The Saturn Ring Effect in Nematic Liquid Crystals with External Field: Effective Energy and Hysteresis

Dominik Stantejsky

Centre de Mathématiques Appliquées, UMR CNRS 7641, École Polytechnique, Institut Polytechnique de Paris

Setting

- A spherical droplet of radius r_0 immersed into a nematic liquid crystal
- A homogeneous external magnetic field $\mathbf{H} = h\mathbf{e}_3$
- Transition between singularities of Saturn ring or dipole type depending on h and r_0

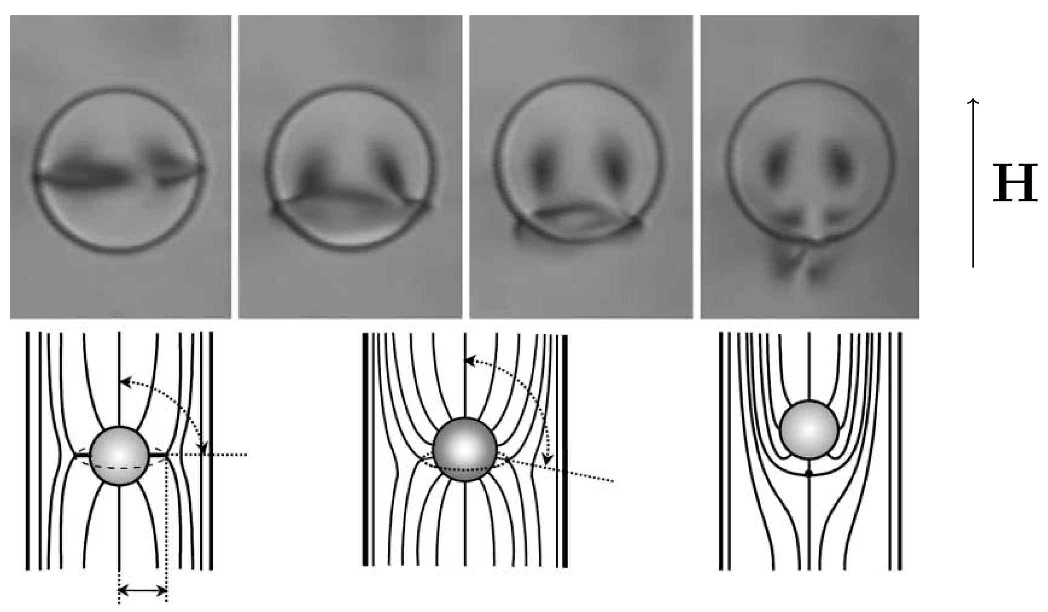
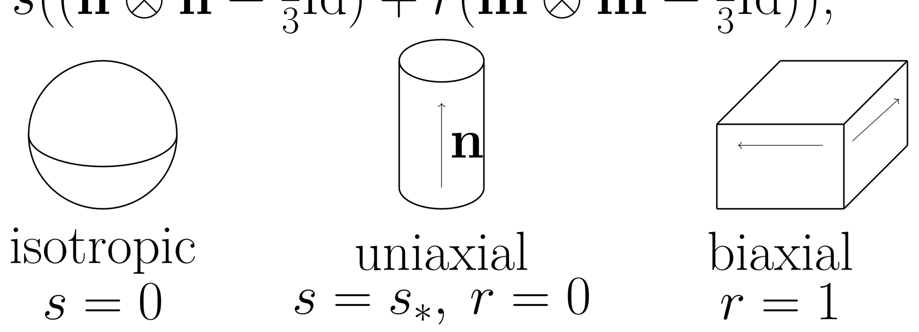


Figure 1: Shrinking of a Saturn ring defect to a dipole by changing the applied field. Figure from [5].

- Rescaled Landau-de Gennes model on $\mathbb{R}^3 \setminus B_1(0)$

$$\mathcal{E}_{\eta,\xi}(Q) = \int_\Omega \underbrace{\frac{1}{2}|\nabla Q|^2}_{\text{elastic}} + \underbrace{\frac{1}{\xi^2}f(Q)}_{\text{bulk}} + \underbrace{\frac{1}{\eta^2}g(Q)}_{\text{magnetic}} \,\mathrm{d}x$$

▷ Q symmetric trace-free matrix of form $Q = s((\mathbf{n}\otimes\mathbf{n} - \frac{1}{3}\mathrm{Id}) + r(\mathbf{m}\otimes\mathbf{m} - \frac{1}{3}\mathrm{Id}))$,



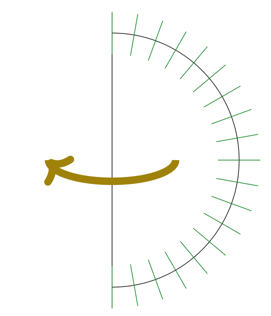
▷ $f \geq 0$ and $f(Q) = 0$ iff Q uniaxial

▷ $g(Q) = c_*^2(1 - \mathbf{n}_3^2)$ for Q uniaxial

- Parameters $\xi \sim r_0^{-1}$ and $\eta \sim (r_0 h)^{-1}$, see [4].

