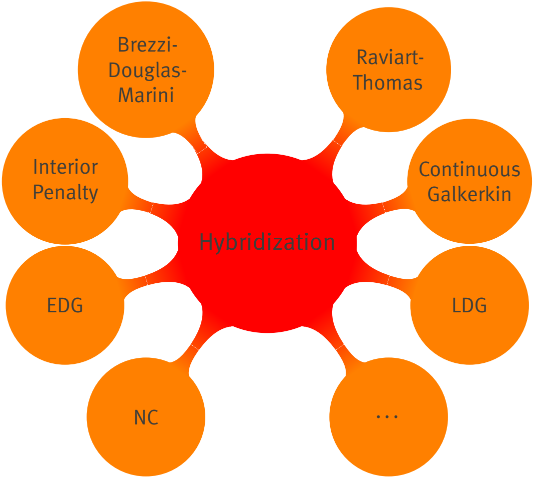
Westfälische
Wilhelms-Universität
Münster

Unified software framework for hybridized discretization methods

Stefan Girke

June 19th, 2012

Motivation [Cockburn2009]



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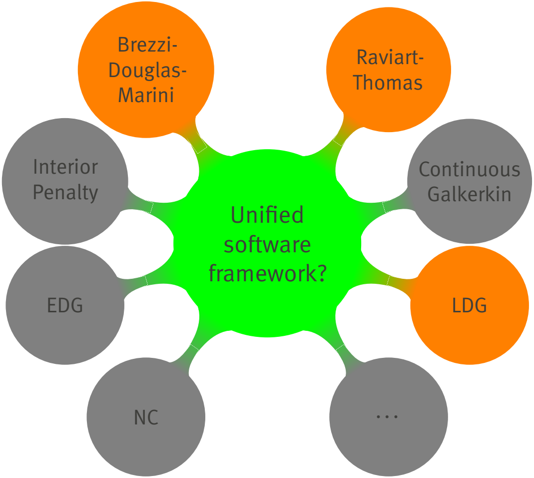

Outline

Hybridization

Unifying framework: Implementation in DUNE

- DUNE-FEM-LOCALFUNCTIONS
- DUNE-HYBRIDFEM

Numerical tests

- Implementation
- Numerical results

Hybridization

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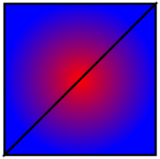
Hybridization I

Let $u \in X$ the solution of

$$\mathcal{L}(u) = f \qquad \text{on } \Omega.$$

- $\Omega \subset \mathbb{R}^2$ domain,
- X Hilbert space,
- $\mathcal{L} : X \to X'$ partial differential operator,
- $f \in X'$ source term
- $\mathcal{T}_h$ triangulation of Ω,
- $\mathcal{E}_h$ edges, $\mathcal{E}_h^\circ$ inner edges

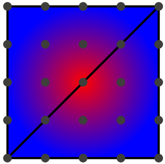


Hybridization II

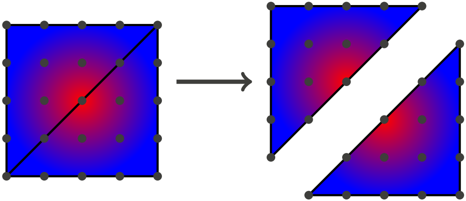
Let $u_h \in X_h$ be the solution of

$$\mathcal{L}_h(u_h) = f \qquad \text{on } \Omega.$$

- $X_h \subset X$ finite dimensional subspace,
- $\mathcal{L}_h : X_h \to X_h'$ discretization operator,

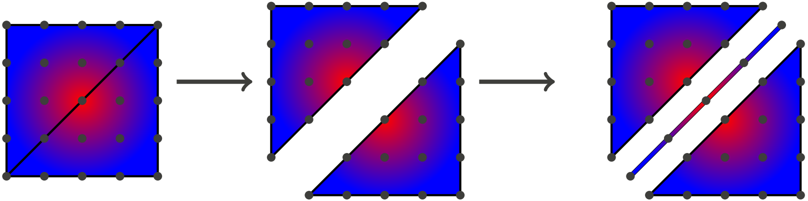


Hybridization III



1. neglect continuity conditions ("transmission condition"),

Hybridization III



1. neglect continuity conditions ("transmission condition"),
2. reinforce them by new unkowns ("Lagrange multipliers")

