Introduction to Linear Transport Theory

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Foreword

These lecture notes built upon the PhD thesis [17] and the research articles [13, 12, 11]. In these references the interested reader might find further references and more general statements of the results presented in this lecture. To preserve the introductory style we will often treat only simplified cases.
Since this is the first time this lecture has been held, there might be some typos in the script. The author is happy if you report these to schlottbom@wwu.de .

Contents

Chapter 1

The radiative transfer equation

The motion of particles is guided by different physical phenomena on different scales of interest. There are basically three different types of scales [10].

1. **Microscopic level: equations of motion:** Newton's law, Heisenberg equations.
2. **Mesoscopic level:** Liouville equation, Boltzmann equation, neutron transport equation, Vlasov equation.
3. **Macroscopic level:** Conservation laws or transport equations, e.g., Navier-Stokes equations, Euler equations, neutron diffusion equations.

In this lecture, we are interested in quantities which can be observed on the macroscopic level but which are substantially influenced by phenomena taking place on the mesoscopic level, i.e., we are interested in calculating the total photon flux leaving the medium of interest, but this flux is strongly influenced by physical phenomena on the mesoscopic level. In order to account for these phenomena on the mesoscopic level, we will investigate the *radiative transfer equation.* The structure of the radiative transfer equation can be seen as a prototype for many other transport (or kinetic) equations. Therefore, besides in radiative transfer through stellar atmospheres [7], this type of equation has many practically relevant applications, e.g., neutron transport [9, 5], or biomedical optics [18]; see also [10] for several other applications.

In the following, we will introduce the basic notation which is necessary to formulate a transport equation for photon propagation.

1.1 Basic definitions in radiative transfer

Although the following considerations are valid for different types of particles (neutrons, photons, etc.), we will define all quantities in terms of photons.

Before introducing the necessary notation, let us state some assumptions in order to clarify the physical phenomena we take into account. These assumptions are standard in linear transport theory [9, 5, 14].

(i) Photons are considered as points, i.e., the wave nature of photons is neglected.

(ii) Photons (particles) do not interact with each other.

(iii) Between interactions with the background medium photons travel along straight lines (with constant energy which is proportional to their frequency).

(iv) The material is isotropic, i.e., no distinguished direction exists, and time-independent, i.e., photons travel much faster than the background medium changes.

Let us begin with the basic quantity of interest.

Photon density: The function $\phi(r,s,t)$ describes the *density of photons* at a point r with direction s, at time t.

Let $\mathrm{d}A$ denote a small area with unit normal n. Moreover, let $\mathrm{d}r$ denote a small spatial volume element, $\mathrm{d}s$ a small angular volume/surface element and $\mathrm{d}t$ a small time interval. Then the expected number of photons in $\mathrm{d}r$ about r with directions lying in $\mathrm{d}s$ about s at time t is given by

$$\phi(r,s,t)\,\mathrm{d}r\,\mathrm{d}s.$$

Photon flux: The number of photons with directions in $\mathrm{d}s$ about s crossing $\mathrm{d}A$ in time t to $t+\mathrm{d}t$ is given by

$$n\cdot j(r,s,t)\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}A,$$

where $j(r,s,t) := cs\phi(r,s,t)$ is called the *photon flux* and c is the speed of light in the medium. Two physically relevant, i.e., measurable quantities are given next.

Total photon density: The function

$$\Phi(r,t) := \int_{\mathcal{S}} \phi(r,s,t)\,\mathrm{d}s \tag{1.1}$$

is called *total photon density* and $\Phi(r,t)\,\mathrm{d}r$ describes the total number of photons located in $\mathrm{d}r$ around r, at time t.

Total photon flux: The function

$$J(r,t) := \int_{\mathcal{S}} s\,\phi(r,s,t)\,\mathrm{d}s \tag{1.2}$$

is called *total photon flux*. The effective number of photons crossing a small area $\mathrm{d}A$ in time t to $t+\mathrm{d}t$ is given by $n\cdot J(r,t)\ \mathrm{d}A\,\mathrm{d}t$.

Sources: The density function $f(r,s,t)$ describes the number density of photons with direction s gained in r at t, i.e., $f(r,s,t)\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}t$ is the number of photons with directions in $\mathrm{d}s$ about s inserted into the medium at position $\mathrm{d}r$ about r between t and $t+\mathrm{d}t$.

Mean free path: Let us shortly describe the interaction phenomena of photons with the background medium. We denote by $l(r)$ the *mean free path* between interactions for a photon at position r, which by isotropy of the material, is independent of s. Thus, on average a photon will suffer $c/l(r)$ interactions per second at a point r.

Interaction rates: The inverse mean free path is called *transport* or *attenuation coefficient*, and is denoted by

$$\mu_t(r) := \frac{1}{l(r)}.$$

The transport coefficient $\mu_t(r)$ models the probability of particle interactions per unit distance traveled by a photon at position r. We will distinguish between two types of interactions, namely absorption and elastic scattering

$$\mu_t = \mu_a + \mu_s.$$

For instance, the scattering rate per unit distance for photons of velocity s at r is described by $\mu_s(r)$. The average number of photons after an interaction event is described by the fraction μ_s/μ_t. The behavior of the scattering events is specified next.

Scattering kernel: Since we have assumed an isotropic material, the probability for a photon with direction s' to be scattered into direction s only depends on $s\cdot s'$, the cosine of the angle between s and s', i.e., the collisions are rotationally invariant. Thus, the probability for a photon at position r with direction s' to be scattered into direction s can be described by a function $\theta(r, s \cdot s')$.

1.2 Derivation of the radiative transfer equation

A derivation of the radiative transfer equation based on balance laws can be found for instance in [5, 10, 14, 18]; for another derivation based on linearization of the Boltzmann equation see [6]. We will closely follow the presentation of [5].

In the previous section we have introduced different quantities which can affect the photon density $\phi(r, s, t)$. Either a photon is transported or it undergoes an interaction event. In order establish a relation between these effects, let us fix some small volume $V \subset \mathbb{R}^d$ with surface ∂V. The temporal change of the number of photons with a certain direction s within V in a small time interval $[t, t + \mathrm{d}t]$ is due to the number of photons which

(1.) leave or enter V through ∂V in $\mathrm{d}t$,

(2.) are absorbed or scattered into a different direction s' in $\mathrm{d}t$,

(3.) are gained due to a scattering event from a direction s' to s in $\mathrm{d}t$,

(4.) are emitted by source terms f in V in $\mathrm{d}t$.

Besides these balance considerations, also the relation

$$\text{change of number of photons in } V \text{ with direction } s \text{ about } \mathrm{d}s \text{ in } [t, t + \mathrm{d}t] = \mathrm{d}s\,\mathrm{d}t \int_V \frac{\partial \phi}{\partial t}\,\mathrm{d}r$$

holds true. Let us discuss the specific items in detail.

Item (1.) accounts for photons which leave or enter V with no change in velocity. Mathematically, this can be described by the surface integral of the angular flux

$$(1.) \;\; = \mathrm{d}s\,\mathrm{d}t \int_{\partial V} j(r, s, t) \cdot n \,\mathrm{d}\sigma$$

where n is the unit outward normal to ∂V at r. By the divergence theorem and the definition of j we obtain that

$$(1.) \;\; = \mathrm{d}s\,\mathrm{d}t \int_V c\, s \cdot \nabla \phi(r, s, t)\,\mathrm{d}r.$$

Item (2.) accounts for photons being absorbed or changing direction (without changing position). Thus, by definition of the collision rate we have that

$$(2.) \;\; = \mathrm{d}s\,\mathrm{d}t \int_V \frac{c}{l(r)}\, \phi(r, s, t)\,\mathrm{d}r.$$

Item (3.) accounts for photons scattered from directions $s' \in \mathcal{S}$ into direction $s \in \mathcal{S}$ (without changing position). Since this takes place with rate $c\,\mu_s$, we obtain by definition of θ the following relation

$$(3.) \;\; = \mathrm{d}s\,\mathrm{d}t \int_V c\,\mu_s(r) \int_{\mathcal{S}} \theta(r, s \cdot s')\phi(r, s', t)\,\mathrm{d}s'\,\mathrm{d}r.$$

Item (4.) accounts for photons emitted in V with direction s at time t

$$(4.) \;=\; \mathrm{d}s\,\mathrm{d}t \int_V f(r,s,t)\,\mathrm{d}r.$$

Since the control volume V was arbitrary, we conclude that the photon density ϕ must fulfill the following equation

$$\frac{\partial\phi(r,s,t)}{c\,\partial t} + s\cdot\nabla\phi(r,s,t) + \mu_t(r)\phi(r,s,t) = \mu_s(r)\int_{\mathcal{S}} \theta(r, s\cdot s')\phi(r,s',t)\,\mathrm{d}s' + \frac{1}{c}f(r,s,t). \qquad (1.3)$$

This is the *time-dependent one-speed (mono-chromatic) radiative transfer equation*; sometimes this equation is called *linear Boltzmann equation* or *linear transport equation.*

Remark 1.1. The one-speed (mono-chromatic) transport equation is also of interest if one can define clearly separated energy levels. If for example the energies lie in the interval $[E_1, E_2]$, the domain of integration on the right-hand side of (1.3) would have been $[E_1, E_2]\times\mathcal{S}$ instead of only $\mathcal{S}$. Using Fubini's theorem in order to split the integral $\int_{[E_1,E_2]\times\mathcal{S}} \;\mathrm{d}(s,E) = \int_{[E_1,E_2]}\int_{\mathcal{S}} \mathrm{d}s \;\mathrm{d}E$ and using an integration rule for the outer integral leads to an equation like (1.3), where the scattering kernel also acts between the different energy groups. This multi-group approximation arises for instances in neutron transport theory where the material properties for fast neutrons differ substantially from those for thermal neutrons [9].

