A Tail of Two Distributions:

(Inverse Problems for Fractional Diffusion Equations)

William Rundell

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Fick's, Darcy's $\ldots$ law $+$ Continuity equation: $J = \rho \nabla u \quad \frac{\partial u}{\partial t} = \nabla \cdot \mathrm{J}$

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- In the time-independent/steady-state case: Laplace's equation: $\triangle u = 0$
- The heat equation $u_t = \nabla \cdot \rho \nabla u$ and with $\rho = 1$
- Fundamental solution $K(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$
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Many ways to argue this.

- Foundation was Brownian motion. Made explicit by Einstein in 1905: verified by Perrin to compute Avogadro's number (Nobel Prize 1926).
- A statistician might argue this as follows: The particle jumps should be independent random variables. In the limit or aggregate, by the *Central Limit Theorem*, these should approach a Gaussian distribution.
- Einstein demonstrated $\langle x^2 \rangle \propto t^2$ gave straight line motion - wave equation.

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So what if we allow the more general version $\langle x^2 \rangle \propto \phi(t)$

– and (somehow) find the ϕ that “best fits the data?”

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Nice (in the sense of beauty) is always in the eyes of the beholder.

If in the sense of mathematical analysis, then, frankly, **no**.

Where do the *fractions* enter?

Anomalous (fractional) Diffusion: $\langle x^2 \rangle \propto t^\alpha, \, \alpha \neq 1$ leads to a continuous time random walk and a fractional derivative in the time variable $D^\alpha_{0,t} u$,

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Anything more general than simple powers?

Of course! How close to a pde do you want? One aim would be to preserve as much structure as possible in the resulting "differential" equations and prevent the analysis from being overly challenging or too abstract.

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> *"Always keep things simple, but no simpler"*

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Will lead to very different physics.

In turn will lead to interesting inverse problems.

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Replace either the time or the space derivative in the heat equation by a derivative of fractional order $J = \rho D_x^\beta u \qquad D_t^\alpha = \nabla \cdot J$

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Fractional differential equation

$$D_{0,t}^\alpha u - D_{0,x}^\beta u + a u_x + bu = f$$

with $\alpha \in (0,1)$, $\beta \in (1,2)$ being the order of "differentiation" at a microscopical level: (Can also have $1 < \alpha < 2$ for "fractional wave equation").

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In the case of space fractional derivatives we might need

$$\sum_{i=1}^n a_i D_{x_i}^{\alpha_i} u + \sum_{i=1}^n b_i D_{x_i}^{\beta_i} u + cu = f \qquad 1 < \alpha_i \leq 2, \quad 0 < \beta_i \leq 1$$

- Leibniz's letter to L'Hospital (1695):

  "*Thus it follows that* $d^{\frac{1}{2}}x$ *will be equal to* $x\sqrt{[2]dx : x}$, *an apparent paradox, from which one day useful consequences will be drawn*."

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- Euler's observation (1738): $n \in \mathcal{N}^+$, $\mu, \alpha \in \mathcal{R}^+$

$$\frac{d^n}{dx^n} x^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-n+1)} x^{\mu-n},$$

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- Abel's integral equation for tautochrone problem (1823): $\mu \in (0,1)$,

$$I_\mu[\varphi] = \int_0^x (x-t)^{-\mu}\varphi(t)dt = f(x)$$

Two limiting cases:

$\lim_{\mu\to 1^-} \frac{1}{\Gamma(1-\mu)} I_\mu[\varphi] = \varphi(x)$ $\qquad$ $\lim_{\mu\to 0^+} \frac{1}{\Gamma(1-\mu)} I_\mu[\varphi] = \int_0^x \varphi(t)\,dt$

$\Rightarrow$ L.H.S. is a *fractional integral* of order $1-\mu$

Some members of the fractional derivative zoo of order $\alpha \in (n-1, n)$

[a] Riemann-Liouville fractional derivative

$$\partial_t^\alpha u(t) = {}^R D_t^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^{\mathrm{t}} (\mathrm{t}-\mathrm{s})^{\mathrm{n}-1-\alpha} \mathrm{u}(\mathrm{s})\, \mathrm{ds}$$

[b] Caputo fractional derivative (Dzherbashyan, 1960, Caputo, 1967)

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- $\alpha \to n$: ${}^R D_t^\alpha u(t) \to u^{(n)}(t)$, ${}^C D_t^\alpha u(t) \to u^{(n)}(t)$

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- Composition of fractional derivatives: $\alpha \in (1,2)$

$$({}^R D_0^{\frac{\alpha}{2}})^2 u(x) = {}^R D_0^\alpha u(x), \quad \text{if } u(0) = 0,$$

$$({}^C D_0^{\frac{\alpha}{2}})^2 u(x) = {}^C D_0^\alpha u(x) - \frac{u'(0)}{\Gamma(2-\alpha)} x^{1-\alpha}, \quad \text{if } u \in C^2[0,1].$$

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$${}^R D_t^\alpha t^\gamma = {}^C D_t^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} t^{\gamma-\alpha}, \quad \gamma > 0, \quad \alpha \in (0,1).$$

but $\ldots$ $\qquad {}^R D_t^\alpha 1 = \dfrac{t^{-\alpha}}{\Gamma(1-\alpha)}, \qquad {}^C D_t^\alpha 1 = 0, \quad \alpha \in (0,1).$

Product rule fails! - even for powers!! (check previous formula)

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- No product rule

$${}^R D_0^\alpha (fg) \neq ({}^R D_0^\alpha f) g + f({}^R D_0^\alpha g),$$
$${}^C D_0^\alpha (fg) \neq ({}^C D_0^\alpha f) g + f({}^C D_0^\alpha g)$$

$\Rightarrow$ no usual integration by parts!! $\Rightarrow$ no Green's Theorem!!

Massive Difference from classical case.

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If t is a time variable, then it appears one can get comfortably along with just these two as the left to right definition corresponds to increasing time.

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If x is a space variable, then we might need further animals; for example,

- $I^\beta_{[0,1]}[u](x) = \frac{1}{2}[{}^R_0 D_x^\beta + {}^R_1 D_x^\beta] u(x)$, with $1<\beta<2$, $x \in [0,1]$
- Fractional Laplacian (with/without various boundary conditions)

  In $\mathbb{R}^d$ use Fourier transforms: $-\widehat{\triangle^\alpha f}(s) = |s|^{2\alpha} \hat{f}(s)$

  In $\Omega \subset \mathbb{R}^d$, A sectorial: $A^\alpha f = \frac{sin(\pi\alpha)}{\pi} \int_0^\infty \lambda^{\alpha-1} A(\lambda I + A)^{-1} f\, d\lambda$

Numerics: the history mechanism changes the game ...

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- If f is bounded, $f^{(j)} \in L^1(\mathbb{R})$ for $j \leq n$ with $n > 1+\alpha$ then $\mathcal{D}_x^\alpha f$ has Fourier transform $(i\xi)^\alpha \hat{f}(\xi)$.

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- Implicit/explicit schemes based on the Grünwald-Letnikov formula are *unstable*. The trick is to use a shifted formula
  - Even here, the scheme is only $O(h)$.

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- Not surprisingly, there is developing a vast literature for all the derivatives (but especially Riemann-Liouville) based on finite elements.

