Universität Münster

# Model order reduction using nonlinear transformations of space-time domains

Reisensburg Workshop

**Hendrik Kleikamp** (University of Münster); based on work with Mario Ohlberger and Stephan Rave

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living.knowledge

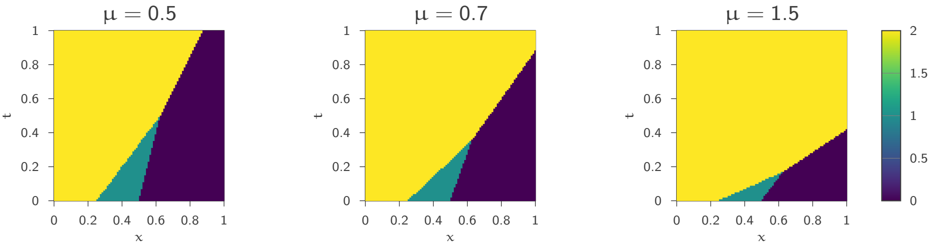
MM Mathematics Münster Cluster of Excellence

# Slowly decaying Kolmogorov $N$-widths

## Burgers' equation as a simple example

- Example: Moving discontinuities with different speeds in a parametrized Burgers' equation

$$\partial_t y_\mu(t,x) + \mu \cdot y_\mu(t,x)\partial_x y_\mu(t,x) = 0, \qquad y_\mu(0,x) = \begin{cases} 2, & \text{if } x \leqslant 0.25, \\ 1, & \text{if } 0.25 \leqslant x \leqslant 0.5, \\ 0, & \text{otherwise} \end{cases}$$





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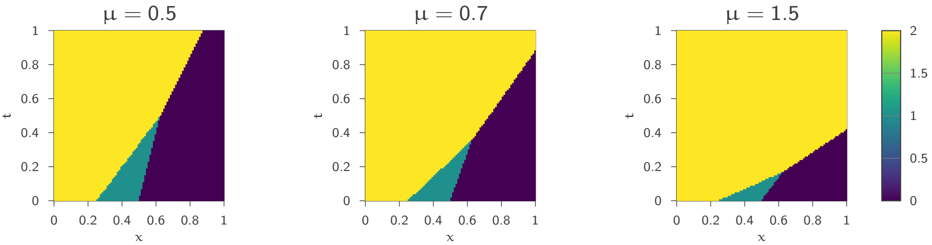


- Transport-dominated problems suffer from a slow Kolmogorov $N$-width decay!


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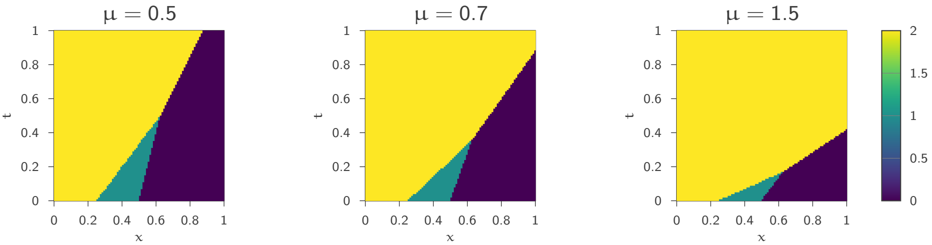


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$\Longrightarrow$ Approximation by linear subspaces not suitable!

# Nonlinear model order reduction strategy

- Approach: Perform nonlinear transformation of solution snapshots


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Main question: How to represent transformations in a reduced manner?

# Landmark matching

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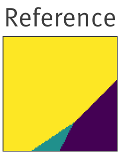




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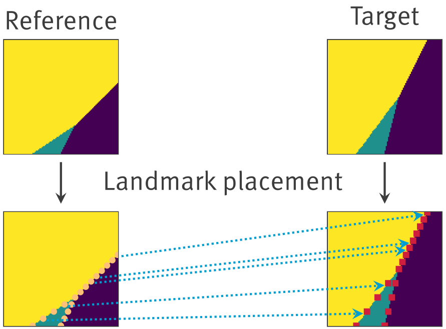

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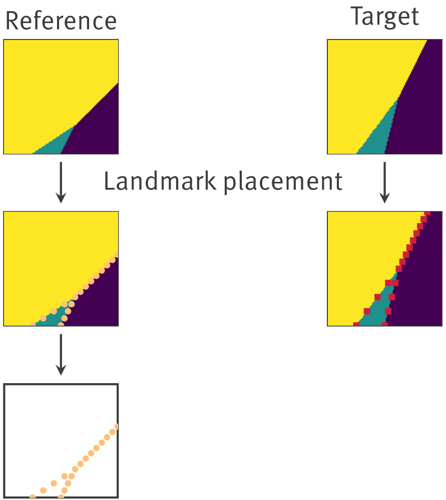