Boundary and initial conditions: The photon density ϕ in a given domain $\mathcal{R}\subset\mathbb{R}^d$ with boundary $\partial\mathcal{R}$ is uniquely determined, if

i) the initial photon density ϕ_0,

ii) the sources f within $\mathcal{R}$, and

iii) the photon density g incident on $\partial\mathcal{R}$

are given [5, Chapter 2]. We therefore impose the following initial and boundary conditions

$$\begin{aligned} \phi(r,s,0) &= \phi_0(r,s) && \text{for all } r\in\mathcal{R},\ s\in\mathcal{S},\\ \phi(r,s,t) &= g(r,s,t) && \text{for all } t>0,\ r\in\partial\mathcal{R},\ s\in\mathcal{S} \text{ such that } s\cdot n<0. \end{aligned}$$

Remark 1.2. One can think of other boundary conditions as well. For example one could consider reflections at the boundary or periodic boundary conditions.

An integral formulation of the time-dependent radiative transfer equation: For later reference let us shortly describe an integral formulation of the transport equation [4, 5, 9, 16, 8]. The basic idea is that the homogeneous transport equation ($\theta = 0$, $q = 0$) decouples into a linear transport equation for each direction $s\in\mathcal{S}$:

$$\begin{aligned} \frac{\partial\phi(r,s,t)}{c\partial t} + s\cdot\nabla\phi(r,s,t) + \mu_t(r,s)\phi(r,s,t) &= 0 && \text{for } t\geq 0,\ r\in\mathcal{R},\\ \phi(r,s,0) &= \phi_0(r,s) && \text{for } r\in\mathcal{R},\\ \phi(r,s,t) &= 0 && \text{for } t>0,\ r\in\partial\mathcal{R} \text{ with } s\cdot n<0. \end{aligned}$$

A solution to the homogeneous case is given by integration along the characteristics $r - st$, i.e., for $r\in\mathcal{R},\ s\in\mathcal{S}$ and $t\geq 0$

$$\phi_{hom}(r,s,t) = \phi_0(r-st,s)\exp\Big(-\int_0^t \mu_t(r+(t'-t)s)\;\mathrm{d}t'\Big),$$

where we extend all functions by zero on $\mathbb{R}^d \setminus \mathcal{R}$ [8]. By the variation of constants formula a solution ϕ of the transport equation thus satisfies [8]

$$\phi(r,s,t) = \phi_{hom}(r,s,t) + \phi_{part}(r,s,t), \tag{1.4}$$

where the particular solution ϕ_{part} is given by the formula

$$\phi_{part}(r,s,t) = \int_0^t \exp\Big(- \int_0^{t'} \mu_t(r + (t''-t)s)\,\mathrm{d}t''\Big) F(r - t's, s, t-t')\,\mathrm{d}t'$$

with

$$F(r,s,t) := \mu_s(r) \int_{\mathcal{S}} \theta(r, s\cdot s')\phi(r,s',t)\,\mathrm{d}s' + \frac{1}{c} f(r,s,t).$$

We will use the integral formulation (1.4) to prove existence of solutions to the stationary radiative transfer equation via Banach's fixed point theorem. The Picard iteration is closely related to the "source iteration" which is one method to obtain unique solvability of the radiative transfer equation.

Let us shortly discuss two simplifications of the time-dependent radiative transfer equation which are of particular interest in optical tomography:

Stationary radiative transfer equation: In optical tomography the typical length scale of an object of interest is 10 to 100 mm. Since the speed of light is approximately 0.3 mm/ps the average residence time of a photon in the object is in the order of nanoseconds. Thus, if the source terms are constant in time, a stationary state $\phi(r,s)$ will be reached immediately. The radiative transfer equation (1.3) then reduces to the following equation

$$s\cdot\nabla\phi(r,s) + \mu_t(r,s)\phi(r,s) = \mu_s(r)\int_{\mathcal{S}} \theta(r, s\cdot s')\phi(r,s')\,\mathrm{d}s' + \frac{1}{c} f(r,s) \tag{1.5}$$

for $r \in \mathcal{R}$ and $s \in \mathcal{S}$. We will complement this equation by the following inflow boundary condition

$$\phi(r,s) = g(r,s) \quad \text{for all } r \in \partial\mathcal{R},\ s \in \mathcal{S} \text{ such that } s\cdot n < 0. \tag{1.6}$$

Time-harmonic radiative transfer equation: Another important case arises in optical tomography when time-harmonic (intensity modulated) source terms are utilized, i.e.,

$$f(r,s,t) = f(r,s)\exp(i\omega t) \quad \text{and} \quad g(r,s,t) = g(r,s)\exp(i\omega t)$$

with *modulation frequency* ω, which is usually given in hundreds of MHz. If we expect the solution to be time-harmonic as well, that is,

$$\phi(r,s,t) = \phi(r,s)\exp(i\omega t),$$

the time-dependent radiative transfer equation (1.3) reduces to

$$s\cdot\nabla\phi(r,s) + (\mu_t(r,s) + ik)\phi(r,s) = \mu_s(r)\int_{\mathcal{S}} \theta(r, s\cdot s')\phi(r,s')\,\mathrm{d}s' + \frac{1}{c} f(r,s), \tag{1.7}$$

where $k = \omega/c$ is the *wave number*. In order to complement (1.7), consider the boundary condition (1.6), which transforms to

$$\phi(r,s) = g(r,s) \quad \text{for all } r \in \partial\mathcal{R},\ s \in \mathcal{S} \text{ such that } s\cdot n < 0. \tag{1.8}$$

Chapter 2

Functions spaces and traces

A main difference to more standard elliptic equations are the functions spaces used to analyze radiative transfer problems. These spaces contain functions with anisotropic regularity, that is, the functions possess derivatives only in certain directions.

In this chapter, we introduce the basic notation used in this work. In Section **??**, we will introduce and investigate function spaces and their corresponding trace spaces. In particular, a Poincaré-Friedrichs type inequality and an integration-by-parts formula are given, cf. Lemma 2.10 and Lemma 2.8, see also [1, 3, 8, 15].

2.0.1 Geometry

Let $\mathcal{R} \subset \mathbb{R}^d$ denote a convex domain with C^1 boundary, and denote by $n(r) \in \mathcal{S}$ the continuous unit outward pointing normal vector for a point $r \in \partial\mathcal{R}$. Furthermore, let $\mathcal{S} = \{s \in \mathbb{R}^d : |s| = 1\}$ denote the unit sphere. We define the product domain $\mathcal{D} := \mathcal{R} \times \mathcal{S}$. The boundary $\partial\mathcal{D} := \partial\mathcal{R} \times \mathcal{S}$ of $\mathcal{D}$, cf. Figure 2.1, can be decomposed into an inflow part

$$\partial\mathcal{D}_- = \Gamma_- = \{(r,s) \in \partial\mathcal{D} : s \cdot n < 0\},$$

an outflow part

$$\partial\mathcal{D}_+ = \Gamma_+ = \{(r,s) \in \partial\mathcal{D} : s \cdot n > 0\},$$

and a remaining tangential part

$$\partial\mathcal{D}_0 = \Gamma_0 = \{(r,s) \in \partial\mathcal{D} : s \cdot n = 0\}.$$

For integration on $\mathcal{D}$ we use the measure $\mathrm{d}(r,s) = \mathrm{d}r\,\mathrm{d}s$ which is the product measure of the d-dimensional Lebesgue measure and the surface measure on $\mathcal{S}$. Similarly, for integation along $\partial\mathcal{D}$ we use $\mathrm{d}\sigma(r)\,\mathrm{d}s$ which is the product measure of the surface measure on $\partial\mathcal{R}$ and the surface measure on $\mathcal{S}$.

Lemma 2.1. *The in- and outflow boundaries $\partial\mathcal{D}_-$ and $\partial\mathcal{D}_+$ are open subsets of $\partial\mathcal{D}$, and $\partial\mathcal{D}_0$ is a closed subset of $\partial\mathcal{D}$ with $(2d-2)$-dimensional measure zero.*

Proof. Due to the regularity of the boundary, the mapping $(r,s) \mapsto s \cdot n$ is continuous, and hence the set $\partial\mathcal{D}_0$ is closed. With the same arguments, $\partial\mathcal{D}_-$ and $\partial\mathcal{D}_+$ are open. Since $\partial\mathcal{R} \in C^1$, it is locally diffeomorphic to a subset of $\mathbb{R}^{d-1}$. A standard parametrization of the sphere $\mathcal{S}$ and the product structure of $\partial\mathcal{D}_-$, $\partial\mathcal{D}_+$, $\partial\mathcal{D}_0 \subset \partial\mathcal{R} \times \mathcal{S}$ yield the assertion. $\square$

Lemma 2.1 allows us to identify measurable functions defined on $\partial\mathcal{D}$ with those defined on $\partial\mathcal{D}_- \cup \partial\mathcal{D}_+$. Here and below, the subscript $\pm$ is used to treat the two cases $+$ (outflow) and $-$ (inflow) simultaneously.

