Crowded Particles - From Ions to Humans

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1. Ions
   - Motivation
   - One-Dimensional Model
   - Entropy

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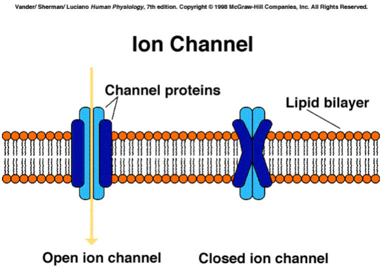
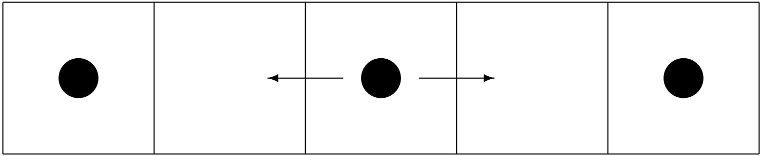
2. Pedestrians
   - Observations
   - Social Force Model and Pedestrian Motion
   - 1D Movement
     - Investiagtions for the 1D Model
     - Limiting Behaviour
   - Further Work

Biological Background



- located in membrane cells
- proteins with a hole in their middle
- regulate movement of inorganic ions through impermeable cell membrane
- exact structure not known
- topic of interest in biophysics, medicine

One-Dimensional Hopping Model



Probabilities

$n(x,t) = P(\text{negatively charged ion at position } x \text{ at time } t)$
$p(x,t) = P(\text{positively charged ion at position } x \text{ at time } t)$

- transitionrate:
  $\Pi_{+}^{n/p}(x,t) =$
  $P(\text{jump of } n/p \text{ from position } x \text{ to } x+h \text{ in } (t, t+\Delta t))\cdot$
  $\frac{1}{\Delta t} P(x+h \text{ is empty})$

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- probability that a negative particle is located at position x at time $t+\Delta t$:
  $n(x, t+\Delta t) = n(x,t)(1 - \Pi_+(x,t) - \Pi_-(x,t))$
  $+n(x+h,t)\Pi_-(x+h,t) + n(x-h,t)\Pi_+(x-h,t)$

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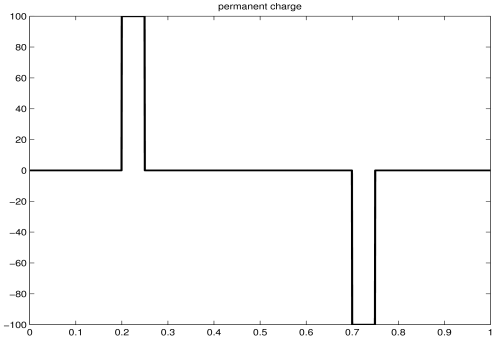
Resulting Model

$$\partial_t n = \nabla\cdot(D(1-m)\nabla n - Dn\nabla m - n(1-m)\nabla V)$$
$$\partial_t p = \nabla\cdot(D(1-m)\nabla p - Dp\nabla m + p(1-m)\nabla V)$$

- D diffusion coefficient, mass $m(x,t) = n(x,t) + p(x,t)$

- V satisfies the Poisson equation:
  $\lambda^2 V_{xx}(x,t) = n(x,t) - p(x,t) + f(x)$
- $f(x)$: Permanent charge on the membrane