Basic tools:

Mittag-Leffler function (1903) $E_{\alpha,\beta}(z)$ with $\alpha > 0$, $\beta \in \mathbb{R}$, $z \in \mathbb{C}$

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}$$ – entire function of z with order $1/\alpha$

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The efficient and accurate numerical computation of the Mittag- Leffler function is delicate. An efficient algorithm relies on partitioning the complex plane where different approximations, i.e., power series, integral representation and asymptotic values for small, intermediate and large values of the argument respectively, are used for efficient numerical computation.

The special (and for fractional diffusion, important) case of the Mittag-Leffler function $E_{\alpha,\beta}(z)$ with real argument z can also be efficiently computed with a combination of Laplace transforms and suitable quadrature rules.

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Asymptotic behaviour: $0<\alpha<2$, $\mu \in (\frac{\alpha\pi}{2}, \min(\pi, \alpha\pi))$

$$E_{\alpha,\beta}(z) = \begin{cases} \frac{1}{\alpha} z^{\frac{1-\beta}{\alpha}} e^{z^{\frac{1}{\alpha}}} - \sum_1^N \frac{z^{-k}}{\Gamma(\beta-\alpha k)} + O\left(\frac{1}{z^{N-1}}\right), & |\arg(z)| \leq \mu \\ -\sum_1^N \frac{z^{-k}}{\Gamma(\beta-\alpha k)} + O\left(\frac{1}{z^{N+1}}\right) & \mu < |\arg(z)| \leq \pi. \end{cases}$$

M. Dzherbashyan, 1960s.

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Important point: for $\alpha<1$,

$E_{\alpha,\beta}$ has exponential [$e^{(z^{1/\alpha}}$] growth for large x

$E_{\alpha,\beta}$ has polynomial decay for large $-x$

Also need the Wright function (1933) $$W_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\,\Gamma(\rho k + \mu)}\,,$$

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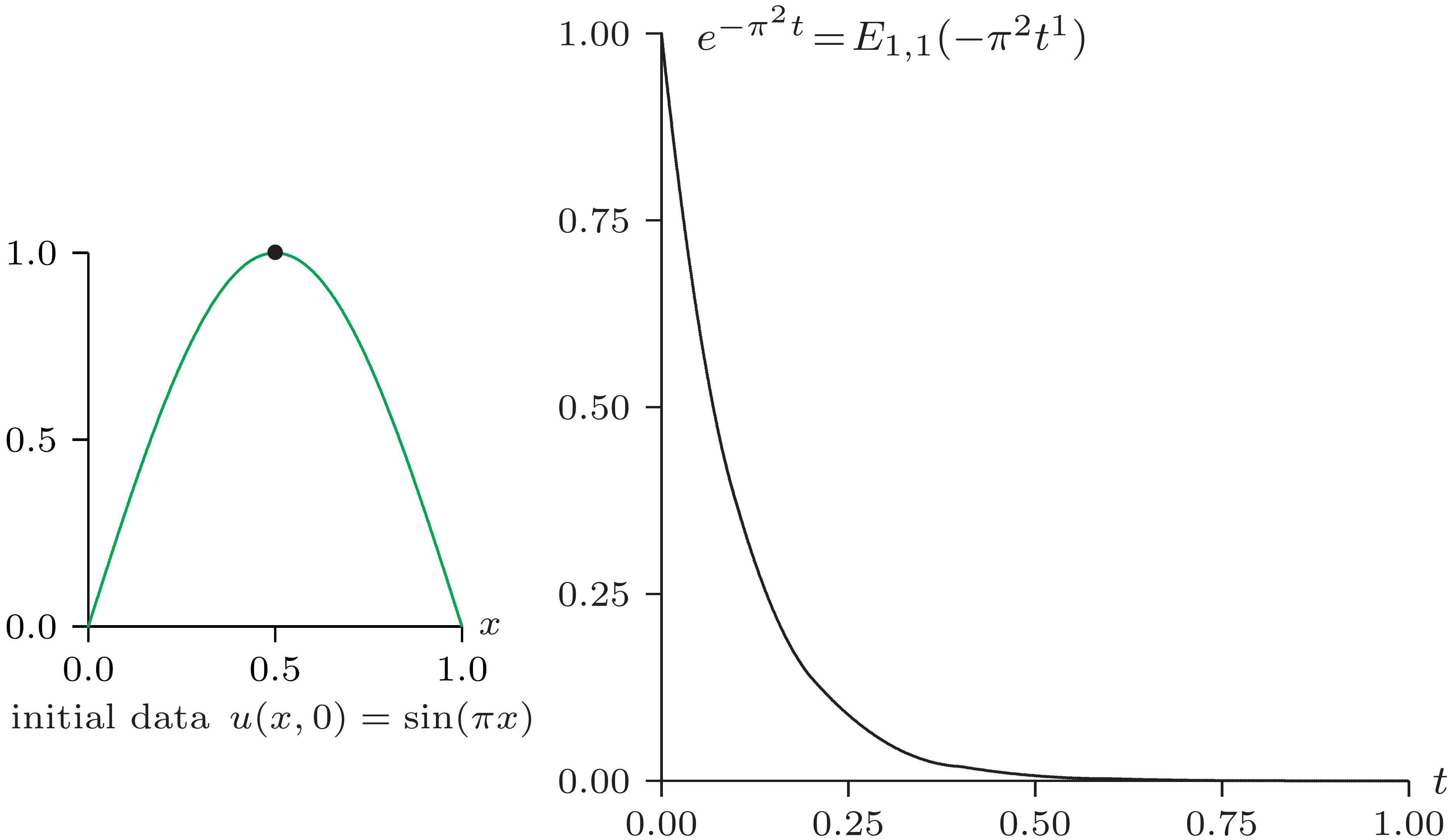
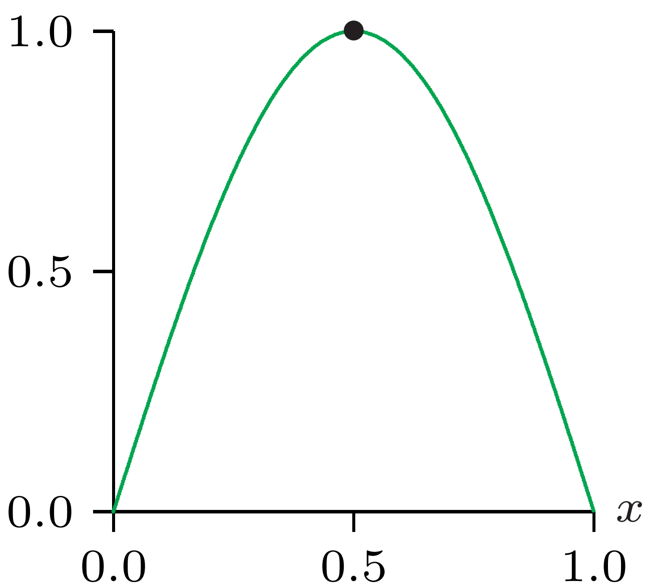
The computation of the Wright function $W(z)$ is even more delicate. An algorithm for the Wright over the whole complex plane with rigorous error analysis is still missing.

For real z a divide and conquer approach for small, intermediate and large values is necessary. The intermediate range (which is very large) is aided by an integral represention formula.

