Optimization and Optimal Control in Banach Spaces

Problem sheet 2 - return on 2018-11-20

Exercise 1. Let $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto \operatorname{abs}(\operatorname{abs}(x) - 1)$ where $\operatorname{abs}(y)$ denotes the absolute value of y.

(i) Determine $\partial f(x)$ for $x \in \mathbb{R}$.

(ii) Determine f^*. *Hint:* Use (i).

(iii) Determine f^{**}.

Exercise 2. Let $f : \mathbb{R}^2 \to \mathbb{R} \cup \{\infty\}$,

$$(x, y) \mapsto \begin{cases} \frac{y^2}{2x} & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ +\infty & \text{else.} \end{cases}$$

(i) Determine the sublevel sets $S_r f$ for $r \in \mathbb{R}$. Is f lower semi-continuous?

(ii) Determine f^*. *Hint:* Proof by case analysis.

(iii) Determine f^{**}. Is f convex?

(iv) Determine $\partial f(x, y)$ for $(x, y) \in \mathbb{R}^2$. *Hint:* Use that f is positively 1-homogeneous.

Exercise 3. Let $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto \frac{\lambda}{4} x^4$ for some $\lambda > 0$.

(i) For $x \in \mathbb{R}$ find an equation that determines $y = \operatorname{Prox}_f(x)$.

(ii) Consider solving this equation for y using Newton's method. For which starting points does the iteration converge? For simplicity, assume $x > 0$.