Solutions for Equilibrium

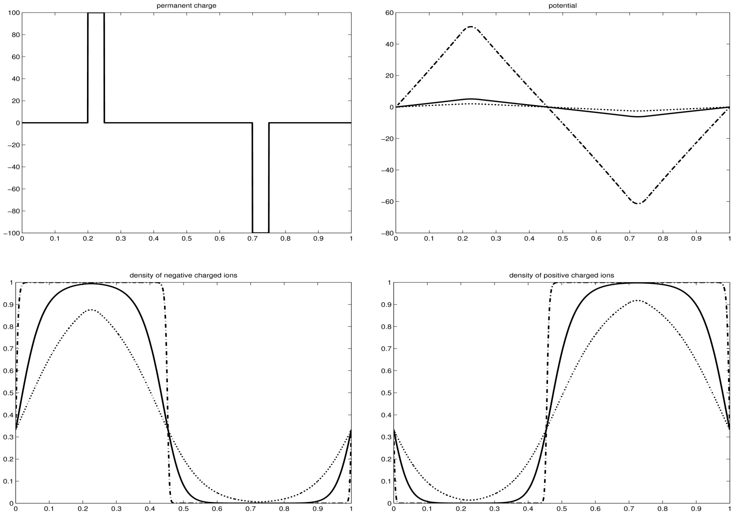
$$\partial_t n = \nabla \cdot (D(1-m)\nabla n - Dn\nabla m - n(1-m)\nabla V)$$
$$\partial_t p = \nabla \cdot (D(1-m)\nabla p - Dp\nabla m + p(1-m)\nabla V)$$

Solutions for Equilibrium

$$n(x,t) = \frac{\exp(V(x,t)/D)}{\exp(V(x,t)/D)+\beta\cdot\exp(-V(x,t)/D)+\alpha}$$
$$p(x,t) = \frac{\beta\cdot\exp(-V(x,t)/D)}{\exp(V(x,t)/D)+\beta\cdot\exp(-V(x,t)/D)+\alpha}$$

- α and β are constants, α, $\beta > 0$

Numerical Results



Entropy

$$E = \int (n \cdot \log(n) + p \cdot \log(p) + (1-m) \cdot \log(1-m) - nV + pV)\, dx$$

- decreasing during the process
- minimal in equilibrium state

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Under normal Condition

- *Fastest* route, not the *shortest* one
- Individual speed: most comfortable one (dependent of age, sex, purpose of the trip...)
  Speeds are Gaussian distibuted (mean value: 1,34 m/s, St-d 0.26 m/s)
- Certain distance from other pedestrians and boundaries
- Resting pedestrians are unifomly distributed about the available space

In Case of Competitive Evacuation (Panic)

- Nervousness
- Pedestrians try to move faster than normal
- Interactions become physical in nature
- Uncoordinated passing of bottlenecks
- Jams build up. Arching and clogging at exits
- Physical interactions add up $\Rightarrow$ dangerous pressures up to 4,5 tons
- Escape is slowed down by injured or fallen people
- Herding behaviour

Social Force Model (Helbing)

Equation of Motion

$$\frac{d\mathbf{r}_i(t)}{dt} = \mathbf{v}_i(t)$$

$$m_i \frac{d\mathbf{v}_i(t)}{dt} = m_i \frac{v_i^0 \mathbf{e}_i^0(t) - \mathbf{v}_i(t)}{\tau_i} + \sum_{j(\neq i)} \mathbf{f}_{ij} + \sum_W \mathbf{f}_{iW}$$

$\mathbf{r}_i$ position of pedestrian i
$\mathbf{v}_i$ velocity
$\frac{d\mathbf{v}_i(t)}{dt}$ acceleration

m_i mass
τ_i acceleration time
v_i^0 desired velocity, $\mathbf{e}_i^0$ desired direction

Forces

$$\mathbf{f}_{ij} = \mathbf{f}_{ij}^{interact} + \mathbf{f}_{ij}^{body} + \mathbf{f}_{ij}^{slid}$$

$$\mathbf{f}_{ij}^{interact} = A_i \cdot e^{\left[\left(R_{ij}-d_{ij}\right)/B_{ij}\right]}\mathbf{n}_{ij}$$

$$\mathbf{f}_{ij}^{body} = k(R_{ij} - d_{ij})\mathbf{n}_{ij}$$

$$\mathbf{f}_{ij}^{slid} = \kappa(R_{ij} - d_{ij})\Delta v_{ji}^{t}\mathbf{t}_{ij}$$