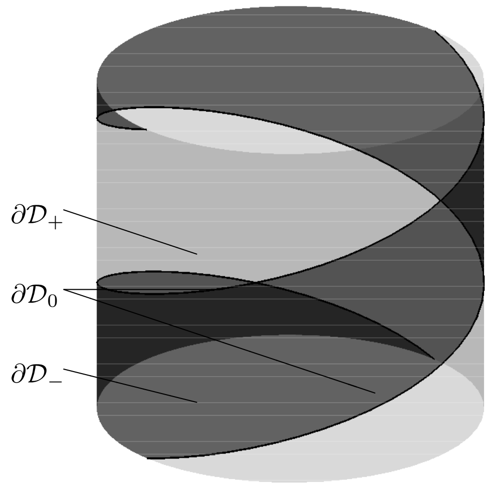

Figure 2.1: Schematic diagram of the domain $\mathcal{D} = \mathcal{R} \times \mathcal{S}$, with $\mathcal{R}$ being the two-dimensional unit circle. The angular domain $\mathcal{S}$ is identified with the interval $[0, 2\pi)$ via $s = (\cos(\alpha), \sin(\alpha))^\top$. The inflow part $\partial\mathcal{D}_-$ is shaded in dark-gray, while the outflow boundary $\partial\mathcal{D}_+$ is in light-gray. The separating black curves denote $\partial\mathcal{D}_0$.

2.0.2 The spaces W^p and $\widetilde{W}^p$

We begin with

$$C_0^\infty(\mathcal{D}) = \{\psi \in C^\infty(\mathcal{D}) : \operatorname{supp}(\psi) \subset K \times \mathcal{S} \text{ for some compact set } K \subset \mathcal{R}\}.$$

For $1 \le p < \infty$, we denote by

$$L^p(\mathcal{D}) = L^p(\mathcal{R} \times \mathcal{S}) = \{v : \mathcal{D} \to \mathbb{R} : v \text{ is measurable and } \int_\mathcal{R} \int_\mathcal{S} |v(r, s)|^p \,\mathrm{d}s \,\mathrm{d}r < \infty\}$$

the space of (equivalence classes) of measurable functions, whose pth power is integrable. The space $L^\infty(\mathcal{D})$ consists of measurable functions, which are essentially bounded. The spaces $L^p(\mathcal{D})$, $1 \le p \le \infty$ are Banach spaces (Theorem of Riesz-Fischer). $L^2(\mathcal{D})$ is a Hilbert space when endowed with the scalar product

$$(v, w)_{L^2(\mathcal{D})} := (v, w)_\mathcal{D} := \int_\mathcal{R} \int_\mathcal{S} v(r, s) w(r, s) \,\mathrm{d}s \,\mathrm{d}r.$$

Similar notation is used for scalar products defined as integrals over other domains, and the norm associated with a scalar product $(\,\cdot\,,\,\cdot\,)_*$ is always denoted by $\|v\|_* := \sqrt{(v, v)_*}$.

Lemma 2.2. *$C_0^\infty(\mathcal{D})$ is dense in $L^p(\mathcal{D})$ for $1 \le p < \infty$.*

Proof. Via convolution and parametrization of $\mathcal{S}$. □

For $u \in C^1(\mathcal{R})$ and $s \in \mathcal{S}$ the directional derivative is defined as

$$s \cdot \nabla u(r) = \frac{\partial}{\partial t} u(r + ts)_{|t=0}, \quad r \in \mathcal{R}.$$

Similarly, we define $s \cdot \nabla v(r, s)$ for a smooth function $v : \mathcal{D} \to \mathbb{R}$. We define

$$W^p = \overline{C^\infty(\overline{\mathcal{D}})}^{\|\cdot\|_{W^p}}, \qquad \|v\|_{W^p}^p = \|v\|_{L^p(\mathcal{D})}^p + \|s \cdot \nabla v\|_{L^p(\mathcal{D})}^p.$$

One can show that $W^p = W^p(\mathcal{D}) = \{v \in L^p(\mathcal{D}) : s \cdot \nabla v \in L^p(\mathcal{D})\}$. Here, we say that a function $v \in L^p(\mathcal{D})$ has a weak directional derivative $w \in L^p(\mathcal{D})$ if

$$\int_{\mathcal{D}} w(r,s)\psi(r,s)\,\mathrm{d}(r,s) = -\int_{\mathcal{D}} v(r,s)s \cdot \nabla\psi(r,s)\,\mathrm{d}(r,s) \quad \text{for all } \psi \in C_0^\infty(\mathcal{D}).$$

We write $w = s \cdot \nabla v$.

If $v \in W^1$, then we can define for a.e. $(r,s) \in \partial\mathcal{D}_-$, the function $u(t) = v(r+ts,s)$, $0 \leq t \leq \tau(r,s)$ defined in (2.2). Since $u'(t) = s \cdot \nabla v(r+ts,s)$, we have $u \in W^{1,1}(0,\tau(r,s))$. Hence, $u \in C^0([0,\tau(r,s)])$, and for a.e. $s \in \mathcal{S}$ and any $t_1, t_2 \in [0,\tau(r,s)]$

$$v(r+t_2s,s) - v(r+t_1s,s) = \int_{t_1}^{t_2} s \cdot \nabla v(r+ts,s)\,\mathrm{d}t. \tag{2.1}$$

In view of (2.1) a function $v \in W^p$ is defined a.e. on $\partial\mathcal{D}$. In order to characterize the function space containing these traces we need some more tools, see also [3, 15, 8, 1].

For a point $(r,s) \in \partial\mathcal{D}_\pm$ we define the *time of travel* $\tau(r,s)$ by

$$\tau(r,s) := \sup\{t > 0 : r \mp t's \in \mathcal{R} \text{ for all } 0 < t' < t\}, \tag{2.2}$$

i.e., $\tau(r,s)$ is the length of the longest line segment through r with direction s lying completely in $\overline{\mathcal{R}}$ [8, 15], cf. Figure 2.2. Furthermore, we set $\tau(r+ts,s) = \tau(r,s)$ for $0 \leq t \leq \tau(r,s)$ and $(r,s) \in \partial\mathcal{D}_-$, i.e.

$$s \cdot \nabla\tau = 0. \tag{2.3}$$

For fixed $s \in \mathcal{S}$, there holds

$$\overline{\mathcal{R}} = \{r + ts : \ r \in \partial\mathcal{R} \text{ such that } s \cdot n(r) < 0, \text{ and } 0 \leq t \leq \tau(r,s)\}.$$

Lemma 2.3. *For any $u \in L^1(\mathcal{R})$ there holds*

$$\int_{\mathcal{R}} u(r)\,\mathrm{d}r = \int_{\{r\in\partial\mathcal{R}: s\,\cdot\, n(r)<0\}} \int_0^{\tau(r,s)} u(r+ts)|s \cdot n(r)|\,\mathrm{d}t\,\mathrm{d}\sigma(r), \tag{2.4}$$

and

$$\int_{\mathcal{R}} u(r)\,\mathrm{d}r = \int_{\{s\,\cdot\, n>0\}} \int_0^{\tau(r,s)} u(r-ts)|s \cdot n|\,\mathrm{d}t\,\mathrm{d}\sigma. \tag{2.5}$$

Proof. Let $s \in \mathcal{S}$ be fixed. Let $\tilde{r} \in \mathcal{R}$ be arbitrary. Choose $r \in \partial\mathcal{R}$ such that $n(r) \cdot s < 0$ and $\tilde{r} = r + \tilde{t}s$ for some $0 \leq \tilde{t} \leq \tau(r,s)$. Therefore, we can define for a.e. $\tilde{r} \in \mathcal{R}$

$$v(\tilde{r},s) = v(r+\tilde{t}s,s) = \int_{\tilde{t}}^{\tau(r,s)} u(r+ts)\,\mathrm{d}t.$$

Then, by definition of the directional derivative,

$$\begin{aligned} s \cdot \nabla v(\tilde{r},s) &= -u(\tilde{r}) \quad \text{in } \mathcal{D}, \\ v(\tilde{r},s) &= 0 \quad \text{for } \tilde{r} \in \partial\mathcal{R} \text{ with } n(\tilde{r}) \cdot s > 0. \end{aligned}$$

Since $s \cdot \nabla v = \operatorname{div}(sv) \in L^1(\mathcal{R})$, the divergence theorem yields

$$\int_{\mathcal{R}} u(\tilde{r})\,\mathrm{d}\tilde{r} = -\int_{\partial\mathcal{R}} v(\tilde{r},s)s \cdot n(\tilde{r})\,\mathrm{d}\sigma(\tilde{r}) = \int_{\{s\,\cdot\, n<0\}} v(\tilde{r},s)|s \cdot n(\tilde{r})|\,\mathrm{d}\sigma(\tilde{r}).$$

If $\tilde{r} \in \partial\mathcal{R}$ is such that $n(\tilde{r}) \cdot s < 0$, then $\tilde{t} = 0$ and $r = \tilde{r}$. Therefore, we conclude with

$$\int_{\{s\,\cdot\, n<0\}} v(\tilde{r},s)|s \cdot n(\tilde{r})|\,\mathrm{d}\sigma(\tilde{r}) = \int_{\{s\,\cdot\, n<0\}} \int_0^{\tau(r,s)} u(r+ts)\,\mathrm{d}t|s \cdot n(r)|\,\mathrm{d}\sigma(r).$$

Eq. (2.5) follows similarly. ☐

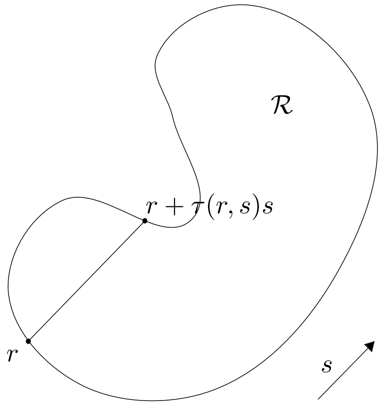

Figure 2.2: Travel time $\tau(r,s)$ for $r \in \partial\mathcal{R}$ and $s \in \mathcal{S}$ where $\mathcal{R}$ is the bean-shaped object.