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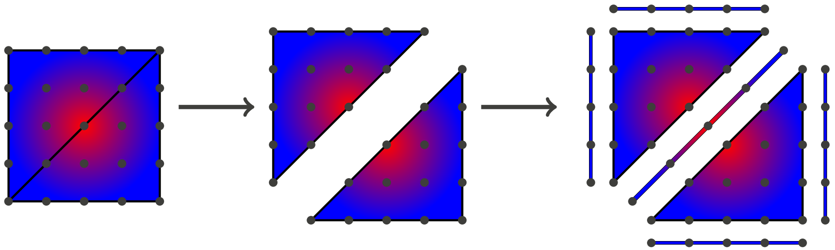

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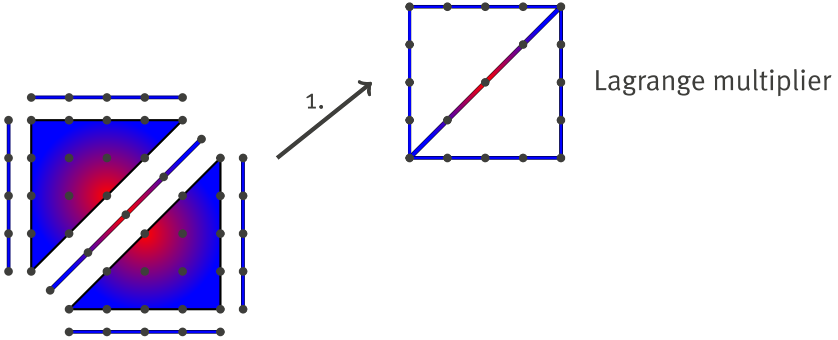
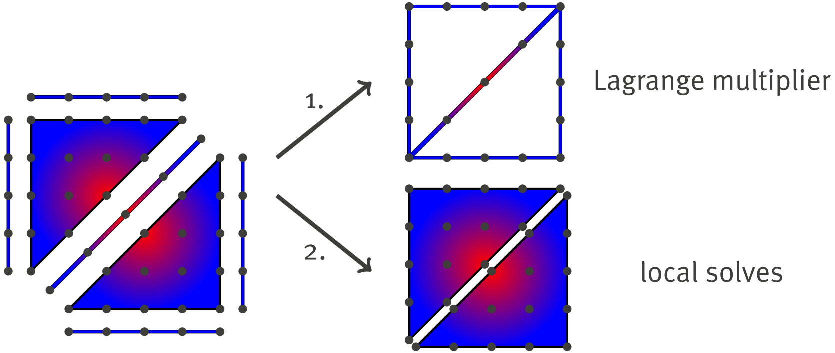
Hybridization IV



1. solve global system for the Lagrange multipliers,

Hybridization IV



1. solve global system for the Lagrange multipliers,
2. solve local systems approximating the solution u_h element-wise

Hybridization V

To hybridize a discretization method "N" with operator $\mathcal{L}_h$ we need to:

1. define local solvers by using $\mathcal{L}_h|_K$ for each element $K \in \mathcal{T}_h$,

Hybridization V

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We call the hybridized method "N-Hybrid" or "N-H".

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Finally we need to check:

- Is the hybridized method equivalent to the non hybridized method, or is it a new method?

Advantages of hybridization

1. smaller (easier) global system,

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   3.2 *p*-adaptivity...
4. new methods can be obtained

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- local assembler/solvers,
- assembler/solver for global Lagrange multiplier

Hybridization

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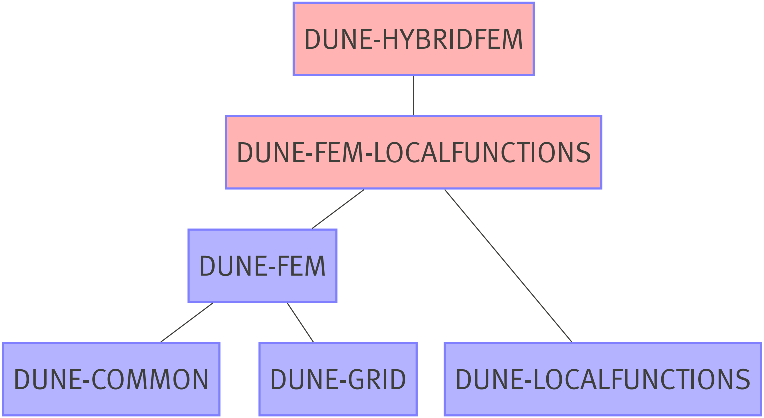
Implementation in DUNE

