Optimization and Optimal Control in Banach Spaces

Problem sheet 3 - return on 2018-12-04

Exercise 1. For convex functions $f, g : H \to \mathbb{R} \cup \{\infty\}$ show that the primal-dual algorithm (Prop. 1.141) for $\tau = 1$ can be rewritten as Douglas–Rachford algorithm (Def. 1.124) for suitable choices of γ and λ.
More precisely, from the iterates $(x^{(\ell)})_\ell$, $(y^{(\ell)})_l$ generated by the primal-dual algorithm construct sequences $(\hat{y}^{(\ell)})_\ell$, $(\hat{z}^{(\ell)})_\ell$, $(\hat{x}^{(\ell)})_\ell$ that correspond to the Douglas–Rachford iterations.
Hints: The roles of f and g may be swapped in both algorithms, as compared to the lecture notes. Use the Moreau decomposition.

Exercise 2. Let $f : H \to \mathbb{R} \cup \{\infty\}$ be proper, convex and lsc. For $\varepsilon > 0$ the *Moreau envelope* of f for parameter ε is given by

$$f_\varepsilon : x \mapsto \min_{y \in H} \left[\tfrac{1}{2\varepsilon} \|x - y\|^2 + f(y) \right].$$

The minimizer exists since the proximal operator of f is well-defined.

(i) Show that f_ε is convex. *Hint:* Use existence of minimizers and the equality introduced in the beginning of the proof for Prop. 1.132.

(ii) Show that $\lim_{\varepsilon \nearrow \infty} f_\varepsilon(x) = \inf_H f$. *Hint:* For a minimizing sequence $(x_k)_k$ of $\inf_H f$ construct a suitable sequence $(\varepsilon_k)_k$ such that $\lim_{k \to \infty} \varepsilon_k = \infty$ and

$$\lim_{\varepsilon \nearrow \infty} f_\varepsilon(x) = \lim_{k \to \infty} f_{\varepsilon_k}(x) \le \lim_{k \to \infty} f(x_k) = \inf_H f.$$

(iii) Assume $\partial f(x) \neq \emptyset$. Show that $\lim_{\varepsilon \searrow 0} f_\varepsilon(x) = f(x)$. *Hint:* For a decreasing, strictly positive sequence $(\varepsilon_k)_k$, $\varepsilon_k \to 0$, consider a corresponding sequence $(x_k)_k$ of minimizers such that $f_{\varepsilon_k}(x) = \frac{1}{2\varepsilon} \|x - x_k\|^2 + f(x_k)$.