As we will see below, we can define traces for functions in the space

$$\widetilde{W}^p = \overline{C^\infty(\overline{\mathcal{D}})}^{\|\cdot\|_{\widetilde{W}^p}}, \quad \|v\|^p_{\widetilde{W}^p} = \|v\|^p_{L^p(\mathcal{D})} + \|\tau s\cdot\nabla v\|^p_{L^p(\mathcal{D})}.$$

Since $\tau \in L^\infty(\mathcal{D})$, we have that $W^p \subset \widetilde{W}^p$.

Lemma 2.4. *The partial trace operators $v \mapsto v_{|\partial\mathcal{D}_\pm}$ defined for continuous functions can be extended to surjective, bounded linear operators $\gamma_\pm : \widetilde{W}^p \to L^p(\partial\mathcal{D}_\pm; \tau|s\cdot n|)$ with*

$$\|\gamma_\pm(v)\|_{L^p(\mathcal{D}_\pm;\tau|s\cdot n|\,\mathrm{d}\sigma\,\mathrm{d}s)} \le p^{\frac{1}{p}}\|v\|_{\widetilde{W}^p}.$$

Proof. Let $p=1$ and $v \in C^\infty(\overline{\mathcal{D}})$ and $(r,s) \in \partial\mathcal{D}_-$, let $\tilde{r} = r + \tilde{t}s \in \overline{\mathcal{R}}$, $\tilde{t} \in [0,\tau(r,s)]$, be the first point where $|v(\tilde{r},s)|$ is minimal. In particular, there holds

$$|v(\tilde{r},s)|\tau(r,s) \le \int_0^{\tau(r,s)} |v(r+ts,s)|\,\mathrm{d}t. \tag{2.6}$$

By (2.1) we obtain that

$$v(r,s) = v(\tilde{r},s) - \int_0^{\tilde{t}} s\cdot\nabla v(r+ts,s)\,\mathrm{d}t. \tag{2.7}$$

Integrating (2.7) over $\partial\mathcal{D}_-$ with measure $\tau(r,s)|s\cdot n|\,\mathrm{d}\sigma\,\mathrm{d}s$, and using (2.4), (2.6) yields

$$\begin{aligned}
\int_{\partial\mathcal{D}_-} |v(r,s)|\tau(r,s)|s\cdot n|\,\mathrm{d}\sigma\,\mathrm{d}s &\le \int_{\partial\mathcal{D}_-} |v(\tilde{r},s)|\tau(r,s)|s\cdot n|\,\mathrm{d}\sigma\,\mathrm{d}s \\
&\quad + \int_{\mathcal{S}}\int_{\{s\cdot n<0\}}\int_0^t |s\cdot\nabla v(r+ts,s)||s\cdot n|\tau(r,s)\,\mathrm{d}t\,\mathrm{d}\sigma\,\mathrm{d}s \\
&\le \int_{\mathcal{S}}\int_{\{s\cdot n<0\}}\int_0^{\tau(r,s)} |v(r+ts,s)||s\cdot n|\,\mathrm{d}t\,\mathrm{d}\sigma\,\mathrm{d}s \\
&\quad + \int_{\mathcal{S}}\int_{\{s\cdot n<0\}}\int_0^{\tau(r,s)} |s\cdot\nabla v(r+ts,s)|\tau(r,s)|s\cdot n|\,\mathrm{d}t\,\mathrm{d}\sigma\,\mathrm{d}s \\
&= \|v\|_{L^1(\mathcal{D})} + \|\tau s\cdot\nabla v\|_{L^1(\mathcal{D})}.
\end{aligned}$$

For the general case $p>1$, set $\tilde{v} = |v|^p$, and observe that

$$\tau|s\cdot\nabla\tilde{v}| = \tau|p|v|^{p-2}vs\cdot\nabla v| = p\tau|v|^{p-1}|s\cdot\nabla v| \le (p-1)|v|^p + |\tau s\cdot\nabla v|^p.$$

Therefore,

$$\int_{\partial\mathcal{D}_-} |v(r,s)|^p \tau(r,s)|s\cdot n|\,\mathrm{d}\sigma\,\mathrm{d}s \le p\|v\|^p_{L^p(\mathcal{D})} + \|\tau s\cdot\nabla v\|^p_{L^p(\mathcal{D})}.$$

To show surjectivity of the trace mapping, let $g \in L^p(\partial\mathcal{D}_-;\tau|s\cdot n|)$ and define

$$v(r+ts,s) = g(r,s) \qquad \text{for a.e. } (r,s)\in\partial\mathcal{D}_-, \quad 0\le t\le \tau(r,s).$$

By construction v is constant along the line $r+ts$, i.e. $s\cdot\nabla v = 0$. Moreover, by (2.4) there holds

$$\begin{aligned}\|g\|^p_{L^p(\partial\mathcal{D}_-;\tau|s\cdot n|)} &= \int_{\partial\mathcal{D}_-} |g(r,s)|^p\tau(r,s)|s\cdot n|\,\mathrm{d}\sigma\,\mathrm{d}s\\ &= \int_{\mathcal{S}}\int_{\{s\cdot n<0\}}\int_0^{\tau(r,s)} |v(r+ts,s)|^p|s\cdot n|\,\mathrm{d}t\,\mathrm{d}\sigma\,\mathrm{d}s = \|v\|^p_{L^p(\mathcal{D})}.\end{aligned}$$

The result for the outflow part follows in the same way due to (2.5). □

Lemma 2.5 (Poincaré-Friedrichs inequality 1)**.** *For $v\in\widetilde{W}^p$ there holds*

$$\|v\|^p_{L^p(\mathcal{D})} \le 2^{p-1}\left(\|v\|^p_{L^p(\partial\mathcal{D}_-;\tau|s\cdot n|)} + \|\tau s\cdot\nabla v\|^p_{L^p(\mathcal{D})}\right).$$

Proof. Let $s\in\mathcal{S}$, $r\in\partial\mathcal{R}$ such that $s\cdot n<0$, and $0\le t\le\tau(r,s)$. For $v\in C^\infty(\overline{D})$ we obtain by the fundamental theorem of calculus

$$|v(r+ts,s)|^p \le 2^{p-1}\left(|v(r,s)|^p + \tau(r,s)^{p-1}\int_0^{\tau(r,s)} |s\cdot\nabla v(r+ts,s)|^p\,\mathrm{d}t\right).$$

Integrating gives with (2.4)

$$\begin{aligned}\|v\|^p_{L^p(\mathcal{D})} &= \int_{\partial\mathcal{D}_-}\int_0^{\tau(r,s)} |v(r+ts,s)|^p\,\mathrm{d}t|s\cdot n|\,\mathrm{d}\sigma\,\mathrm{d}s\\ &\le 2^{p-1}\int_{\partial\mathcal{D}_-}\left(|v(r,s)|^p\tau(r,s) + \tau(r,s)^p\int_0^{\tau(r,s)}|s\cdot\nabla v(r+ts,s)|^p\,\mathrm{d}t\right)|s\cdot n|\,\mathrm{d}\sigma\,\mathrm{d}s\\ &= 2^{p-1}\left(\|v\|^p_{L^p(\partial\mathcal{D}_-;\tau|s\cdot n|)} + \|\tau s\cdot\nabla v\|^p_{L^p(\mathcal{D})}\right).\end{aligned}$$

□

Corollary 2.6. *Let $v:\mathcal{D}\to\mathbb{R}$ be a measurable function such that $\tau s\cdot\nabla v\in L^p(\mathcal{D})$. Then the following statements are equivalent:*
(i) $v\in L^p(\mathcal{D})$.
(ii) $v\in L^p(\partial\mathcal{D}_-;\tau|s\cdot n|)$.
(iii) $v\in L^p(\partial\mathcal{D}_+;\tau|s\cdot n|)$.

Proof. (i) implies (ii) and (iii) according to Lemma 2.4. According to Lemma 2.5 (ii) implies (i). Similarly, (iii) implies (i). □

Since by Lemma 2.1, the boundary $\partial\mathcal{D}$ coincides with the union of $\partial\mathcal{D}_+$ and $\partial\mathcal{D}_-$ up to a set of measure zero, we have

$$W^p = \{v\in L^p(\mathcal{D}) : s\cdot\nabla v\in L^p(\mathcal{D}) \text{ and } \gamma_\pm(v)\in L^p(\partial\mathcal{D}_\pm;\tau|s\cdot n|)\}.$$

and W^2 is a Hilbert space when equipped with the inner product

$$(v,w)_{W^2} = (v,w)_{\mathcal{D}} + (s\cdot\nabla v, s\cdot\nabla w)_{\mathcal{D}} + (\tau|s\cdot n|\,v,w)_{\partial\mathcal{D}}.$$

2.0.3 The spaces $\mathcal{V}^p$ and $\mathcal{W}^p$

The spaces W^p defined in the previous section are not sufficient for our analysis. To see this let $w, v \in C^\infty(\overline{\mathcal{D}})$. Then, integration by parts shows that

$$\int_{\mathcal{S}} \int_{\mathcal{R}} s \cdot \nabla v w \, \mathrm{d}r \, \mathrm{d}s = -\int_{\mathcal{S}} \int_{\mathcal{R}} v s \cdot \nabla w \, \mathrm{d}r \, \mathrm{d}s + \int_{\mathcal{S}} \int_{\partial\mathcal{R}} v w s \cdot n \, \mathrm{d}\sigma(r) \, \mathrm{d}s.$$

The term, involving the boundary integral is not well-defined for functions in W^2. Therefore, for smooth functions $w, v \in C^\infty(\overline{\mathcal{D}})$ let us define a norm by

$$\|v\|_{\mathcal{V}^p}^p = \|\tau^{-\frac{1}{p}} v\|_{L^p(\mathcal{D})}^p + \|\tau^{1-\frac{1}{p}} s \cdot \nabla v\|_{L^p(\mathcal{D})}^p, \tag{2.8}$$

and let $\mathcal{V}^p$ denote the completion of $C^\infty(\overline{\mathcal{D}})$ with respect to the associated norm.