Main result

- Regime $\eta|\ln(\xi)| \to \beta \in (0,\infty)$ for $\eta,\xi \to 0$ (large particles and weak field)
- Boundary condition: strong radial anchoring
- Rotational equivariance of Q around the $\mathbf{e}_3-$axis



Theorem [2]. The energy $\eta\,\mathcal{E}_{\eta,\xi}$ converges to $\mathcal{E}_0$ in a variational sense, where

$$\mathcal{E}_0(F) = 2s_*c_*\int_F (1-\cos(\theta))\,\mathrm{d}\omega + 2s_*c_*\int_{F^c}(1+\cos(\theta))\,\mathrm{d}\omega + \frac{\pi}{2}s_*^2\beta|D\chi_F|(\mathbb{S}^2)$$

for a set $F \subset \mathbb{S}^2$ and θ is the angle between the normal ν on $\mathbb{S}^2$ and $\mathbf{e}_3$.

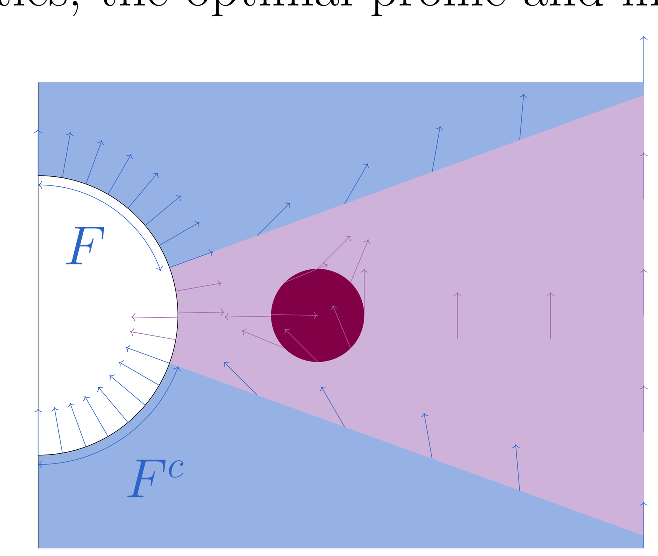
- *Compactness:* $\forall\ Q_{\eta,\xi}$ with $\eta\,\mathcal{E}_{\eta,\xi}(Q_{\eta,\xi}) \leq C \quad \exists\ \mathbf{n}_{\eta,\xi} : \Omega \to \mathbb{S}^2$ and $F \subset \mathbb{S}^2$ of finite perimeter such that $Q_{\eta,\xi} - s_*(\mathbf{n}_{\eta,\xi}\otimes\mathbf{n}_{\eta,\xi} - \frac{1}{3}\mathrm{Id}) \to 0$ in L^2_{loc} and $\{\nu(\omega)\cdot\mathbf{n}_{\eta,\xi} = 1\} \to F$ in BV.
- $\Gamma-$*liminf:* $\forall\ Q_{\eta,\xi} \quad \exists\ F \subset \mathbb{S}^2$ with $\liminf_{\eta,\xi\to 0} \eta\,\mathcal{E}_{\eta,\xi}(Q_{\eta,\xi}) \geq \mathcal{E}_0(F)$.
- $\Gamma-$*limsup:* $\forall\ F \subset \mathbb{S}^2 \quad \exists\ Q_{\eta,\xi}$ such that $\limsup_{\eta,\xi\to 0} \eta\,\mathcal{E}_{\eta,\xi}(Q_{\eta,\xi}) \leq \mathcal{E}_0(F)$.

Lower bound

- Introduce an approximative sequence Q minimizing $\mathcal{E}_{\eta,\xi}(Q) + \xi^{-\alpha}\|Q - Q_{\eta,\xi}\|^2_{L^2}$.
- Establish upper bounds on the size of singularities of Q where f is large.
- Use methods from [3] to derive a lower bound in regions where $\eta|\nabla Q|^2 + \eta\xi^{-2}f(Q)$ dominates.
- Conclude that if f is small, there exists a lifting $\mathbf{n}$ of Q with values in $\mathbb{S}^2$.
- Prove that $\mathbf{n}$ turns to $\mathbf{e}_3$ at infinity, define approximations of F and F^c using $\mathbf{n}|_{\partial\Omega}$.
- A radial auxiliary problem as in [1] determines the contribution from $\eta|\nabla Q|^2 + \eta^{-1}g(Q)$ in terms of $\mathbf{n}$.