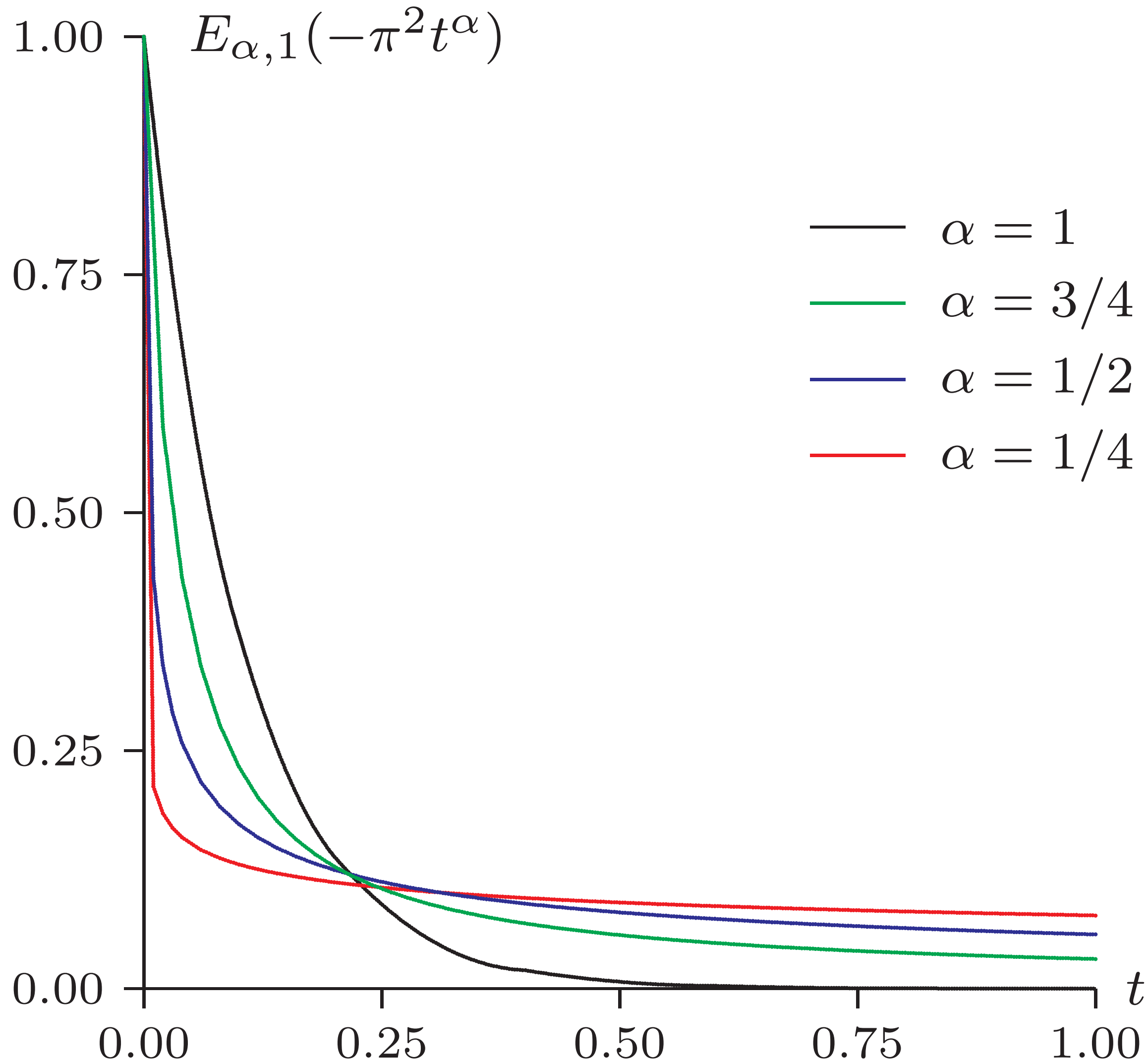
Unfortunately, this has a singular kernel but a transformation allows Gauss-Jacobi quadrature to be used effectively [Luchko, 2008; Jin-R, 2015]



Time trace of the point $u(0.5, t)$



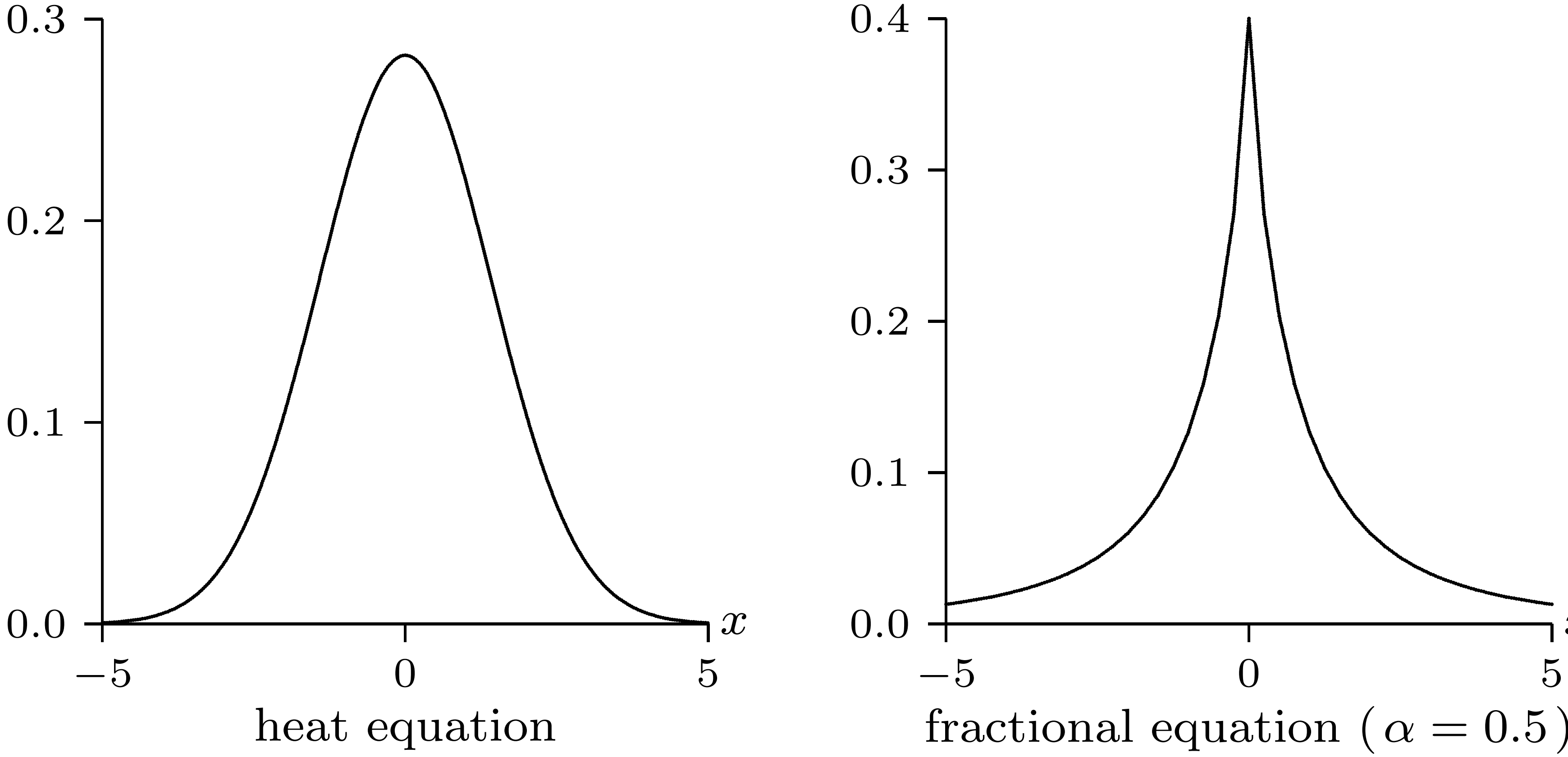
initial data $u(x,0) = \sin(\pi x)$



Time trace of the point $u(0.5, t)$

]

Fundamental solutions at $t = 1$:



- **What about inverse problems???**

- **What about inverse problems???**
  - The analysis is definitely going to be more challenging,
  - But will there be new physics?.
  - Or just tedius analysis leading to the same conclusions?.