Since $0 \le \tau(r,s) \le \operatorname{diam}(\mathcal{R}) < \infty$ on $\partial\mathcal{D}$, integrability with respect to $\tau(r,s)|s \cdot n| \, \mathrm{d}s \, \mathrm{d}\sigma$ is a weaker condition than integrability with respect to $|s \cdot n| \, \mathrm{d}s \, \mathrm{d}\sigma$. Next, we will see that traces of functions in $\mathcal{V}^p$ have the required regularity. We denote by $L^p(\partial\mathcal{D}_\pm; |s \cdot n|)$ the completion of $L^p(\partial\mathcal{D}_\pm)$ with respect to $\|\cdot\|_{L^p(\partial\mathcal{D}_\pm; |s \cdot n|)}$.

Lemma 2.7. *The trace mappings $v \mapsto v|_{\partial\mathcal{D}}$ defined for $v \in C^1(\overline{\mathcal{D}})$ can be extended by continuity to bounded linear operators $\gamma_\pm : \mathcal{V}^p \to L^p(\partial\mathcal{D}_\pm; |s \cdot n|)$ with $\|\gamma_\pm(v)\|_{L^p(\partial\mathcal{D}_\pm; |s \cdot n|)} \le p^{1/p} \|v\|_{\mathcal{V}^p}$. The mappings $\gamma_\pm$ are surjective.*

Proof. The assertion follows from Lemma 2.4 and (2.3). To see this, let $v \in \mathcal{V}^p$ and set $\tilde{v} = \tau^{-\frac{1}{p}} v$. Then, using (2.3) we see that

$$\tilde{v} \in L^p(\mathcal{D}) \quad \text{and} \quad \tau s \cdot \nabla \tilde{v} = s \cdot \nabla(\tau^{1-\frac{1}{p}} v) = \tau^{1-\frac{1}{p}} s \cdot \nabla v \in L^p(\mathcal{D}),$$

i.e. $\tilde{v} \in \widetilde{W}^p$ and $\|\tilde{v}\|_{\widetilde{W}^p} = \|v\|_{\mathcal{V}^p}$. By Lemma 2.7, $\tilde{v} \in L^p(\partial\mathcal{D}_\pm; \tau|s \cdot n|)$. The assertion follows from

$$\int_{\partial\mathcal{D}_\pm} |\tilde{v}|^p \tau |s \cdot n| \, \mathrm{d}\sigma \, \mathrm{d}s = \int_{\partial\mathcal{D}_\pm} |v|^p |s \cdot n| \, \mathrm{d}\sigma \, \mathrm{d}s.$$

□

Lemma 2.8 (Poincaré-Friedrichs inequality)**.** *For $v \in \mathcal{V}^p$ there holds*

$$\|\tau^{-\frac{1}{p}} v\|_{L^p(\mathcal{D})}^p \le 2^{p-1} \left(\|v\|_{L^p(\partial\mathcal{D}_-; |s \cdot n|)}^p + \|\tau^{1-\frac{1}{p}} s \cdot \nabla v\|_{L^p(\mathcal{D})}^p \right).$$

Proof. The assertion follows from Lemma 2.5 and (2.3) with $\tilde{v} = \tau^{-\frac{1}{p}} v$ for $v \in \mathcal{V}^p$. □

Corollary 2.9. *Let $v : \mathcal{D} \to \mathbb{R}$ be a measurable function such that $\tau^{1-\frac{1}{p}} s \cdot \nabla v \in L^p(\mathcal{D})$. Then the following statements are equivalent:*
(i) $\tau^{-\frac{1}{p}} v \in L^p(\mathcal{D})$.
(ii) $v \in L^p(\partial\mathcal{D}_-; |s \cdot n|)$.
(iii) $v \in L^p(\partial\mathcal{D}_+; |s \cdot n|)$.

Proof. (i) implies (ii) and (iii) according to Lemma 2.7. According to Lemma 2.8 (ii) implies (i). Similarly, (iii) implies (i). □

For further details, we refer to [1, 3, 8, 12, 15]. The following integration-by-parts formula will be a central tool in the derivation of a variational framework for the radiative transfer equation. Note, that it does not hold for arbitrary functions in space W^2, since the boundary values do not have the required regularity.

Lemma 2.10 (Integration by parts)**.** *For any pair of functions v, $w \in \mathcal{V}^2$ there holds*

$$(s \cdot \nabla v, w)_{\mathcal{D}} = -(v, s \cdot \nabla w)_{\mathcal{D}} + (s \cdot n\, v, w)_{\partial\mathcal{D}}. \tag{2.9}$$

Proof. For smooth functions $v \in C^\infty(\overline{\mathcal{D}})$, the formula is a direct consequence of Green's theorem, and the result then follows by density of $C^\infty(\overline{\mathcal{D}}) \subset \mathcal{V}^2$ and Lemma 2.7. □

Remark 2.11. It is easy to see that $\mathcal{V}^1 \subset W^1$ and $W^\infty \subset \mathcal{V}^\infty$, and $\widetilde{W}^\infty = \mathcal{V}^\infty$. A further function space is

$$\mathcal{W}^p = \{v \in W^p : \gamma_-(v) \in L^p(\partial\mathcal{D}_-; |s \cdot n|)\}.$$

Since $\tau \in L^\infty(\mathcal{D})$, we have that $\tau^{1-\frac{1}{p}} s \cdot \nabla v \in L^p(\mathcal{D})$ for any $s \cdot \nabla v \in L^p(\mathcal{D})$. Using Corollary 2.9, we see that $\mathcal{W}^p \subset \mathcal{V}^p$ for all $1 \leq p \leq \infty$ and $\mathcal{W}^1 = \mathcal{V}^1$. In particular, the following Poincaré-Friedrichs inequality holds true

$$\|v\|_{L^p(\mathcal{D})} \leq C(\|v\|_{L^p(\partial\mathcal{D}_-; |s \cdot n|)} + \|s \cdot \nabla v\|_{L^p(\mathcal{D})}), \qquad \text{for } v \in \mathcal{W}^p. \tag{2.10}$$

The constant C depends on $\operatorname{diam}(\mathcal{R})$ and p only.

Chapter 3

Existence theory

By means of a fixed-point argument we can show under certain conditions on the data that the stationary radiative transfer equation has a unique solution. We consider

$$s \cdot \nabla\phi(r,s) + \mu_t(r)\phi(r,s) = \mu_s(r)\int_{\mathcal{S}} \theta(r, s\cdot s')\phi(r,s')\,\mathrm{d}s' + f(r,s) \quad \text{for } (r,s) \in \mathcal{D} \tag{3.1}$$

$$\phi(r,s) = g(r,s) \qquad \text{where } n(r)\cdot s < 0. \tag{3.2}$$

We make the following assumptions:

(A1) $\mu_t, \mu_s : \mathcal{R} \to \mathbb{R}$ are non-negative and $\mu_t, \mu_s \in L^\infty(\mathcal{R})$ and

$$\mu_t - \mu_s \geq 0.$$

(A2) $\theta : \mathcal{R} \times [-1,1] \to \mathbb{R}$ is non-negative and measurable and for each $(r,s) \in \mathcal{D}$

$$\int_{\mathcal{S}} \theta(r, s\cdot s')\,\mathrm{d}s' = 1.$$

In Section 3.4 we will prove well-posedness of the radiative transfer equation (3.1)–(3.2).

Theorem 3.1. *Let (A1)–(A2) hold. Then for all* $1 \leq p \leq \infty$ *and for* $f \in L^p(\mathcal{D}; \tau^{p-1})$, $g \in L^p(\partial\mathcal{D}_-; |s\cdot n|)$, *the radiative transfer problem* (3.1)–(3.2) *admits a unique solution* $\phi \in \mathcal{V}^p$ *that satisfies*

$$\|\phi\|_{\mathcal{V}^p} \leq C\big(\|\tau^{1-\frac{1}{p}} f\|_{L^p(\mathcal{D})} + \|g\|_{L^p(\Gamma_-; |n\cdot s|)}\big), \tag{3.3}$$

with $C = (3 + 2\|\tau\mu_t\|_{L^\infty(\mathcal{D})})e^{\|\tau\mu_s\|_{L^\infty}}$.

