Hausaufgabe 5 (Abgabe bis Mittwoch, 14. Mai, 12 Uhr)

1. Consider the following model for traffic flow. Let $u(t,x)$ be the density of cars on a stretch of road, where x is distance along the road and t is time. Suppose that the speed v of each car is related to the local density by $v=(1-u)$.

   Given a segment $U=(a,b)$ along the road, let $C_U(t)$ be the total number of cars on U at time t. Express the conservation of cars, "temporal change of $C_U(t)$=flux of cars into U – flux of cars out of U", as an equation in the form $\frac{\mathrm{d}C_U}{\mathrm{d}t}=\int_U\ldots\mathrm{d}x$. Since a and b are arbitrary, what is the PDE satisfied by u and what is its appropriate divergence form? (3 pt)

   Write down the Rankine–Hugoniot conditions for a shock (a suddenly occurring traffic jam) and show that the rate at which cars enter the shock from one side equals the rate at which they exit the other. Show that this is not true for the Rankine–Hugoniot conditions based on the divergence form $(\frac{u^2}{2})_t+(\frac{u^2}{2}-\frac{2}{3}u^3)_x=0$. (3 pt)

2. Let $u\in C^0(\Omega)$ be a viscosity solution of $H(x,u(x),\nabla u(x))=0$ and $\varphi\in C^1(\mathbb{R})$ be a function with $\varphi'>0$. Show that $v=\varphi(u)$ is a viscosity solution of $K(x,v,\nabla v)=0$ on Ω, where $K(x,z,p)=H(x,\varphi^{-1}(z),(\varphi^{-1})'(z)p)$. (2 pt)

3. Given the homogeneous Hamilton–Jacobi equation $H(\nabla u)=0$ in $\Omega\subset\mathbb{R}^n$, compute the optical distance δ for
   - $H(p)=\|p\|_1-1\,,$ (2 pt)
   - $H(p)=\|p\|_\infty-1\,.$ (2 pt)

4. Let $A\subset\mathbb{R}^n$ closed, $A\neq\mathbb{R}^n$, and $a\in C^1(\mathbb{R}^n)$. Define
$$d_A(x)=\inf\left\{\int_0^1 a(\gamma(t))|\dot\gamma(t)|\,\mathrm{d}t\;\middle|\;\gamma\text{ smooth curve, }\gamma(0)\in A,\,\gamma(1)=x\right\}.$$
   Show that d_A is a viscosity solution of $\frac{1}{a(x)}|\nabla u|-1=0$. (3 pt)