The backwards Heat problem

Given

$$u_t = u_{xx} \qquad 0 < x < 1, \quad 0 < t < T$$

$$u(0,t) = u(1,t) = 0 \qquad u(x,0) = u_0(x)$$

We measure $g(x) := u(x,T)$ and wish to recover the initial value $u_0(x)$.

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$\phi_n(x) = \sqrt{2}\,\sin n\pi x$, $c_n = \langle u_0, \phi_n \rangle$, $d_n = \langle u(\cdot, T), \phi_n \rangle$, $\lambda_n = n^2 \pi^2$.

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$$\boxed{\begin{array}{l} \quad u_t = u_{xx} \\ u(x,t) = \sum c_n e^{-\lambda_n t} \phi_n(x) \\ u_0(x) = \sum d_n e^{\lambda_n T} \phi_n(x) \\ \text{Recover } \{c_n\}\text{:} \ \ c_n = e^{n^2\pi^2 T} d_n \\ \\ g \Rightarrow \{d_n\} \Rightarrow \{c_n\} \rightarrow u_0 \end{array}}$$

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Uniqueness holds !!

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$$D_t^\alpha u = u_{xx}$$
$$u(x,t) = \sum c_n E_{\alpha,1}(-\lambda_n t^\alpha)\phi_n(x)$$
$$u_0(x) = \sum d_n [E_{\alpha,1}(-\lambda_n T^\alpha)]^{-1}\phi_n(x)$$
$$\text{Recover } \{c_n\}: \ c_n = \frac{1}{E_{\alpha,1}(-\lambda_n T^\alpha)} d_n$$

How ill-posed?

Look at the asymptotic: if $\lambda_n T >> 1,\ E_{\alpha,1}(-\lambda_n T^\alpha) \approx C\lambda_n T^\alpha$.
The n^{th} Fourier mode of u_0 equals that of g multiplied by $\lambda_n \approx n^2\pi^2$
– a two derivative loss in Fourier space.

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$$c\|u(T)\|_{H^2} \leq \|u(0)\|_{L^2} \leq C\|u(T)\|_{H^2}$$

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Conjecture:

Reconstructing u_0 from $u(x,T)$ is always easier in the fractional case

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Conjecture:

Reconstructing u_0 from $u(x,T)$ is always easier in the fractional case

The answer is no, and the difference can be substantial.

To illustrate the point, let J be the highest frequency mode required of the initial data u_0 and assume that we believe we are able to multiply the first J modes $\{g_j := \langle g, \phi_j \rangle\}_1^J$, by a factor no larger than M.

By monotonicity of $E_{\alpha,1}(-t)$ in t, it suffices to examine the Jth mode.

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For a fixed J, let $T_\alpha^\star$ denote the point where $e^{-\lambda_J T_\alpha^\star} = E_{\alpha,1}(-\lambda_J T_\alpha^\star)$. Then in the fractional case for $T < T_\alpha^\star$ the growth factor on g_J will exceed M for any $T < T_\alpha^\star$.

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The critical values T_α^*

$\alpha \backslash J$	3	5	10
$1/4$	0.0442	0.0197	0.0059
$1/2$	0.0387	0.0163	0.0049
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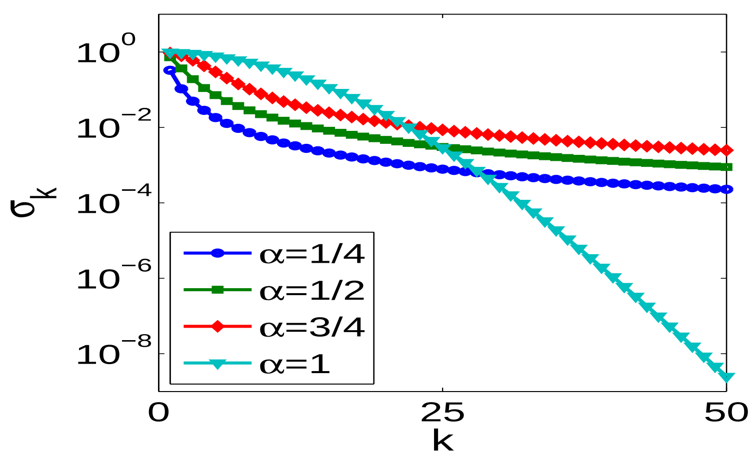

For example, at $T = 0.001$, the first twenty singular values for the heat equation are larger than the fractional counterpart.

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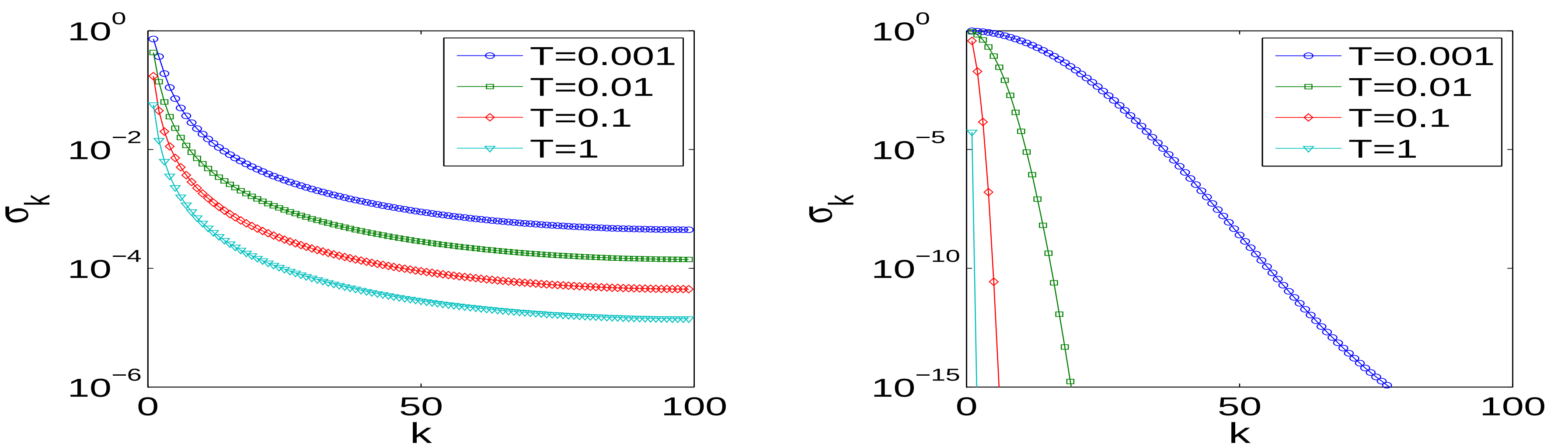

The singular value spectrum of the map $F : u_0 \to g$ for $\alpha = \frac{1}{2}$ and $\alpha = 1$

Recovery of unknown source terms

$$\partial_t^\alpha u - \triangle u = f \quad \text{in } \Omega \times (0,T],$$

$\partial_t^\alpha u$ denotes the Djrbashian-Caputo fractional derivative of order $\alpha \in (0,1)$.

Assume an initial condition $u(0) = u_0$ $+$ suitable boundary conditions.

Goal: Recover the source term f from lateral boundary or final time data.