3.1 Reformulation as fixed-point equation

Let us start by reformulating the radiative transfer problem as an equivalent integral equation [4]. We define the scattering operator by

$$\mathcal{K}\phi(r,s) = \mu_s(r)\int_{\mathcal{S}} \theta(r, s\cdot s')\phi(r,s')\,\mathrm{d}s', \qquad (r,s) \in \mathcal{D}, \tag{3.4}$$

further denote by

$$(\mathcal{J}g)(r_- + ts, s) = e^{-\int_0^t \mu_t(r_- + t's)\,\mathrm{d}t'} g(r_-, s), \qquad (r_-, s) \in \Gamma_- \tag{3.5}$$

the extension of boundary values, and define a lifting

$$\mathcal{L}f(r_- + ts, s) = \int_0^t e^{-\int_{t'}^t \mu_t(r_- + t''s)\,\mathrm{d}t''} f(r_- + t's, s)\,\mathrm{d}t', \qquad (r_-, s) \in \Gamma_-, \tag{3.6}$$

where $0 < t < \tau(r_-, s)$. By elementary calculations (exercise) one can verify that

$$(s\cdot\nabla + \mu_t)\mathcal{J}g = 0, \qquad \mathcal{J}g|_{\Gamma_-} = g, \quad \text{and} \tag{3.7}$$

$$(s\cdot\nabla + \mu_t)\mathcal{L}f = f, \qquad \mathcal{L}f|_{\Gamma_-} = 0. \tag{3.8}$$

This means that the extension $\mathcal{J}g$ of the boundary values lies in the kernel of the differential operator and that the lifting $\mathcal{L}$ is a right inverse of $s\cdot\nabla + \mu_t$. Note, that each of the latter two equations can be interpreted as an ordinary differential equation with parameter s. Using the variation of constants formula, the radiative transfer problem can then be seen to be equivalent to the following operator equation in integral form [4]

$$\phi = \mathcal{L}\mathcal{K}\phi + \mathcal{L}f + \mathcal{J}g. \tag{3.9}$$

The unique solvability for (3.1)–(3.2) is therefore equivalent to the existence of a unique fixed-point for (3.9). To show the existence of a unique fixed-point, we will in the following sections select appropriate solution spaces, provide conditions on the data such that $\mathcal{L}f$ and $\mathcal{J}g$ lie in this space, and show that $\mathcal{L}\mathcal{K}$ is a contraction.

3.2 Solvability in L^∞

We will assume throughout that (A1)–(A2) hold and use the fact that for every point $r \in \mathcal{R}$ and any velocity $s \in \mathcal{S}$ we can find a point (r_-, s) on the inflow boundary Γ_- such that

$$r = r_- + ts \qquad \text{with} \quad 0 < t < \tau(r, s). \tag{3.10}$$

Also note that $\tau(r, s) = \tau(r_-, s)$. We show first that $\mathcal{L}\mathcal{K}$ is a contraction on $L^\infty(\mathcal{R}\times\mathcal{S})$.

Lemma 3.2. *For any $\phi \in L^\infty(\mathcal{D})$ there holds*

$$\|\mathcal{L}\mathcal{K}\phi\|_{L^\infty(\mathcal{D})} \le \big(1 - e^{-\|\mu_s\tau\|_{L^\infty(\mathcal{D})}}\big)\|\phi\|_{L^\infty(\mathcal{D})}.$$

Proof. Using $f = \mathcal{K}\phi$ in (3.6) and the assumption that $\mu_s \le \mu_t$, we obtain for $0 < t < \tau(r_-, s)$

$$\begin{aligned} |(\mathcal{L}\mathcal{K}\phi)(r_- + ts, s)| &\le \int_0^t e^{-\int_{t'}^t \mu_s(r_- + t''s)\,\mathrm{d}t''} \mu_s(r_- + t's)\,\mathrm{d}t' \|\phi\|_{L^\infty(\mathcal{D})} \\ &\le \big(1 - e^{-\|\mu_s\tau\|_{L^\infty}}\big)\ \|\phi\|_{L^\infty(\mathcal{D})}. \end{aligned}$$

For the last estimate see the exercises. □

Applying Banach's fixed-point theorem, we see that (3.9) has a unique solution $\phi \in L^\infty(\mathcal{D})$ whenever $\mathcal{L}f$ and $\mathcal{J}g$ are in $L^\infty(\mathcal{D})$. This can be guaranteed by the following two results.

Lemma 3.3. *Assume that $\tau f \in L^\infty(\mathcal{D})$. Then*

$$\|\mathcal{L}f\|_{L^\infty(\mathcal{D})} \le \|\tau f\|_{L^\infty(\mathcal{D})}.$$

Proof. Using the definition of $\mathcal{L}$, we obtain

$$|\mathcal{L}f(r_- + ts, s)| \le \int_0^t e^{-\int_{t'}^t \mu_t(r_- + t''s)\,\mathrm{d}t''} \tau^{-1}|\tau f(r_- + t's, s)|\,\mathrm{d}t' \le \|\tau f\|_{L^\infty(\mathcal{D})}.$$

□

Lemma 3.4. *For any $g \in L^\infty(\Gamma_-)$ there holds*

$$\|\mathcal{LKJ}g\|_{L^\infty(\mathcal{D})} \leq \|\mathcal{J}g\|_{L^\infty(\mathcal{D})} \leq \|g\|_{L^\infty(\Gamma_-)}.$$

Proof. Since $\mu_t \geq 0$ we immediately obtain $|\mathcal{J}g(r_- + ts, s)| \leq |g(r_-, s)|$, which yields the second estimate. The first one follows from Lemma 3.2. □

Combining the three previous Lemmas and the equivalence of the fixed-point equation (3.9) with the radiative transfer problem, we obtain

Theorem 3.5. *For any $g \in L^\infty(\Gamma_-)$ and $\tau f \in L^\infty(\mathcal{D})$, problem* (3.1)–(3.2) *has a unique solution $\phi \in L^\infty(\mathcal{D})$ which satisfies the a-priori bound*

$$\|\phi\|_{L^\infty(\mathcal{D})} \leq e^{\|\mu_s\tau\|_{L^\infty}} \big(\|\tau f\|_{L^\infty(\mathcal{D})} + \|g\|_{L^\infty(\Gamma_-)}\big).$$

Proof. Define the mapping $\mathcal{F}_\infty : L^\infty(\mathcal{D}) \to L^\infty(\mathcal{D})$ by $\mathcal{F}_\infty(\phi) = \mathcal{LK}\phi + \mathcal{L}f + \mathcal{J}g$. In view of Lemma 3.2, Lemma 3.3 and Lemma 3.4, $\mathcal{F}_\infty$ is well-defined. Moreover, by Lemma 3.2, for $\phi, \psi \in L^\infty(\mathcal{D})$, we have

$$\|\mathcal{F}_\infty(\phi) - \mathcal{F}_\infty(\psi)\|_{L^\infty(\mathcal{D})} = \|\mathcal{LK}(\phi - \psi)\|_{L^\infty(\mathcal{D})} \leq (1 - e^{-\|\mu_s\tau\|_{L^\infty}})\|\phi - \psi\|_{L^\infty(\mathcal{D})}.$$

Thus, $\mathcal{F}_\infty$ is a contraction, and the existence of a fixed-point follows from Banach's fixed point theorem. For the a-priori estimate, we employ again the above Lemmata, i.e.

$$\begin{aligned}\|\phi\|_{L^\infty(\mathcal{D})} = \|\mathcal{F}_\infty(\phi)\|_{L^\infty(\mathcal{D})} &\leq \|\mathcal{LK}\phi\|_{L^\infty(\mathcal{D})} + \|\mathcal{L}f\|_{L^\infty(\mathcal{D})} + \|\mathcal{J}g\|_{L^\infty(\mathcal{D})} \\ &\leq (1 - e^{-\|\mu_s\tau\|_{L^\infty}})\|\phi\|_{L^\infty(\mathcal{D})} + \|\tau f\|_{L^\infty(\mathcal{D})} + \|g\|_{L^\infty(\Gamma_-;|s\cdot n|)}.\end{aligned}$$

Rearrangement of the terms yields the assertion.

□

3.3 Solvability in L^1

Setting $w = s \cdot \nabla\phi + \mu_t\phi$ allows us to express the solution as $\phi = \mathcal{J}g + \mathcal{L}w$. The fixed-point problem (3.9) can then be stated equivalently as

$$w = \mathcal{KL}w + f + \mathcal{KJ}g, \qquad \phi = \mathcal{L}w + \mathcal{J}g. \tag{3.11}$$

We want to show existence of a unique fixed-point for (3.11) in $L^1(\mathcal{D})$. To do so, we will first establish the contraction property for the operator $\mathcal{KL}$.

Lemma 3.6. *For any $w \in L^1(\mathcal{D})$ there holds*

$$\|\mathcal{KL}w\|_{L^1(\mathcal{D})} \leq (1 - e^{-\|\mu_s\tau\|_{L^\infty}})\|w\|_{L^1(\mathcal{D})}.$$

Proof. By the definitions of $\mathcal{K}$ and μ_s, we get

$$\begin{aligned}\|\mathcal{KL}w\|_{L^1(\mathcal{D})} &\leq \int_\mathcal{R}\int_\mathcal{S} \mu_s(r) \int_\mathcal{S} \theta(r, s\cdot s')|(\mathcal{L}w)(r, s')|\,\mathrm{d}s'\,\mathrm{d}s\,\mathrm{d}r \\ &= \int_\mathcal{R}\int_\mathcal{S} \mu_s(r)|(\mathcal{L}w)(r, s')|\,\mathrm{d}s'\,\mathrm{d}r = (*).\end{aligned}$$