$\mathbf{f}_{ij}^{body}$ and $\mathbf{f}_{ij}^{slid}$ only if pedestrians i and j touch each other (panic)
A_i, B_i, k, κ constants
d_{ij} distance
$\mathbf{n}_{ij} = (n_{ij}^1, n_{ij}^2) = (\mathbf{r}_i - \mathbf{r}_j)/d_{ij}$ normalized vector from j to i
$\mathbf{t}_{ij} = (-n_{ij}^2, n_{ij}^1)$ tangential direction
$\Delta v_{ji}^t = (\mathbf{v}_j - \mathbf{v}_i) \cdot \mathbf{t}_{ij}$ tangential velocity difference
$R_{ij} = (R_i + R_j)$ sum of radii

Additional Forces

- Walls are treated analogously
- Anisotropy:

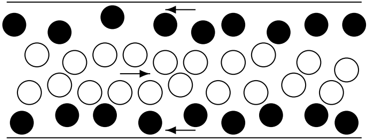
$$\mathbf{f}_{ij} = A_i \exp[(R_{ij} - d_{ij})/B_i]\mathbf{n}_{ij} \cdot \left(\lambda_i + (1-\lambda_i)\frac{1+\cos\varphi_{ij}}{2}\right) \quad (1)$$

  $\lambda < 1$: happenings in front more weighted than events behind
  φ_{ij}: angle between $\mathbf{e}_i$ and $-\mathbf{n}_{ij}$
- Sights: forces of type (1) (B larger, A smaller and negative)
- Groups: $\mathbf{f}_{ij}^{att} = C_{ij}\mathbf{n}_{ij}$
- Fluctuation term ξ_i

Self-Organization Phenomena

Helbing's Model describes the following phenomena quite realistically:

- **Segregation**
  Lane Formation



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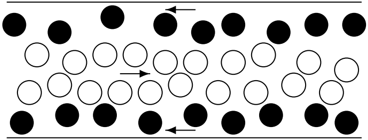

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  Bottlenecks: Passing of direction

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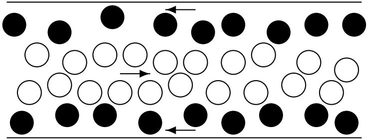


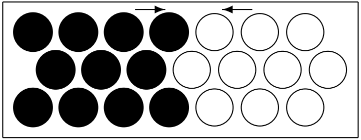
- **Segregation**
  Lane Formation
- **Oscillations**
  Bottlenecks: Passing of direction
- **Intersections**
  Unstable traffic

Behaviour in Competitive Evacuation (Panic)

- People are getting **nervous** $\Rightarrow$ Higher level of fluctuations
- **Higher desired velocity**
- **Herding behaviour:**

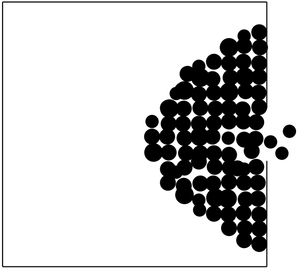
$$\mathbf{e}_i^0 = \frac{(1-n_i)\mathbf{e}_i + n_i \left\langle \mathbf{e}_j^0 \right\rangle_i}{\left\| (1-n_i)\mathbf{e}_i + n_i \left\langle \mathbf{e}_j^0 \right\rangle_i \right\|} \qquad n_i \text{ nervousness}$$

- **Freezing by heating:**



Behaviour in Competitive Evacuation (Panic)

- 'Faster is slower Effect':



- 'Phantom Panics':

$$v_i^0(t) = [1 - n_i(t)]\, v_i^0(0) + n_i(t) v_i^{max} \qquad n_i(t) = 1 - \frac{\bar{v}_i(t)}{v_i^0(0)}$$

Optimization

- Series of columns in the middle of a corridor
- Funnelshaped geometrie at bottlenecks
- Two small doors better than a large one
- Intersections: Guidance arrangements which lead to a roundabout, an obstacle in the centre
- Slim queues
- Zigzag shape
- Avoidance of staircases
- Escape routes should not have a constant width
- Column placed asymetrically in front of exits