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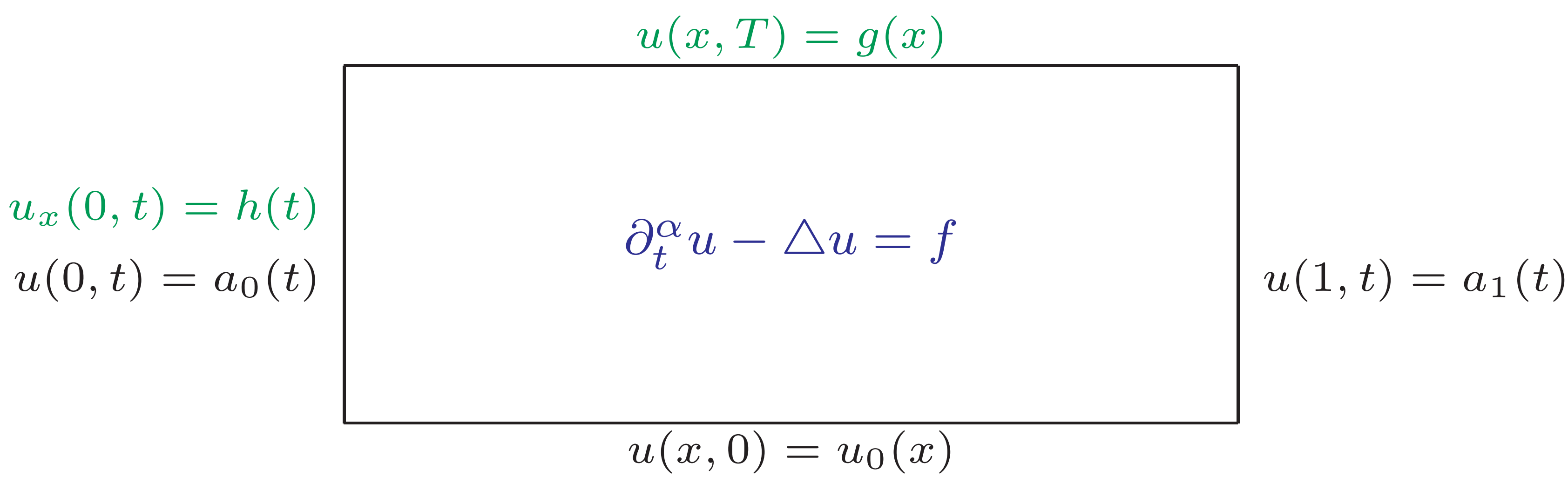
We consider only the one-dimensional model; the analysis and computation can be extended into the general multi-dimensional case.



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Clearly, one piece of boundary data or final time data alone is insufficient to uniquely determine a general source term $f(x,t)$ and so we break things down into two cases; a spatially unknown term and a time-dependent one.

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We can include additional terms without much additional theoretical difficulties so, for example our base model might become

$$\partial_t^\alpha u - (a(x)u_x)_x + q(x) = \phi(t)f(x)$$

where $a(x)$, $q(x)$, $\phi(t)$ are given while we seek $f(x)$.
[However we won't clutter the discussion with these additions.]

Also, by linearity of the problem, w.l.o.g. we can assume initial data, $u_0 = 0$.

Space-dependent source, final time data

$$\partial_t^\alpha u - u_{xx} = f(x) \quad \text{in } [0,1] \times (0,T], \qquad u(x,T) = g(x)$$

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The solution u to the forward problem is given by

$$\begin{aligned} u(t) &= \sum_{j=1}^\infty \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j (t-\tau)^\alpha)(f,\phi_j)\phi_j d\tau \\ &= \sum_{j=1}^\infty \lambda_j^{-1}(1 - E_{\alpha,1}(-\lambda_j t^\alpha))(f,\phi_j)\phi_j. \end{aligned}$$

Hence the measured data $g = u(T)$ is given by

$$g = \sum_{j=1}^{\infty} \frac{1}{\lambda_j} (1 - \mathrm{E}_{\alpha,1}(-\lambda_\mathrm{j} \mathrm{T}^\alpha))(\mathrm{f}, \phi_\mathrm{j})\phi_\mathrm{j}.$$

By taking the inner product with ϕ_j on both sides, we obtain the representation

$$f = \sum_{j=1}^{\infty} \lambda_j \frac{(g,\phi_j)}{1 - E_{\alpha,1}(-\lambda_j T^\alpha)} \phi_j.$$

Note: $E_{1,1}(x) = e^x$

$$f = \sum_{j=1}^{\infty} \lambda_j \frac{(g, \phi_j)}{1 - E_{\alpha,1}(-\lambda_j T^\alpha)} \phi_j. \qquad (*)$$

By the complete monotonicity of the Mittag-Leffler function $E_{\alpha,1}(-t)$ on the positive real axis, we deduce $1 > E_{\alpha,1}(-\lambda_1 T^\alpha) > E_{\alpha,1}(-\lambda_2 T^\alpha)$, Thus (*) is well defined for any $T > 0$, and gives the precise condition for the existence of a source term.

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Even with a modest value of the terminal time T, the factor

$$1 - E_{\alpha,1}(-\lambda_j T^\alpha) \approx 1 \quad \text{for all } \alpha \approx 0$$

Each frequency component (f, ϕ_j) differs from (g, ϕ_j) essentially by a factor λ_j, which amounts to a two derivative loss in space.

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Each frequency component (f, ϕ_j) differs from (g, ϕ_j) essentially by a factor λ_j, which amounts to a two derivative loss in space.

Actually one can show $\|f\|_{L^2(\Omega)} \leq c\|g\|_{H^2(\Omega)}$.

This behavior is identical to that for the backward fractional diffusion.

Holds also for the inverse source problem for the classical diffusion case.

This is not surprising, since with a space dependent source term f, the solution u to the forward problem can be split into the steady solution u_s and the decaying transient solution u_d: $u = u_s + u_d$, where u_s and u_d solve respectively

$$-u_s'' = f, \;\; u_s(0) = u_s(1) = 0,$$

and

$$\partial_t^\alpha u_d - u_{d,xx} = 0, \;\; u_d(0,x) = f(x), \;\; u_d(0,t) = u_d(1,t) = 0,$$

The steady state component u_s is dominating, which amounts to a two spatial derivative loss and this is confirmed by numerical experiments where

- The condition number is almost independent of the fractional order α.
- For large T, the singular value spectra are almost identical for all fractional orders, decaying to zero at an algebraic rate.

Unknown time-dependent f, final data at time T

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We have a source term f of form $f(x,t) = p(t)q(x)$, with $q(x)$ known and seek to recover $p(t)$ from data $u(x,T) = g$.

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We have a source term f of form $f(x,t) = p(t)q(x)$, with $q(x)$ known and seek to recover $p(t)$ from data $u(x,T) = g$.

The inclusion of a nontrivial term $q(x)$ is essential to retain uniqueness. even in the classical case. To see this, take u to satisfy

$$u_t - u_{xx} = f(t), \quad (x,t) \in (0,1) \times (0,T)$$

$$u(x,0) = 1, \qquad -u_x(0,t) = u_x(1,t) = 0.$$

with

$$u(x,T) = g(x) = 1.$$

Then one solution is given by $u(x,t) = 1$ and $f \equiv 0$,

but another is $u(x,t) = \cos(2\pi t/T)$ and $f = (-2\pi/T)\sin(2\pi t/T)$.

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In the fractional case, take $u = \cos(2\pi t/T)$ for the second solution and define f to be its αth order Djrbashian-Caputo fractional derivative in time.