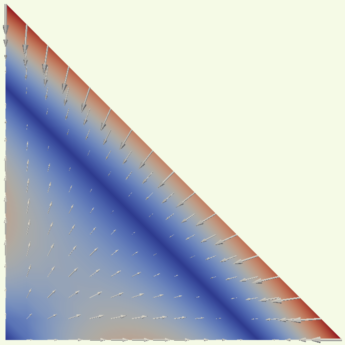


Dune
Distributed and Unified Numerics Environment

DUNE-FEM-LOCALFUNCTIONS - Motivation

1. want to use Raviart-Thomas finite elements in DUNE-FEM,

Example (first basis function for Raviart-Thomas, order 1)



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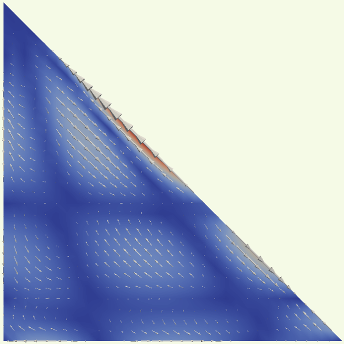
Example (42nd basis function for Raviart-Thomas, order 5)



DUNE-FEM-LOCALFUNCTIONS - Motivation

1. want to use Raviart-Thomas finite elements in DUNE-FEM,
2. some implementation work in DUNE-FEM was already done, but not flexible enough,
3. want to have a discrete function space similar to DUNE-PDELAB.

DUNE-FEM-LOCALFUNCTIONS I

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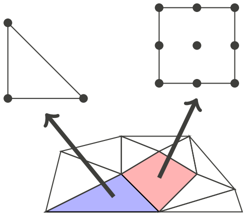
The finite element may be independent of geometry and polynomial order on each entity.

- No *h*-adaptiv and parallel version implemented, yet.

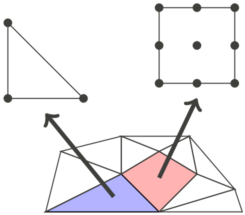
Basis function set maps



task: return correct basis function set depending on

1. polynomial degree,

Basis function set maps



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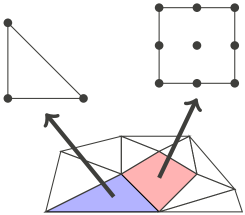
Basis function set maps



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Main challenges

1. fulfill interfaces in DUNE-FEM

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1. fulfill interfaces in DUNE-FEM
2. intersections $\neq$ codim-1-entities
3. no single basis functions in DUNE-LOCALFUNCTIONS

DUNE-FEM-LOCALFUNCTIONS II

Example (Lagrange finite element)

```
using namespace Dune;
using namespace Dune::FemLocalFunctions;

typedef LagrangeLocalFiniteElement< EquidistantPointSet,
                                    dimension,
                                    double,
                                    double >
  LagrangeElementType;
```

DUNE-FEM-LOCALFUNCTIONS III

Example (Linear Lagrange space)

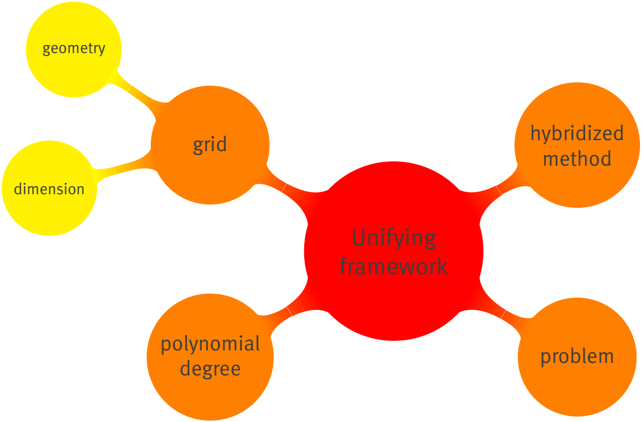
```
typedef LagrangeElementType
  FiniteElementType;

typedef BaseFunctionSetMap< GridPartType, FiniteElementType,
            NoTransformation, SimpleStorage, 1 >
  BaseFunctionSetMapType;

typedef DiscreteFunctionSpace< BaseFunctionSetMapType >
  DiscreteFunctionSpaceType;

BaseFunctionSetMapType      fem(gridPart);
DiscreteFunctionSpaceType   space(gridPart, fem);
```