Using the definition of $\mathcal{L}$ and applying the integral formula (2.4) further yields

$$\begin{aligned}
(*) &\leq \int_{\Gamma_-}\int_0^{\tau(r_-,s)} \mu_s(r_-+ts)\int_0^t e^{-\int_{t'}^t \mu_t(r_-+t''s)\,\mathrm{d}t''}|w(r_-+t's,s)|\,\mathrm{d}t'\,\mathrm{d}t|n\cdot s|\,\mathrm{d}(r_-,s)\\
&= \int_{\Gamma_-}\int_0^{\tau(r_-,s)}\int_{t'}^{\tau(r_-,s)} \mu_s(r_-+ts)e^{-\int_{t'}^t \mu_t(r_-+t''s)\,\mathrm{d}t''}\,\mathrm{d}t|w(r_-+t's,s)|\,\mathrm{d}t'|n\cdot s|\,\mathrm{d}(r_-,s)\\
&\leq \int_{\Gamma_-}\int_0^{\tau(r_-,s)} \big(1-e^{-\int_{t'}^{\tau(r_-,s)} \mu_s(r_-+t''s)\,\mathrm{d}t''}\big)|w(r_-+t's,s)|\,\mathrm{d}t'|n\cdot s|\,\mathrm{d}(r_-,s).
\end{aligned}$$

Here we used $\mu_s \leq \mu_t$ and applied Fubini's theorem again to exchange the order of integrals with respect to $\mathrm{d}t'$ and $\mathrm{d}t$ and explicitly computed the latter. The assertion now follows from (2.4). □

To establish the existence of a fixed-point, we additionally have to require that f and $\mathcal{K}\mathcal{J}g$ are in $L^1(\mathcal{D})$. For the latter term, we use

Lemma 3.7. *For any $g\in L^1(\Gamma_-;|s\cdot n|)$ there holds*

$$\|\mathcal{K}\mathcal{J}g\|_{L^1(\mathcal{D})} \leq \|\mu_s\mathcal{J}g\|_{L^1(\mathcal{D})} \leq (1-e^{-\|\mu_s\tau\|_{L^\infty}})\|g\|_{L^1(\Gamma_-;|s\cdot n|)}.$$

Proof. By the definition of $\mathcal{K}$ and exchanging the order of integration, we obtain

$$\|\mathcal{K}\mathcal{J}g\|_{L^1(\mathcal{D})} \leq \int_{\mathcal{R}}\int_{\mathcal{S}}\mu_s(r)\int_{\mathcal{S}}\theta(r,s\cdot s')|\mathcal{J}g(r,s')|\,\mathrm{d}s'\,\mathrm{d}s\,\mathrm{d}r = \|\mu_s\mathcal{J}g\|_{L^1(\mathcal{D})}.$$

Employing the definition of $\mathcal{J}$ and the integral formula (2.4), yields

$$\begin{aligned}
\|\mu_s\mathcal{J}g\|_{L^1(\mathcal{D})} &\leq \int_{\Gamma_-}\int_0^{\tau(r_-,s)} \mu_s(r_-+ts)e^{-\int_0^t \mu_t(r_-+t's)\,\mathrm{d}t'}\,\mathrm{d}t|g(r_-,s)||s\cdot n|\,\mathrm{d}(r_-,s)\\
&\leq (1-e^{-\|\mu_s\tau\|_{L^\infty}})\|g\|_{L^1(\Gamma_-;|s\cdot n|)},
\end{aligned}$$

where in the last step, we used $\mu_s\leq\mu_t$ and a direct computation of the integral similar as in the proof of Lemma 3.6. □

By Banach's fixed-point theorem and the previous estimates, we now obtain

Lemma 3.8. *For any $f\in L^1(\mathcal{D})$ and $g\in L^1(\Gamma_-;|s\cdot n|)$, the fixed-point problem* (3.11) *has a unique solution $w\in L^1(\mathcal{D})$ and there holds*

$$\|w\|_{L^1(\mathcal{D})} \leq e^{\|\mu_s\tau\|_{L^\infty}}\big(\|f\|_{L^1(\mathcal{D})} + (1-e^{-\|\mu_s\tau\|_{L^\infty}})\|g\|_{L^1(\Gamma_-;|s\cdot n|)}\big).$$

Proof. Define the mapping $\mathcal{F}_1 : L^1(\mathcal{D}) \to L^1(\mathcal{D})$ by $\mathcal{F}_1(w) = \mathcal{K}\mathcal{L}w + f + \mathcal{K}\mathcal{J}g$. In view of Lemma 3.6 and Lemma 3.7, $\mathcal{F}_1$ is well-defined. Moreover, by Lemma 3.6, for $w,v\in L^1(\mathcal{D})$, we have

$$\|\mathcal{F}_1(w)-\mathcal{F}_1(v)\|_{L^1(\mathcal{D})} = \|\mathcal{K}\mathcal{L}(w-v)\|_{L^1(\mathcal{D})} \leq (1-e^{-\|\mu_s\tau\|_{L^\infty}})\|w-v\|_{L^1(\mathcal{D})}.$$

Thus, $\mathcal{F}_1$ is a contraction, and the existence of a fixed-point follows from Banach's fixed point theorem. For the a-priori estimate, we employ again Lemma 3.6 and Lemma 3.7, i.e.

$$\begin{aligned}
\|w\|_{L^1(\mathcal{D})} &= \|\mathcal{F}_1(w)\|_{L^1(\mathcal{D})} \leq \|\mathcal{K}\mathcal{L}w\|_{L^1(\mathcal{D})} + \|f\|_{L^1(\mathcal{D})} + \|\mathcal{K}\mathcal{J}g\|_{L^1(\mathcal{D})}\\
&\leq (1-e^{-\|\mu_s\tau\|_{L^\infty}})\|w\|_{L^1(\mathcal{D})} + \|f\|_{L^1(\mathcal{D})} + (1-e^{-\|\mu_s\tau\|_{L^\infty}})\|g\|_{L^1(\Gamma_-;|s\cdot n|)}.
\end{aligned}$$

Rearrangement of the terms yields the assertion. □

To establish an L^1 estimate for the solution $\phi = \mathcal{L}w + \mathcal{J}g$ of problem (3.1)–(3.2), we have to establish additional bounds for $\mathcal{L}w$ and $\mathcal{J}g$.

Lemma 3.9. *For any $w \in L^1(\mathcal{D})$ and any $g \in L^1(\Gamma_-; |s \cdot n|)$ there holds*

$$\|\tau^{-1}\mathcal{L}w\|_{L^1(\mathcal{D})} \leq \|w\|_{L^1(\mathcal{D})} \qquad \text{and} \qquad \|\tau^{-1}\mathcal{J}g\|_{L^1(\mathcal{D})} \leq \|g\|_{L^1(\Gamma_-;|s\cdot n|)}.$$

Proof. Using integral formula (2.4) and writing $\tau = \tau(r_-, s)$, we obtain

$$\begin{aligned}\|\tau^{-1}\mathcal{L}w\|_{L^1(\mathcal{D})} &= \int_{\Gamma_-}\int_0^\tau \frac{1}{\tau}\int_0^t e^{-\int_{t'}^t \mu_t(r_-+t''s)\,\mathrm{d}t''} w(r_-+t's,s)\,\mathrm{d}t'|s\cdot n|\,\mathrm{d}t\,\mathrm{d}(r_-,s)\\ &\leq \int_{\Gamma_-}\int_0^\tau \frac{1}{\tau}\int_0^\tau w(r_-+t's,s)\,\mathrm{d}t'|s\cdot n|\,\mathrm{d}t\,\mathrm{d}(r_-,s) = \|w\|_{L^1(\mathcal{D})}.\end{aligned}$$

Similarly,

$$\|\tau^{-1}\mathcal{J}g\|_{L^1(\mathcal{D})} \leq \int_{\Gamma_-}\int_0^\tau \frac{1}{\tau} e^{-\int_0^t \mu_t(r_-+t's)\,\mathrm{d}t'}|g(r_-,s)||s\cdot n|\,\mathrm{d}t\,\mathrm{d}(r_-,s) \leq \|g\|_{L^1(\Gamma_-;|s\cdot n|)}.$$

□

A combination of the previous estimates now yields

Theorem 3.10. *For any $f \in L^1(\mathcal{D})$ and $g \in L^1(\Gamma_-; |s\cdot n|)$, the boundary value problem* (3.1)–(3.2) *has a unique solution $\phi \in L^1(\mathcal{D})$ which satisfies*

$$\|\tau^{-1}\phi\|_{L^1(\mathcal{D})} \leq e^{\|\mu_s\tau\|_{L^\infty}}\big(\|f\|_{L^1(\mathcal{D})} + \|g\|_{L^1(\Gamma_-;|s\cdot n|)}\big).$$

Proof. The result follows from the representation $\phi = \mathcal{L}w + \mathcal{J}g$ of the solution by applying the triangle inequality and using the estimates of Lemmas 3.8 and 3.9. □

3.4 Solvability in L^p

For establishing solvability in L^p we will need the following classical result, a proof can be found e.g. in [2, Chapter 4, Theorem 2.2].

Theorem 3.11 (Riesz-Thorin convexity theorem)**.** *Let $(\mathcal{M}, \mathrm{d}\mu)$ and $(\Lambda, \mathrm{d}\nu)$ be σ-finite measure spaces. Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and let $T : L^{p_0}(\mathcal{M}) + L^{p_1}(\mathcal{M}) \to L^{q_0}(\Lambda) + L^{q_1}(\Lambda)$ be a linear operator such that*

$$\|T\|_{L^{p_i}(\mathcal{M})\to L^{q_i}(\Lambda)} \leq M_i, \quad i = 0, 1.$$

Then $T \in L(L^{p_\theta}(\mathcal{M}), L^{q_\theta}(\Lambda))$ for

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad 0 < \theta < 1.$$

Moreover, if $p_i \leq q_i$, $i = 0, 1$, then

$$\|T\|_{L^{p_\theta}(\mathcal{M})\to L^{q_\theta}(\Lambda)} \leq M_0^{1-\theta}M_1^\theta.$$

Remark 3.12. The restriction $p_i \leq q_i$ in Theorem 3.11 is not necessary for complex L^p-spaces. For real L^p-spaces and $p_i \nleq q_i$, there holds $\|T\|_{L^{p_\theta}(\mathcal{M})\to L^{q_\theta}(\Lambda)} \leq 2M_0^{1-\theta}M_1^\theta$, see [2].