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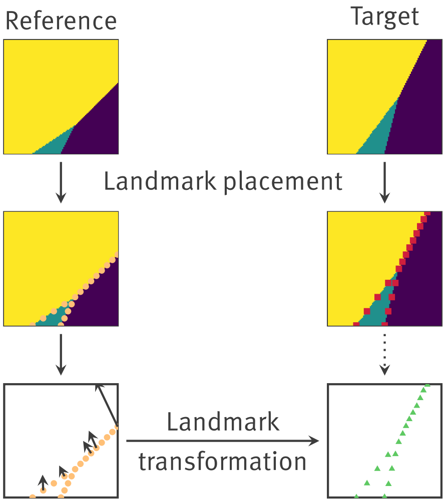

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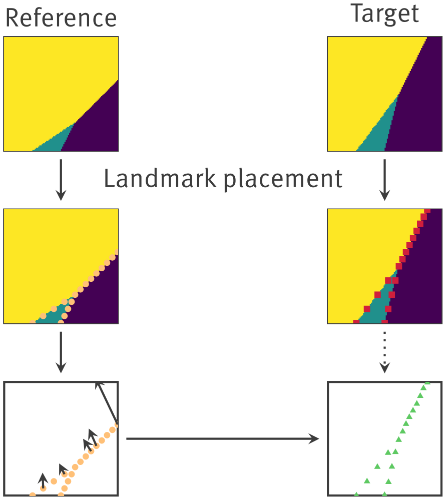
$$v_t(x) = \sum_{i=1}^{n} k(x, q_i(t))\, p_i(t)$$

momentum

position



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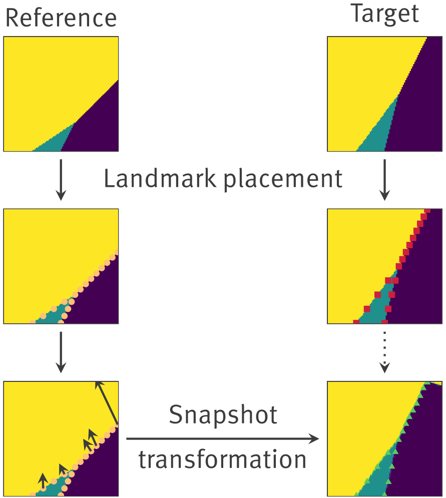



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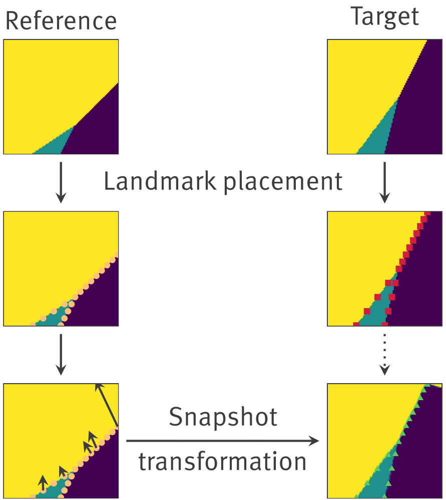



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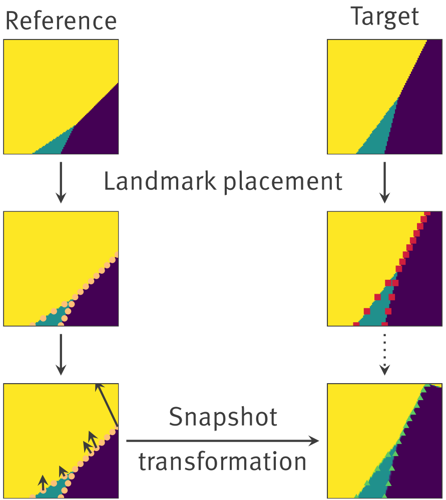
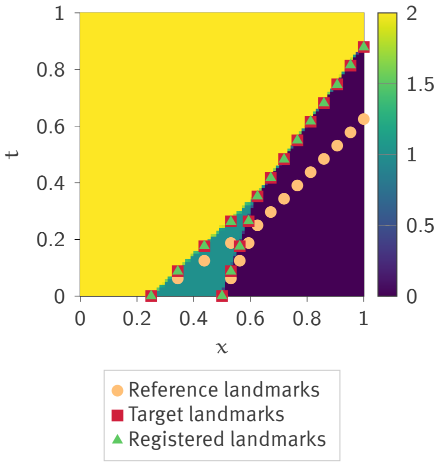
- Approximate initial momenta as a function of the parameter
- Offline/online phase: Similar procedure as for the vector fields (but without additional reduction of initial momenta)

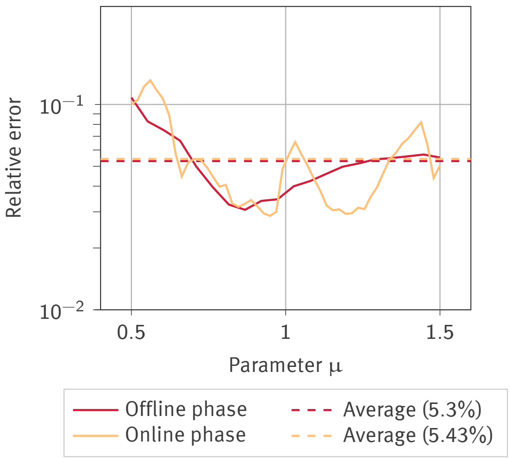


# Numerical results for Burgers' equation
## Landmark matching

Example for landmark registration




Matching errors



# Open problems

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- Intrusive online computations? Residual minimization?
- Error estimates (a priori and a posteriori)?
- Efficient online evaluations of the solution?
- Snapshot reduction and online selection of solution “topology”?