Model for hard Bodies with Remote Action

General Model (Helbing)

$$m_i \frac{dv_i}{dt} = f_i$$

$$f_i = m_i \frac{v_i^0 - v_i}{\tau_i} - \sum_{j \neq i} A_i'(\|r_j - r_i\| - d_i)$$

- A' potential
- $d_i = a_i + b_i v_i$ safety margin; a_i and b_i are constants

Seyfried's Model

The Model

$$f_i(t) = \begin{cases} G_i(t) & ;\ \text{if } v_i(t) > 0 \\ \max(0, G_i(t)) & ;\ \text{if } v_i(t) \leq 0 \end{cases}$$

$$G_i(t) = \frac{v_i^0 - v_i(t)}{\tau_i} - e_i \left(\frac{1}{r_{i+1}(t) - r_i(t) - d_i(t)} \right)^{B_i}$$

- e_i and B_i: strength and the range of the force

Investigations for the Model

Old Model

$$\frac{dv_i(t)}{dt} = \begin{cases} G_i(t) & , \text{ if } v_i(t) > 0 \\ \max(0, G_i(t)) & , \text{ if } v_i(t) \leq 0 \end{cases}$$

$$G_i(t) = \frac{v_i^0 - v_i(t)}{\tau_i} - e_i \left(\frac{1}{r_{i+1}(t) - r_i(t) - d_i(t)} \right)^{g_i}$$

Problem:
Do pedestrians go backwards?

Acceleration Term

Seyfried's term:

$$G_i(t) = \frac{v_i^0 - v_i(t)}{\tau_i} - e_i \left(\frac{1}{r_{i+1}(t) - r_i(t) - d_i(t)} \right)^{g_i}$$

New Acceleration Term

$$\tilde{G}_i(t) = \begin{cases} \frac{v_i^0 - v_i(t)}{\tau_i} - \frac{e_i}{m_i} \left(\frac{h}{r_{i+1}(t) - r_i(t) - d_i(t)} \right)^{g_i} & , \ r_{i+1} - r_i - d_i > 0 \\ -\infty & , \ \text{else} \end{cases}$$

Improved Model

New Model

$$\frac{dv_i}{dt} = \tilde{G}_i(t) \cdot H(v_i(t), \tilde{G}_i(t))$$

$$H(v_i(t), \tilde{G}_i(t)) = \begin{cases} 0 & , \text{ if } \quad v_i(t) \leq 0 \text{ and } \tilde{G}_i \leq 0 \\ 1 & , \text{ if } \quad \tilde{G}_i(t) \geq \delta \text{ or } v_i(t) \geq \epsilon \end{cases}$$

Old model:

$$\frac{dv_i(t)}{dt} = \begin{cases} G_i(t) & , \text{ if } v_i(t) > 0 \\ \max(0, G_i(t)) & , \text{ if } v_i(t) \leq 0 \end{cases}$$

Main difference: **Interpolation**
Interpolation decreases the deceleration in case the velocity comes close to zero.

Results

Pedestrians do not go backwards, if their initial safety margin is large enough, because

- The interpolation we used lessened the deceleration in case the velocity came close to zero
- Pedestrians start deceleration sufficiently long before they approach an obstacle
- The more a person decelerates, the less becomes the velocity and the required safety margin

$\Rightarrow$ Model works well!

Existence and Uniqueness

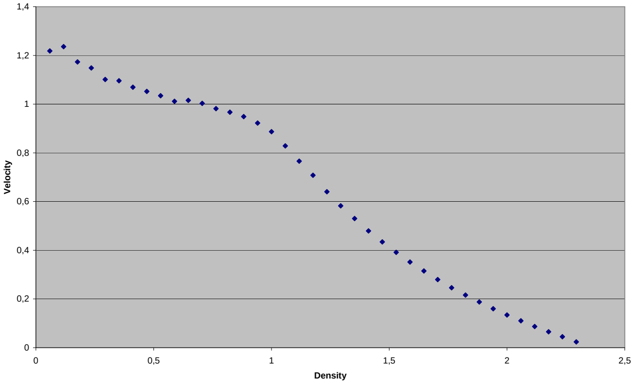
Lemma 1:

Let $r_{i+1}(0) - r_i(0) > d_i(0) \quad \forall i$. *Then the ODE-System*

$$\begin{pmatrix} \dot{r}_1 \\ \dot{v}_1 \\ \dot{r}_2 \\ \dot{v}_2 \\ \vdots \\ \dot{r}_N \\ \dot{v}_N \end{pmatrix} = \begin{pmatrix} v_1 \\ \tilde{G}_1(t, r_1, r_2, v_1) \cdot H(v_1(t), \tilde{G}_1(t)) \\ v_2 \\ G_2(tr_2, r_3, v_2) \cdot H(v_2(t), \tilde{G}_2(t)) \\ \vdots \\ v_N \\ \tilde{G}_N(t, r_N, r_1, v_N) \cdot H(v_N(t), \tilde{G}_N(t)) \end{pmatrix} \Leftrightarrow \dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y})$$

has a local solution. The solution is unique.

Velocity Density Relation



Limiting Behaviour

What happens if the number of pedestrians N tends to ∞?

We investigate the general formulation:

$$\frac{dv_i}{dt} = \frac{v_i^0 - v_i}{\tau_i} + \frac{1}{m} A'(N(x_{i+1} - x_i) - d_i)$$

General Distribution of Particles

- Equation for particle distribution function $f^N = f^N(x_1, x_2, ..., x_N, t)$:

$$\partial_t f^N + \sum_i \partial_{x_i} \left(\frac{dx_i}{dt} f^N \right) = 0$$

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- In our case

$$\partial_t f^N + \sum_i \partial_{x_i} \left(\left[v_i^0 - \frac{1}{m} A'(N(x_{i+1} - x_i) - d_i) \right] f^N \right) = 0$$

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- Analogon of BBGKY hierarchy (classical kinetic theory):

$$m_k^N(x_k, x_{k-1}, ..., x_1, t) = \int_{\mathbb{R}^{N-k}} f^N(x_1, ..., x_N, t) dx_{k+1} dx_{k+2} ... dx_N$$

General Distribution of Particles

- Equation for k-th marginal:

$$\partial_t m_k^N + \sum_{i\leq k} \int_{\mathbb{R}} \partial_{x_i} \left(\left[v_i^0 - \frac{1}{m} \nabla A(N(x_{i+1} - x_i) - d_i)^{-B_i} \right] m_{k+1}^N \right) dx_{k+1} = 0$$

General Distribution of Particles

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- We assume m_k^N to be dependent of m_1^N and the distances $x_i - x_{i-1}$ $\forall i \leq k$, furthermore

$$m_1^N(x_1) \xrightarrow{N\to\infty} \rho(x)$$

ρ denotes density function.

Results

Equation from k-th particle marginal in the limit $N \to \infty$

$$\partial_t \rho(x,t) + \partial_x \left(\left[v^0 - \frac{1}{m} A' \left(\frac{1}{\rho(x,t)} - d \right) \right] \rho(x,t) \right) = 0$$

Results

Equation from k-th particle marginal in the limit $N \to \infty$

$$\partial_t \rho(x,t) + \partial_x \left(\left[v^0 - \frac{1}{m} A' \left(\frac{1}{\rho(x,t)} - d \right) \right] \rho(x,t) \right) = 0$$

- **Continuity equation:**

$$\partial_t \rho + \partial_x (v\rho) = 0$$

$$v = v^0 - \frac{1}{m} A' \left(\frac{1}{\rho(x,t)} - d \right)$$

Conclusions

- In the limits $N \to \infty$ and $\tau \to 0$ all stochastic fluctuations disappear
- Motion can be described by a deterministic equation!
- Motion does not depend on initial distribution

Further Work

- Analyze existing data
- New experiments (train evacuation)
- Group dynamics
- Influence of audible signals (message, voice)
- Influence of fitness

Thank you for your Attention!