3.4.1 Existence and L^p estimate

As a first step, let us establish the a-priori estimate for data that simultaneously satisfy the requirements of Theorems 3.5 and 3.10. Noting that

$$\|\phi\|_{L^p(\mathcal{D};\tau^{-1})} = \|\tau^{-\frac{1}{p}}\phi\|_{L^p(\mathcal{D})},$$

the a-priori bounds of these previous results can be written as

$$\|\phi\|_{L^p(\mathcal{D};\tau^{-1})} \le e^{\|\mu_s\tau\|_{L^\infty}}\big(\|\tau f\|_{L^p(\mathcal{D};\tau^{-1})} + \|g\|_{L^p(\Gamma_-;|s\cdot n|)}\big) \quad \text{for } p\in\{1,\infty\}.$$

Using the linearity of the problem, we can decompose $\phi = \phi_g + \phi_f$, where ϕ_g and ϕ_f are the solutions of (3.1)–(3.2) with $f\equiv 0$ and $g\equiv 0$, respectively. An application of the Riesz-Thorin theorem 3.11 to the solution operator then yields

$$\|\phi_f\|_{L^p(\mathcal{D};\tau^{-1})} \le e^{\|\mu_s\tau\|_{L^\infty}}\|\tau f\|_{L^p(\mathcal{D};\tau^{-1})} \quad \text{and} \quad \|\phi_g\|_{L^p(\mathcal{D};\tau^{-1})} \le e^{\|\mu_s\tau\|_{L^\infty}}\|g\|_{L^p(\Gamma_-;|s\cdot n|)}$$

for any $1\le p\le\infty$. From this estimate the a-priori estimate is derived via the triangle inequality. The unique solvability for all admissible data follows by a density argument.

3.4.2 Estimates for the derivatives

Using the a-priori estimates of Section 3.4.1 and the fixed-point equation (3.9), it is straightforward to obtain also estimates for the directional derivatives $s\cdot\nabla\phi$. Let us first consider the case $p=1$, where we have

Lemma 3.13. *Under the assumptions of Theorem 3.10 one has*

$$\|s\cdot\nabla\phi\|_{L^1(\mathcal{D})} \le 2e^{\|\mu_s\tau\|_{L^\infty}}\big(\|f\|_{L^1(\mathcal{D})} + \|g\|_{L^1(\Gamma_-;|s\cdot n|)}\big).$$

Proof. By $\phi = \mathcal{L}w + \mathcal{J}g$ and the properties of the operators $\mathcal{L}$ and $\mathcal{J}$, we obtain

$$\|s\cdot\nabla\phi\|_{L^1(\mathcal{D})} \le \|w\|_{L^1(\mathcal{D})} + \|\mu_t\phi\|_{L^1(\mathcal{D})}.$$

The first term can be estimated by Lemma 3.8, and for the second, we use

$$\|\mu_t\phi\|_{L^1(\mathcal{D})} \le \|\mu_t\mathcal{L}w\|_{L^1(\mathcal{D})} + \|\mu_t\mathcal{J}g\|_{L^1(\mathcal{D})} \le \|w\|_{L^1(\mathcal{D})} + \|g\|_{L^1(\Gamma_-;|s\cdot n|)}.$$

The second estimate for the boundary term is obtained as in Lemma 3.7. □

For the case $p=\infty$, we have

Lemma 3.14. *Under the assumptions of Theorem 3.5 there holds*

$$\|\tau s\cdot\nabla\phi\|_{L^\infty(\mathcal{D})} \le \big(1+2\|\mu_t\tau\|_{L^\infty}\big)e^{\|\mu_s\tau\|_{L^\infty(\mathcal{D})}}\big(\|\tau f\|_{L^\infty(\mathcal{D})} + \|g\|_{L^\infty(\Gamma_-;|s\cdot n|)}\big).$$

Proof. The identity $s\cdot\nabla\phi = \mathcal{K}\phi - \mu_t\phi + f$ yields

$$\|\tau s\cdot\nabla\phi\|_{L^\infty(\mathcal{D})} \le (\|\tau\mu_s\|_{L^\infty(\mathcal{D})} + \|\tau\mu_t\|_{L^\infty(\mathcal{D})})\|\phi\|_{L^\infty(\mathcal{D})} + \|\tau f\|_{L^\infty(\mathcal{D})}.$$

The estimate then follows from the bounds of Theorem 3.5 and the condition $\mu_s'\le\mu_t$. □

Arguing as in the proof of Theorem 3.1, the case $1\le p\le\infty$ is then covered by

Theorem 3.15. *Under the assumptions of Theorem 3.1 there holds*

$$\|\tau^{1-\frac{1}{p}}s\cdot\nabla\phi\|_{L^p(\mathcal{D})} \le 2(1+\|\mu_t\tau\|_{L^\infty})e^{\|\tau\mu_s\|_{L^\infty}}\big(\|\tau^{1-\frac{1}{p}}f\|_{L^p(\mathcal{D})} + \|g\|_{L^p(\Gamma_-;|s\cdot n|)}\big).$$

Using the results of Sections 3.2–3.4, we also obtain that

$$\|\tau^{1-\frac{1}{p}}\big(s\cdot\nabla\phi+\mu_t\phi-\mathcal{K}\phi\big)\|_{L^p(\mathcal{D})}\leq(1+2\|\mu_t\tau\|_{L^\infty})\|\phi\|_{\mathcal{V}^p}.$$

The individual operators could be estimated in the same way. Summarizing, we obtain

Theorem 3.16. *Let (A1)–(A2) hold. Then the mapping*

$$\mathcal{V}^p\to L^p(\mathcal{D};\tau^{p-1})\times L^p(\Gamma_-;|s\cdot n|),\quad \phi\mapsto(s\cdot\nabla\phi+\mu_t\phi-\mathcal{K}\phi,\ \gamma_-\phi)$$

is continuous and boundedly invertible.

This result shows that the assumptions on the data cannot be relaxed when searching for solutions in $\mathcal{V}^p$.

Remark 3.17. Using (A1)–(A2), we see from the proof of Lemma 3.14 that $s\cdot\nabla\phi\in L^\infty(\mathcal{D})$ as long as $f\in L^\infty(\mathcal{D})$, i.e.

$$\|s\cdot\nabla\phi\|_{L^\infty(\mathcal{D})}\leq(\|\mu_s\|_{L^\infty(\mathcal{D})}+\|\mu_t\|_{L^\infty(\mathcal{D})})\|\phi\|_{L^\infty(\mathcal{D})}+\|f\|_{L^\infty(\mathcal{D})}.$$

Therefore, $\phi\in\mathcal{W}^\infty$, cf. Remark 2.11. According to Remark 2.11 we also have $\mathcal{W}^1=\mathcal{V}^1$. Hence, employing similar arguments as above, we conclude that the mapping

$$\mathcal{W}^p\to L^p(\mathcal{D})\times L^p(\Gamma_-;|s\cdot n|),\quad \phi\mapsto(s\cdot\nabla\phi+\mu_t\phi-\mathcal{K}\phi,\ \gamma_-\phi)$$

is continuous and boundedly invertible.

3.5 Spectral estimates and convergence of the fixed-point iterations

The solvability results of the previous sections were based on Banach's fixed-point theorem. The corresponding fixed-point iteration reads

$$\phi_{n+1}=\mathcal{L}\mathcal{K}\phi_n+\mathcal{L}f+\mathcal{J}g. \tag{3.12}$$

We show now that under our general assumptions (A1)–(A2), the spectral radius of the fixed-point operator $\mathcal{L}\mathcal{K}$ is always uniformly bounded away from one.

Theorem 3.18. *Let (A1)–(A2) hold. Then for all $1\leq p\leq\infty$*

$$\rho_p(\mathcal{L}\mathcal{K}):=\lim_{n\to\infty}\sqrt[n]{\|(\mathcal{L}\mathcal{K})^n\|_{L^p(\mathcal{D};\tau^{-1})}}\leq1-e^{-\|\tau\mu_s\|_\infty}.$$

Proof. The case $p=\infty$ follows immediately from Lemma 3.2. For $p=1$, on the other hand, we can estimate the powers of the fixed-point operator by

$$\|(\mathcal{L}\mathcal{K})^n\|_{L^1(\mathcal{D};\tau^{-1})}=\|\tau^{-1}(\mathcal{L}\mathcal{K})^n\tau\|_{L^1(\mathcal{D})}\leq\|\tau^{-1}\mathcal{L}\|_{L^1(\mathcal{D})}\|\mathcal{K}\mathcal{L}\|^{n-1}_{L^1(\mathcal{D})}\|\mathcal{K}\tau\|_{L^1(\mathcal{D})}.$$

The first two terms can be bounded by Lemma 3.7 and 3.6, and for the third term we use the estimate $\|\mathcal{K}\tau\|_{L^1(\mathcal{D})}\leq\|\mu_s\tau\|_{L^\infty(\mathcal{D})}$. From this we obtain the estimate for the spectral radius for $p=1$. The general case then follows again by interpolation arguments. □

We conclude, that under the weak sub-criticality assumptions (A1), the source iteration (3.12) converges in L^p for any $1\leq p\leq\infty$ with a contraction factor $1-e^{-\|\tau\mu_s\|_\infty}$. Note that no positive lower bound on the absorption is needed for the convergence.

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