} q = 0 & \text{if } \; \beta(1-p) - \alpha p > \delta(1-p) - \gamma p \\ q = 1 & \text{if } \; \beta(1-p) - \alpha p < \delta(1-p) - \gamma p \\ q \in [0,1] & \text{if } \; \beta(1-p) - \alpha p = \delta(1-p) - \gamma p \end{array}$$

Note, that in the last case all Nato strategies would lead to the same expected gain. Hence, if Putin plays according to p such that

$$\beta(1-p) - \alpha p = \delta(1-p) - \gamma p$$

then all Nato strategies would have the same expected Nato payoff. The above equation is solved for

$$p^* = \frac{\beta - \delta}{\alpha - \gamma + \beta - \delta} = \frac{\phi}{1 + \phi} = \frac{1}{1 + \psi}$$

with

$$\phi = \frac{\beta - \delta}{\alpha - \gamma} \quad , \quad \psi = \frac{1}{\phi}.$$

Now we change our side of view. If the Nato chooses a strategy according to the probability q, then Putin would try to take a strategy p that maximises his expected payoff

$$u(p, q) = -capq + ap(1-q) + da(1-p)q - ab(1-p)(1-q) = a\big(p((1-q)-cq) + (1-p)(dq - b(1-q))\big).$$

Hence he

$$\max_{p\in[0,1]} p((1-q) - cq) + (1-p)(dq - b(1-q))$$

and a maximizer is given by

$$\begin{array}{cl} p = 0 & \text{if } dq - b(1-q) > (1-q) - cq \\ p = 1 & \text{if } dq - b(1-q) > (1-q) - cq \\ p \in [0, 1] & \text{if } dq - b(1-q) = (1-q) - cq \end{array}$$

Note, that in the last case all Putin strategies would lead to the same expected gain. Hence, if Nato plays according to q such that

$$dq - b(1-q) = (1-q) - cq$$

then all Putin strategies would have the same expected Putin payoff. The above equation is solved for

$$q^* = \frac{1 + b}{d + c + 1 + b}.$$

With these considerations we have determined a Nash-Equilibrium (p^*, q^*) that corresponds to a specific strategy of Putin and one of the Nato. The meaning is that no player has an incentive to deviate from the equilibrium state since he would only downgrade his expected payoff.

This computed value p^* can be seen as Putin's attack probability. It is interesting to note that this attack probability does not depend on the magnitude of Putin's payoff matrix as long as the principle structure of gains and losses are maintained.

Now we look at specific examples of payoff matrices.

For Putin

	Attack	*Fallback*
Weak	-1	5
Hard	1	-5

and for the Nato

	Attack	*Fallback*
Weak	-1	1
Hard	-2	2

Then

$$(p^*, q^*) = (0.5, 0.5)$$

which corresponds to an equal chance of attack and fallback.

If we increase β hence choose

	Attack	*Fallback*
Weak	-1	1
Hard	-2	4

as Nato's payoff, then

$$(p^*, q^*) = (0.75, 0.5)$$

and the attack probability increases .

If we lower β to

	Attack	*Fallback*
Weak	-1	1
Hard	-2	1.1

then

$$(p^*, q^*) = (0.09, 0.5)$$

and the attack probability decreases .

An increase of α which means a higher deterrence by the Nato to

	Attack	*Fallback*
Weak	-1	1
Hard	-4	2

leads to

$$(p^*, q^*) = (0.25, 0.5)$$

and the attack probability decreases.

If we lower α and therefore the deterrence to

	Attack	*Fallback*
Weak	-1	1
Hard	-1.1	2

then

$$(p^*, q^*) = (0.90, 0.5)$$

and the attack probability increases.

From a Nato point of view a small difference $\beta - \delta$ compared to $\alpha - \gamma$ would reduce an attack probability, hence

$$\frac{\alpha - \gamma}{\beta - \delta} = 4 \quad \Longrightarrow \alpha^* = \frac{1}{5}$$

resp.

$$\frac{\alpha - \gamma}{\beta - \delta} = 9 \quad \Longrightarrow \alpha^* = \frac{1}{10}.$$

The value $\beta - \delta$ can be seen as difference of Nato gains in the case of fallback decision from Putin.

In the same way $\alpha - \gamma$ can be seen as difference of losses in the case of Putin's attack.

Hence, if the difference of losses is substantial higher than the difference of gains an attack probability would become small.