# Additional slides

# Labeled landmark matching I

- The trajectories of the template landmarks $x_i$ are given as solutions to the ODE

$$\frac{dq_i}{dt}(t) = v_t(q_i(t)), \qquad q_i(0) = x_i$$

for $i = 1, \ldots, n$, where $q_i \colon [0,1] \to \mathbb{R}^r$ denotes the evolution of the $i$-th landmark and $v_t \colon \mathbb{R}^r \to \mathbb{R}^r$ denotes the vector field at time $t \in [0,1]$.

- The time-dependent vector field admits a representation similar to kernel approximants, namely

$$v_t(x) = \sum_{i=1}^{n} k(x, q_i(t))p_i(t),$$

where $k \colon \mathbb{R}^r \times \mathbb{R}^r \to \mathbb{R}^{r \times r}$ is a symmetric matrix-valued kernel and $p_i \colon [0,1] \to \mathbb{R}^r$ are momenta associated to the landmarks for $i = 1, \ldots, n$.


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# Labeled landmark matching II

- The momenta and landmarks evolve in an energy minimizing way by means of a Hamiltonian function defined as

$$H(q(t), p(t)) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} p_i(t)^\top k(q_i(t), q_j(t))p_j(t) = \frac{1}{2}\,\|\, v_t \,\|_V^2\,,$$

  where $q(t) = (q_1(t), \ldots, q_n(t))$ is the set of landmarks and $p(t) = (p_1(t), \ldots, p_n(t))$ denotes the set of momenta. Here, the space $V$ corresponds to the reproducing kernel Hilbert space.
- The evolution of the landmarks and momenta is now in such a way that the Hamiltonian is constant over time, which leads to the system

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}(q, p), \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q}(q, p).$$

- The initial momenta are chosen such that they minimize the energy

$$E_{\{x_i\}\to\{y_j\}}(p(0)) = H(q(0), p(0)) + \frac{1}{\sigma^2}\sum_{i=1}^{n} \|\, q_i(1) - y_i \,\|_2^2\,.$$

# Unlabeled landmark matching



- Given sets of template landmarks $x_1, \dots, x_n \in \mathbb{R}^r$ and target landmarks $y_1, \dots, y_m \in \mathbb{R}^r$, we define the associated signed measures $\nu_{\text{template}}$ and $\nu_{\text{target}}$ as

$$\nu_{\text{template}} := \sum_{i=1}^{n} \delta_{x_i} \qquad \text{and} \qquad \nu_{\text{target}} := \sum_{j=1}^{m} \delta_{y_j}.$$

- Given another kernel $k_{\text{dist}} : \mathbb{R}^r \times \mathbb{R}^r \to \mathbb{R}$ we measure the norm of a linear combination $\nu = \sum_{i=1}^{n} \delta_{x_i}$ as

$$\|\nu\|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} k_{\text{dist}}(x_i, x_j).$$

- We consider as mismatch term $g$ in the energy the squared norm of the difference,

$$g(q(1)) = \sum_{i=1}^{n_{\text{diff}}} \sum_{j=1}^{n_{\text{diff}}} c_i c_j k_{\text{dist}}(z_i, z_j), \qquad \text{where} \sum_{i=1}^{n_{\text{diff}}} c_i \delta_{z_i} = \sum_{i=1}^{n} \delta_{q_i(1)} - \sum_{j=1}^{m} \delta_{y_j}.$$

# Landmark matching

## Efficient point-wise evaluation of the transformation

- In order to compute the evolution of a given point $x \in \mathbb{R}^r$ as

$$\frac{dq}{dt}(t) = v_t(q(t)), \qquad q(0) = x$$

only the time evolutions of landmarks and momenta are required according to

$$v_t(q(t)) = \sum_{i=1}^{n} k(q(t), q_i(t))p_i(t).$$

The time evolution of single points can therefore be computed efficiently by solving the corresponding ODE.

- Computational costs for labeled landmark matching when considering $n \in \mathbb{N}$ landmarks, $n_{\text{eval}} \in \mathbb{N}$ points of interest and $n_t \in \mathbb{N}$ time steps:

$$\mathcal{O}\left(n_t n(n + n_{\text{eval}})\right).$$

- A similar complexity involving the total number $n + m$ of template and target landmarks is obtained in the case of unlabeled landmarks.


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