DUNE-FEM-LOCALFUNCTIONS IV

Example (Linear Lagrange space on intersections)

```
typedef IntersectionLocalFiniteElement< LagrangeElementType >
  FiniteElementType;

typedef IntersectionBaseFunctionSetMap< GridPartType,
      FiniteElementType,NoTransformation,SimpleStorage,1>
  BaseFunctionSetMapType;

typedef DiscreteFunctionSpace< BaseFunctionSetMapType >
  DiscreteFunctionSpaceType;

typedef typename BaseFunctionSetMapType::GridPartType
  IntersectionGridPartType;

IntersectionGridPartType    iGridPart(grid);
BaseFunctionSetMapType      fem(iGridPart);
DiscreteFunctionSpaceType   space(iGridPart, fem);
```

DUNE-HYBRIDFEM - Concept



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Implemented problem I

2nd order elliptic boundary value problem

Find $(\mathbf{q}, u) \in \mathbf{V} \times W$ fulfilling

$$\begin{aligned} \mathbf{q} + a\nabla u &= 0 && \text{on } \Omega, \\ \mathrm{div}\,\mathbf{q} + du &= f && \text{on } \Omega, \\ u &= g && \text{on } \partial\Omega. \end{aligned}$$

Implemented problem II

2nd order elliptic boundary value problem

Find $(\mathbf{q}_h, u_h) \in \mathbf{V}_h \times W_h$ fulfilling

$$\begin{aligned} \mathbf{q}_h + a\nabla u_h &= 0 && \text{on } \Omega, \\ \mathrm{div}\,\mathbf{q}_h + d u_h &= f && \text{on } \Omega, \\ u_h &= g && \text{on } \partial\Omega. \end{aligned}$$

Implemented problem III

First local solver

Find for each element $K \in \mathcal{T}_h$ and each function m defined on ∂K a solution $(\mathbf{q}_{h,m}, u_{h,m}) \in \mathbf{V}(K) \times W(K)$ satisfying

$$\int_K c\mathbf{q}_{h,m} \cdot \mathbf{v} - \int_K u_{h,m} \mathrm{div}\mathbf{v} = -\int_{\partial K} m\mathbf{v} \cdot n \quad \forall \mathbf{v} \in \mathbf{V}(K),$$

$$-\int_K \nabla w \cdot \mathbf{q}_{h,m} + \int_{\partial K} w\widehat{\mathbf{q}_{h,m}} \cdot n + \int_K d u_{h,m} w = 0 \quad \forall w \in W(K),$$

where $\widehat{\mathbf{q}_{h,m}}$ depends on the method.

Implemented problem IV

Second local solver

Find for each element $K \in \mathcal{T}_h$ a solution $(\mathbf{q}_{h,f}, u_{h,f}) \in \mathbf{V}(K) \times W(K)$ satisfying

$$\int_K c\mathbf{q}_{h,f} \cdot \mathbf{v} - \int_K u_{h,f} \operatorname{div} \mathbf{v} = 0 \qquad \forall \mathbf{v} \in \mathbf{V}(K),$$

$$-\int_K \nabla w \cdot \mathbf{q}_{h,f} + \int_{\partial K} w\widehat{\mathbf{q}_{h,f}} \cdot n + \int_K d u_{h,f} w = \int_K fw \qquad \forall w \in W(K),$$

where $\widehat{\mathbf{q}_{h,f}}$ depends on the method.