Like previously, the solution u is given by

$$u(t) = \sum_{j=1}^{\infty} \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j (t-\tau)^{\alpha}) p(\tau) d\tau (q, \phi_j) \phi_j.$$

Hence the measured data $g(x) = u(x,T)$ is given by

$$g(x) = \sum_{j=1}^{\infty} \int_0^T (T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j (T-\tau)^{\alpha}) p(\tau) d\tau (q, \phi_j) \phi_j(x).$$

By taking the inner product with ϕ_j on both sides, we deduce

$$(g, \phi_j) = (q, \phi_j) \int_0^T (T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j (T-\tau)^{\alpha}) p(\tau) d\tau.$$

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This resembles a finite-time Laplace transform or moment problem, and thus highly smoothing, which renders the inverse source problem severely ill-posed.

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For $\alpha = 1$, the kernel term is $e^{-\lambda_j (T-t)}$ and can only pick up information for t near T. Otherwise, the information is severely damped, especially for high frequency modes.

Like previously, the solution u is given by

$$u(t) = \sum_{j=1}^{\infty} \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j (t-\tau)^\alpha) p(\tau) d\tau (q, \phi_j) \phi_j.$$

Hence the measured data $g(x) = u(x,T)$ is given by

$$g(x) = \sum_{j=1}^{\infty} \int_0^T (T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j (T-\tau)^\alpha) p(\tau) d\tau (q, \phi_j) \phi_j(x).$$

By taking the inner product with ϕ_j on both sides, we deduce

$$(g, \phi_j) = (q, \phi_j) \int_0^T (T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j (T-\tau)^\alpha) p(\tau) d\tau.$$

In the fractional case, the forward map F from the unknown to the data is clearly compact, and thus the problem is still ill-posed.

However, the kernel $t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j t^\alpha)$ is less smooth and decays much slower.

Thus one might expect that the problem is less ill-posed than the classical counterpart ...

To examine the point, we examine the singular values of the problem.

- Irrespective of the fractional order α, the singular values decay exponentially to zero without a distinct gap in the spectrum.
  - In particular, for the terminal time $T = 1$, the spectrum is almost identical for all fractional orders α.
- For small T, the singular values still decay exponentially, but the rate is different: the smaller is the fractional order α, the faster is the decay.
  - In other words, due to a slower local decay of the exponential function $e^{-\lambda t}$, compared with the Mittag-Leffler function $t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha})$, *more* frequency modes can be picked up by normal diffusion than the fractional counterpart.

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In summary:

Both classical and fractional diffusion paradigms lead to a severely ill-posed problems; in the (colloquial) language they are exponentially ill-conditioned in that the forwards map from $p(t) \to u(x,T)$ is infinitely smoothing - $u(x,T)$ is analytic in x for $p(t)$ continuous.

This price has to be repaid when we invert.

But from a quantitative standpoint, fractional diffusion is always at least as ill-conditoned as the classical counterpart.

Time-dependent data

Overposed data can also be the flux at an end point $-u_x(0,t) = g(t)$.

We seek the recovery of a time dependent component $p(t)$ in the source term $f = q(x)p(t)$ from $g(t)$.

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Previous arguments lead to showing $g(t)$ is related to the unknown $p(t)$ by

$g(t) = -\sum_{j=1}^{\infty} \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j (t-\tau)^{\alpha}) p(\tau) d\tau \langle q(x), \phi_j \rangle \phi_j'(0).$

It can be deduced that ([SakamotoYamamoto:2011])

$$\|p\|_{C[0,T]} \leq c\|\partial_t^{\alpha} g\|_{C[0,T]}.$$

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$$\|p\|_{C[0,T]} \leq c\|\partial_t^{\alpha} g\|_{C[0,T]}.$$

- The inverse problem roughly amounts to taking the αth order Djrbashian-Caputo fractional derivative in time.
- as the fractional order $\alpha \searrow$ from $1 \to 0$, it becomes less and less ill-posed. (for α close to zero, it is nearly well-posed, at least numerically). [More precisely, the condition number of the discrete forward map F decreases monotonically as the fractional order α decreases from $1 \to 0$].

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We omit the details as the have much in common with the previous cases, but here is the outcome:

- The inverse source problem of recovering a space dependent component from the lateral Cauchy data is severely ill-posed for both fractional and normal diffusion. In the simplest case of a space dependent only source term, it is mathematically equivalent to unique continuation.

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Epitaph:

The following “folk theorem” was formulated by John Cannon 50 years ago:

Folk Theorem. *An inverse problem for a partial differential equation where the unknown function and the data are aligned in the same direction is usually only mildly ill-posed; if the directions are different it surely will be severely ill-posed.*

In the case of unknown sources the fractional diffusion equation obeys the same “theorem” - although there may be quantitive differences from the classical case (and in both directions).

Consider the one-dimensional boundary value problem

$$u_t = u_{xx} \qquad 0 < x < 1,\ t > 0$$

$$u(x,0) = 0, \qquad u_x(0,t) = g_0(t), \quad u(L,t) = f_1(t)$$

The solution can be written explicitly as

$$u(x,t) = -2\int_0^t \theta(x,t-\tau)g_0(\tau)\,d\tau + 2\int_0^t \theta_x(x-L,t-\tau)f_1(\tau)\,d\tau \quad (*)$$

where

$$\theta(x,t) = \sum_{m=-\infty}^{\infty} K(x+2m,t), \qquad K(x,t) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}} \qquad (**)$$

Thus one can recover, for example, $f_0(t) = u(0,t)$.

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The *sideways heat problem* turns this around:

Given f_0, g_0 recover f_1.

From (*) we obtain a formula for f_1 as

$$\int_0^t R(t-\tau) f_1(\tau)\, d\tau = \int_0^t \theta_x(-L, t-\tau) f_1(\tau)\, d\tau = \text{known}(f_0, g_0)$$

The bad news: this is a Volterra equation of the first kind whose kernel $R(s)$ satisfies $\frac{d^m}{ds^m}\mathrm{R}(\mathrm{s})\Big|_{\mathrm{s}=0} = 0$ for all $m \geq 0$. Thus severely ill-posed.

$$R(s) = \frac{L}{2\sqrt{\pi}} s^{-\frac{3}{2}} e^{-\frac{L^2}{4s}} \in C^\infty(0,\infty).$$

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What happens for the case of $D_t^\alpha - u_{xx}$?

Same formula (*) holds but with $\theta_\alpha(x,t)$ and $K_\alpha(x,t)$

$$R_\alpha(s) = \frac{1}{2s^\alpha} W_{-\frac{\alpha}{2}, 2-\frac{\alpha}{2}}(-Ls^{-\alpha/2}) = \sum_{k=0}^{\infty} \frac{(-L)^k s^{-k\frac{\alpha}{2}-\alpha}}{k!\,\Gamma(-\frac{\alpha}{2}k + 2 - \frac{\alpha}{2})}$$

Again get a first kind integral equation for f_0 with kernel $R_\alpha(s)$ such that

$$\frac{d^m}{ds^m} R_\alpha(s)\Big|_{s=0} = 0,\ \forall m \geq 0, \qquad \Rightarrow$$ Still severely ill-conditioned.

The solution $f(t)$ can be recovered by an inverse Laplace transform arugment

$$f(t) = \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{Br}} \widehat{\mathrm{f}} \mathrm{e}^{\mathrm{zt}} \mathrm{dz}, \tag{$*$}$$

where $Br = \{z \in \mathbb{C} : \Re z = \sigma, \sigma > 0\}$ is the Bromwich path.