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Implemented problem V

Solver for Lagrange multiplier

Find $\lambda_h \in M_h$ satisfying

$$a_h(\lambda_h, \mu) = b_h(\mu) \qquad \forall \mu \in M_h,$$

where

$$a_h(\eta, \mu) = \sum_{K \in \mathcal{T}_h} \int_K (c\mathbf{q}_{h,\eta} \cdot \mathbf{q}_{h,\mu} + d u_{h,\eta} u_{h,\mu}) + \sum_{e \in \mathcal{E}_h} \int_e [(\widehat{\mathbf{q}_{h,\eta}} - \mathbf{q}_{h,\eta})(u_{h,\mu} - \mu)] \qquad \forall \eta, \mu \in M_h,$$

$$b_h(\mu) = \sum_{e \in \mathcal{E}_h} \int_e g_h [\widehat{\mathbf{q}_{h,\mu}}] + \sum_{K \in \mathcal{T}_h} \int_K f u_{h,\mu} \qquad \forall \mu \in M_h.$$

Implemented methods

	$\mathbf{V}(K)$	$W(K)$	M_h
RT-H	$\mathbb{P}_k(K)^n + x\mathbb{P}_k(K)$	$\mathbb{P}_k(K)$	$\mathbb{P}_k(\mathcal{E}_h^\circ)$
BDM-H	$\mathbb{P}_k(K)^n$	$\mathbb{P}_{k-1}(K)$	$\mathbb{P}_k(\mathcal{E}_h^\circ)$
LDG-H I	$\mathbb{P}_k(K)^n$	$\mathbb{P}_{k-1}(K)$	$\mathbb{P}_k(\mathcal{E}_h^\circ)$
LDG-H II	$\mathbb{P}_k(K)^n$	$\mathbb{P}_k(K)$	$\mathbb{P}_k(\mathcal{E}_h^\circ)$
LDG-H III	$\mathbb{P}_{k-1}(K)^n$	$\mathbb{P}_k(K)$	$\mathbb{P}_k(\mathcal{E}_h^\circ)$

	$\widehat{\mathbf{q}_{h,m}}$	$\widehat{\mathbf{q}_{h,f}}$
RT-H/BDM-H	$\mathbf{q}_{h,m}$	$\mathbf{q}_{h,f}$
LDG-H	$\mathbf{q}_{h,m} + \tau(u_{h,m} - m)n$	$\mathbf{q}_{h,f} + \tau(u_{h,f})n$

Example

```
Hybrid_Test< RT_HybridModel,
             AlbertaGridType,
             PoissonProblem,
             polOrder >( gridFile );
```

Raviart-Thomas-Hybrid, order 1

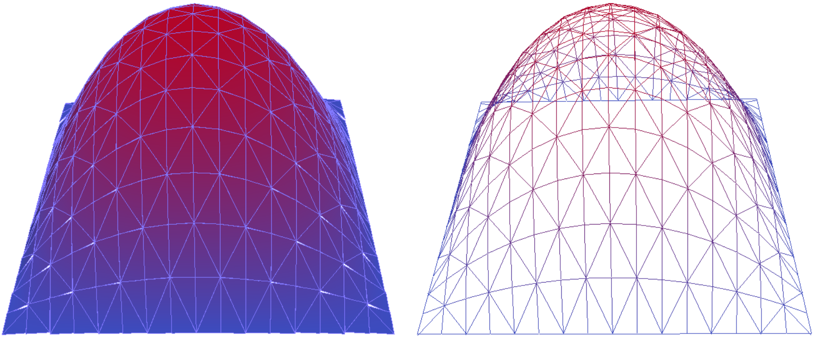


Figure: pressure/Lagrange multiplier

Brezzi-Douglas-Marini-Hybrid, order 1

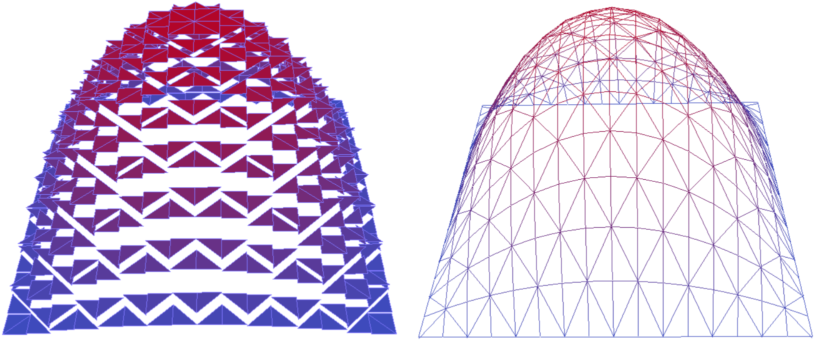


Figure: pressure/Lagrange multiplier

Comparison RT-H vs. BDM-H

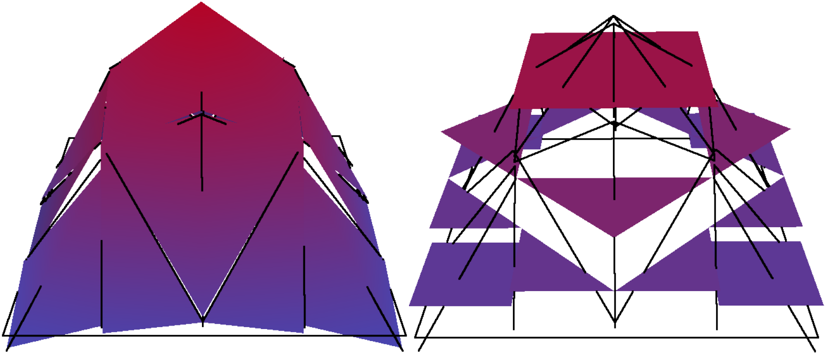


Figure: pressure/Lagrange multiplier

Raviart-Thomas-Hybrid, order 5, 8 elements

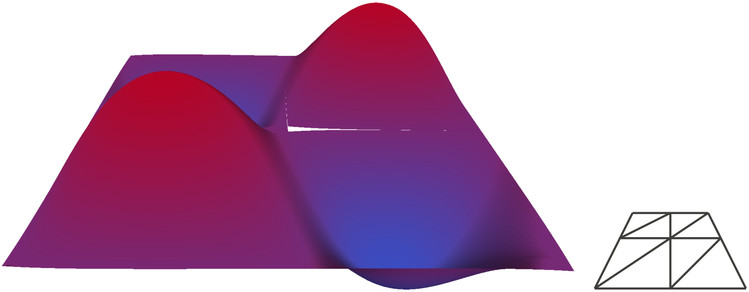
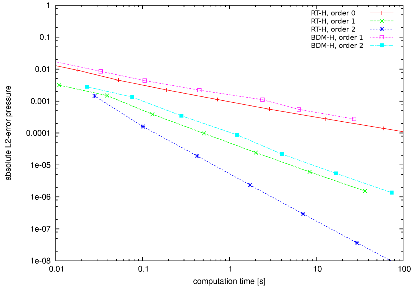
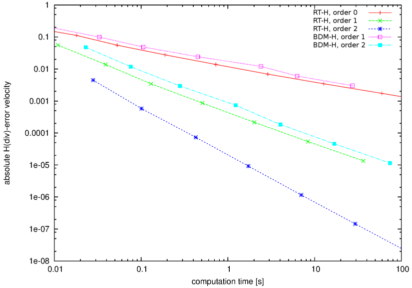


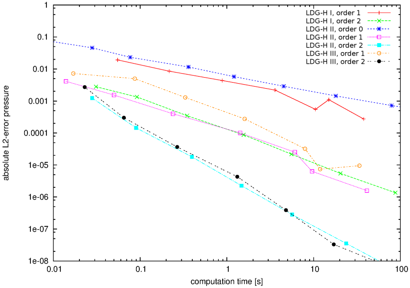
Figure: pressure/grid

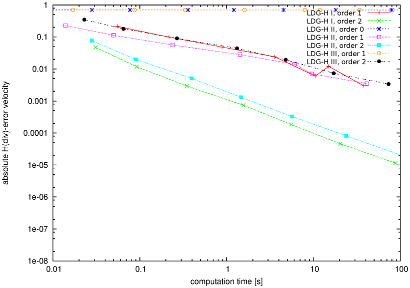
Error vs. computation time RT-H/BDM-H





Error vs. computation time LDG





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- the hybridized Raviart-Thomas method turns out to have the best convergence rates,
- there are still many interesting things to implement,
- we have combined the flexibility of DUNE with the flexibility of hybridization

References

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Thank you for your attention!