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The expression (*) indicates the fractional sideways problem still suffers from severe ill-posedness in theory, since the high frequency modes of the data perturbation are amplified by an exponentially growing multiplier $e^{z^{\alpha/2}}$.

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Numerically, the degree of ill-posedness decreases dramatically as α decreases from $1 \to 0$; as $\alpha \to 0^+$, the multipliers grow at a much slower rate, $\Rightarrow$ better chance of recovering many more modes of f.

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The expression (*) indicates the fractional sideways problem still suffers from severe ill-posedness in theory, since the high frequency modes of the data perturbation are amplified by an exponentially growing multiplier $e^{z^{\alpha/2}}$.

Both classical and fractional sideways problems are severely ill-posed in the sense of norm error estimates between the data and unknown h. But with a fixed frequency range, the time fractional problem can be much less ill-posed.

Both classical and fractional sideways problems are Hence, anomalous diffusion mechanism does help substantially since much more effective reconstructions are possible in the fractional case.

The case of space fractional derivatives

Consider the one-dimensional “sideways heat problem”

$$Du(x,t) = {}^{C}D^{\beta}_{0,x}u(x,t), \qquad x > 0,\ t > 0 \qquad \beta \in (1,2)$$

$$u(x,0) = 0, \qquad u(0,t) = f(t), \qquad u_x(0,t) = g(t),\ t > 0$$

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In the case $\beta = 2$, the model recovers the standard diffusion equation, and we have already discussed the severe ill-conditioning .

Due to the nonlocal nature of the fractional derivative, one might expect that in the space fractional case, the sideways problem is less ill-posed.

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Take Laplace transforms in time to obtain

$$p\hat{u}(x,p) - {}^{C}D^{\beta}_{0,x}\hat{u}(t) \qquad \text{with } \hat{u}(0,p)=\hat{f}(p), \quad \hat{u}_x(0,p)=\hat{g}(p).$$

The solution is given as: $\hat{u}(x,p)=\hat{f}(p)E_{\beta,1}(px^{\beta})+\hat{g}(p)xE_{\beta,2}(p,x^{\beta})$.

Thus

$$\hat{h}(p)=\hat{f}(p)E_{\beta,1}(p)+\hat{g}(p)E_{\beta,2}(p) \ \Rightarrow\ h(t)=\int_{Br} e^{pt}\hat{h}(p)\,dp$$

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If $\beta=2$, this gives $\cosh\sqrt{p}$ and $\sinh\sqrt{p}/\sqrt{p}$ multipliers to the data $\hat{f}(p)$ and $\hat{g}(p)$ resulting in the known exponential ill-conditioning.

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For $\beta \in (1,2)$, the Mittag-Leffler funcion asymptotics shows the problem still suffers from exponentially growing multipliers to the data and these becomes asymptotically *larger* as the fractional order $\beta \to 1$.

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In other words, anomalous diffusion in space does not mitigate the ill-conditioned nature of the sideways problem, but actually *worsens* the conditioning severely.

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We wish to compute the solution at $x = 1$, i.e., $h(t) := u(1,t)$.

In the case $\beta = 2$, one may equally measure the lateral Cauchy data at $x = 1$, and aim at recovering the solution at $x = 0$. Clearly, this does not change the nature of the inverse problem, and it is equally ill-posed.

Due to the directional nature of the Djrbashian-Caputo derivative ${}^C D_{0,x}^{\beta}$, one naturally wonders whether this "directional" feature does influence the ill-posed nature of the sideways problem.

The “ reversed sideways heat problem”

$$Du(x,t) = {}^{C}D^{\beta}_{0,x}u(x,t), \qquad x > 0,\ t > 0 \qquad \beta \in (1,2)$$

$$u(x,0) = 0, \qquad u(1,t) = f(t), \qquad u_x(1,t) = g(t),\ t > 0$$

We now wish to compute the solution at $x = 0$, i.e., $h(t) := u(0,t)$.

The analysis is quite tricky.

However, the key factor is the sign reversal in x translating into evaluating the Mittag-Leffler functions in a direction where they are only *polynomially* growing. This in turns results in only polynomial growth multipliers for $\hat{g}(p)$ and $\hat{f}(p)$.

Thus provided one stays away from $\beta = 2$ $\ldots$

This version of the sideways heat problem is only mildly ill-posed !!

Not every reasonable inverse problem for the parabolic case has a counterpart in the fractional case with very different properties.

Not every reasonable inverse problem for the parabolic case has a counterpart in the fractional case with very different properties.

- Although the work involved in showing this is usually much delicate.
  - Caused by limited tools (there is a weak maximum principle though).
  - Lack of usual parabolic strong regularity results.

Recovering a nonlinear boundary term: find $g(u)$ from

$$
\begin{aligned}
&{}^C D_t^\alpha u - u_{xx} = \gamma(x,t) \qquad x \in (0,1) \quad t > 0 \\
&-u_x(0,t) = k(t) \qquad u_x(1,t) = g(u(1,t)) \quad t > 0 \\
&u(x_0) = u_0(x)
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The functions γ, k and u_0 are given. The overposed data is the profile $u(0,t) = h(t) \qquad t > 0$.

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- Uniqueness and reconstructibility possible (R-Xu-Zuo, 2012)

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&{}^C D_t^\alpha u - \triangle u = f(u) \qquad x \in \Omega \subset \mathbb{R}^n \quad t > 0\\
&\frac{\partial u}{\partial \nu} = \psi \qquad \mathrm{x} \in \partial\Omega \quad \mathrm{t} > 0\\
&u(x_0) = u_0(x) \qquad x \in \Omega
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- Uniqueness and reconstruction algorithm (Luchko-R-Yamamoto-Zuo, 2013)

- With $\alpha < 1$ one can take the same approach although there are now many more technical difficulties and a weaker result follows. The difficulties surround the regularity of the solution and being able to impose conditions that guarantee the maximum temperature range occurs on the left boundary.

- With $\alpha < 1$ one can take the same approach although there are now many more technical difficulties and a weaker result follows. The difficulties surround the regularity of the solution and being able to impose conditions that guarantee the maximum temperature range occurs on the left boundary.

- The parabolic version is only mildly ill-posed: equivalent to a derivative loss.
- The fractional version is even less ill-posed: equivalent to a fractional derivative loss.
- This means that an optimal regularization requires working with a penalty term involving fractional integral norms.

And last, but not least, we have . . .

Inverse Sturm Liouville problem

This is a basic building block of many undetermined coefficient problems as well as important in its own right.

$$-D^{\alpha}_{0,x}u + qu = \lambda u \quad , x \in (0,1) \qquad u(0) = u(1) = 0.$$

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- $1 < \alpha < 2$, space fractional diffusion $\Leftarrow$ Levy motion

Direct problem: Given $q(x)$ determine $\{\lambda_n\}$

	classical, $\alpha = 2$	R-L. case	D-C case
λ_n =zeros of	$E_{2,2}(-\lambda)$	$E_{\alpha,\alpha}(-\lambda)$	$E_{\alpha,2}(-\lambda)$
eigenfunction	$\sin\sqrt{\lambda_n}x$	$x^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n x^{\alpha})$	$xE_{\alpha,2}(-\lambda_n x^{\alpha})$
asymptotic	$\lambda_n = (n\pi)^2$,	$\lvert\lambda_n\rvert \sim (n\pi)^{\alpha}$	$\lvert\lambda_n\rvert \sim (n\pi)^{\alpha}$
asymptotic	$\arg(\lambda_n) = 0$	$\arg(\lambda_n) \sim \frac{2-\alpha}{2}\pi$	$\arg(\lambda_n) \sim \frac{2-\alpha}{2}\pi$

- $q = 0$: Dirichlet eigenvalues of $-D^2$ = zeros of $E_{2,2}(-z) = \frac{\sinh(\sqrt{-z})}{\sqrt{-z}}$,
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- $q \in L^2$: asymptotics: $\lambda_n(0) = (n\pi)^2$, $arg(\lambda_n) = 0$
  $\lambda_n(q) = \lambda_n(0) + \int_0^1 q(x)dx + c_n$ $\{c_n\} \in \ell^2$: for smooth q,
  $lim_{n\to\infty} c_n \to 0$ rapidly. $\phi_n(x) = \frac{1}{\sqrt{\lambda_n}}\sin\sqrt{\lambda_n}x + O\left(\frac{1}{\lambda_n}\right)$.

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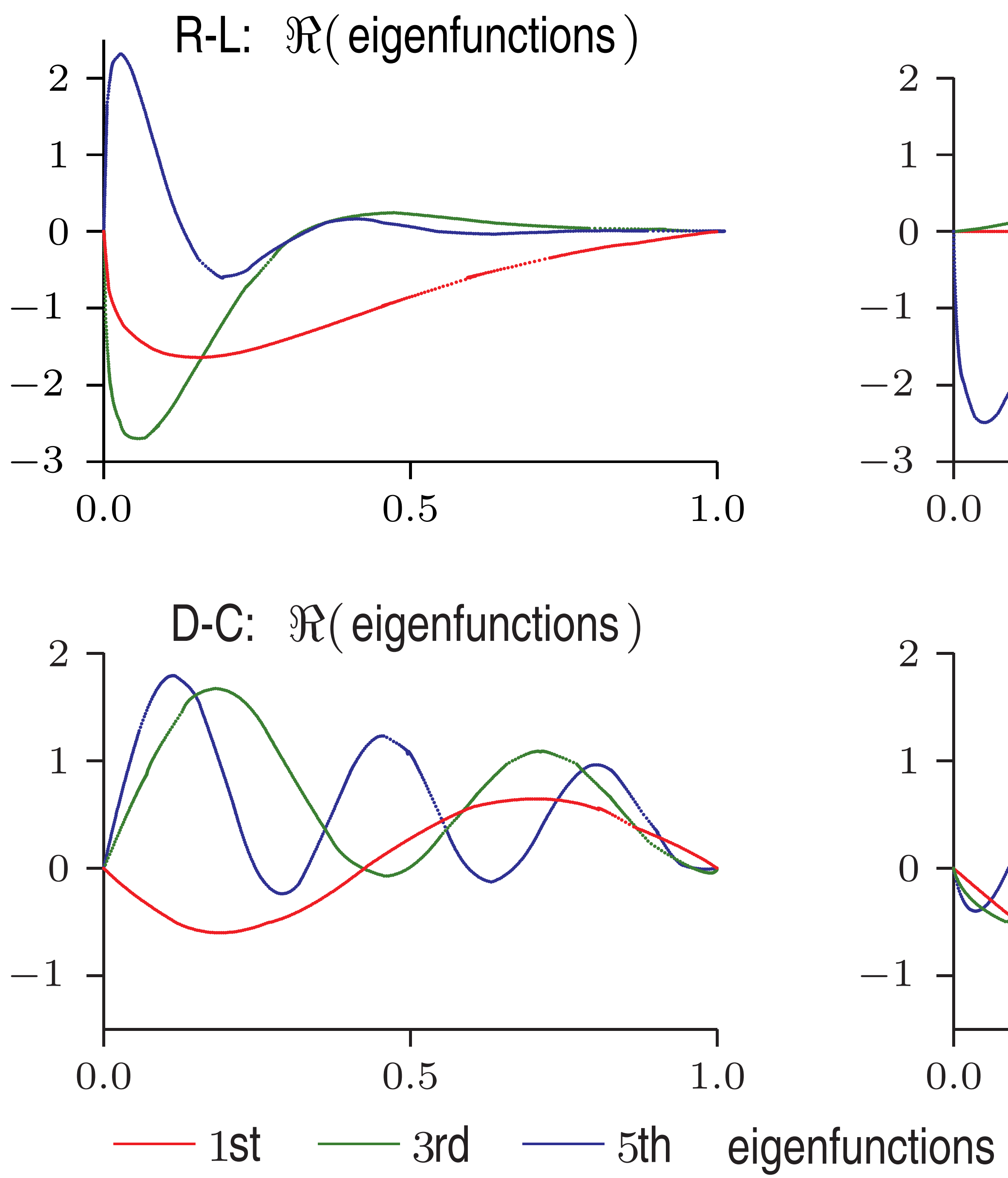
Asymptotically: zeros of $E_{\alpha,\alpha}(z)$ are distributed as

$$z_n^{\frac{1}{\alpha}} = 2n\pi\mathrm{i} - (\alpha-1)\left(\log 2\pi|n| + \frac{\pi}{2}\mathrm{sign}(n)\,\mathrm{i}\right) + \log(\alpha/\Gamma(2-\alpha)) + d_n$$

- Both fractional cases have complex eigenvalues:
- Computation of eigenvalues and eigenvectors quite tricky


R-L: ℜ(eigenfunctions)
R-L: ℑ(eigenfunctions)
D-C: ℜ(eigenfunctions)
D-C: ℑ(eigenfunctions)
2
1
0
−1
−2
−3
0.0
0.5
1.0
1st
3rd
5th
eigenfunctions

Inverse eigenvalue problem: recover $q(x)$ from the spectrum $\{\lambda_k\}$

- For general q this is *insufficient.*
- A second spectrum arising from a change of boundary conditions, or norming constants or endpoint data – in addition to the original, is sufficient.
- If q is symmetric about $x = \frac{1}{2}$ then a single spectrum suffices.
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In the case with the fractional operator $(-u_{xx})^\alpha$ in $H^2 \cap H_0^1(0,1)$:

- Same eigenfunctions sin$\sqrt{\lambda_n}$ - no fundamental change.

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In the Riemann-Liouville case with $1 < \alpha < 2$ we *observe* that

- If q is symmetric about $x = \frac{1}{2}$ then a single spectrum suffices. [Can prove that *at least* this much information is required].
- If q is known on $[0, \frac{1}{2}]$, a single spectrum allows unique recovery on $[\frac{1}{2}, 1]$
- There seems to be no difference in uniqueness in the general case.
- Analysis of uniqueness extremely difficult.
- Computations are much more complex.

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- **A single (Dirichlet) spectra is sufficient to recover a general q!!**

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In the Dzherbashyan-Caputo case with $1<\alpha<2$ we observe that

- **A single (Dirichlet) spectra is sufficient to recover a general q!!**
- The eigenvalues occur in complex-conjugate pairs.
- The real and imaginary parts of the eigenfunctions have resp. odd and even number of zeros and seem to span distinct subspaces and so give distinct information. But proving this . . . .
- Problem is only mildly ill posed for $1<\alpha<\frac{4}{3}$ but ill-conditioning increases rapidly with increasing α. For $\alpha>1.9$ condition numbers exceed 10^{10}.

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The audience for listening

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[†] B. Jin and W. Rundell: "A Tutorial on Inverse Problems for Anomalous Diffusion Processes," *Inverse Problems*, **31**, 035003, (2015).