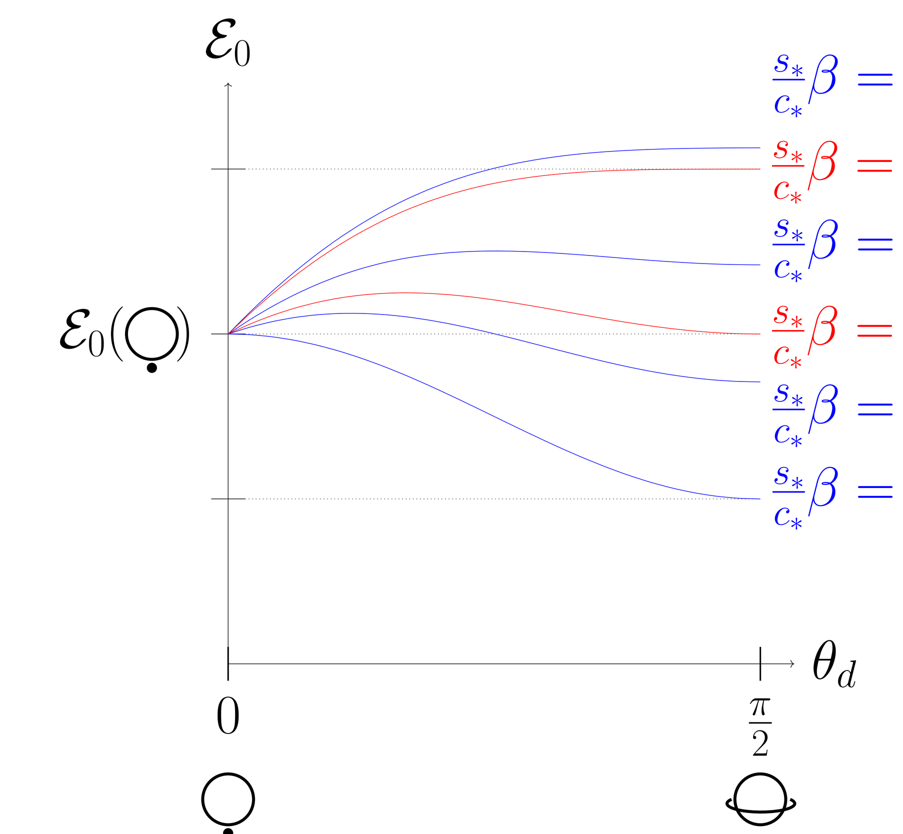
Upper bound

- Use the uniaxial optimal profile of the radial auxiliary problem in the interior of F.
- Place singularities of degree $\pm\frac{1}{2}$ and size ξ at distance η of $\partial\Omega$ to create ∂F.
- Interpolate in the regions between the singularities, the optimal profile and infinity.



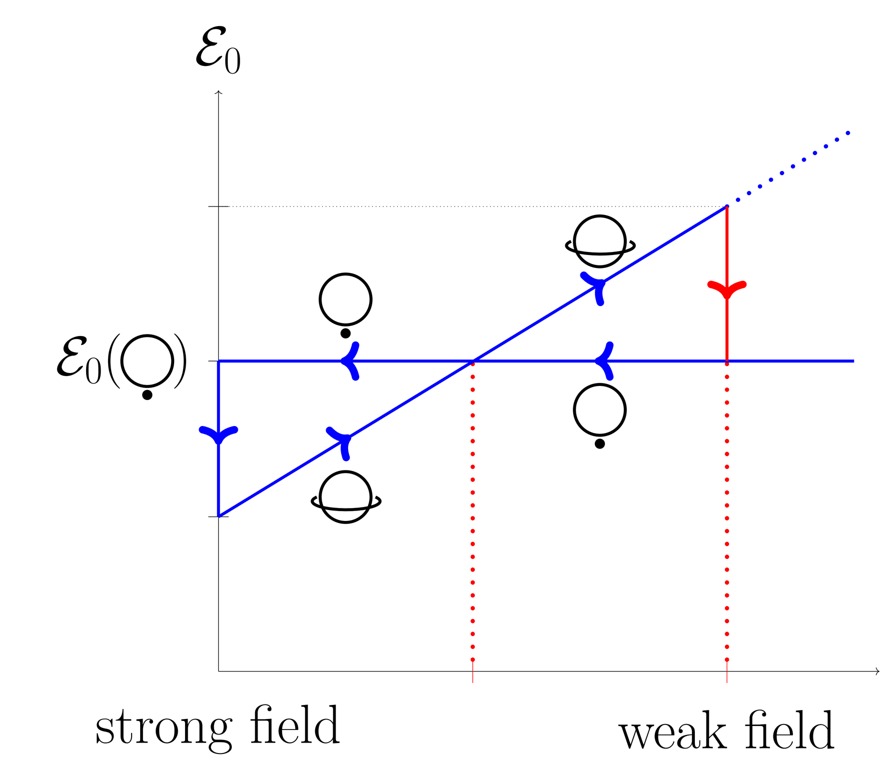
Limit model

- Minimizer of $\mathcal{E}_0$: circle on $\mathbb{S}^2$ at angle θ_d
- Derived numerically in [6, Fig. 11]



Hysteresis

- Predicted in [6]
- Not yet observed experimentally



References

[1] S. Alama, L. Bronsard, and X. Lamy. "Spherical Particle in Nematic Liquid Crystal Under an External Field: The Saturn Ring Regime". In: *J. Nonlinear Sci.* 28.4 (2018), pp. 1443–1465.

[2] F. Alouges, A. Chambolle, and D. Stantejsky. "The saturn ring effect in nematic liquid crystals with external field: effective energy and hysteresis". http://arxiv.org/pdf/2005.06238v1. 2020.

[3] G. Canevari. "Line defects in the small elastic constant limit of a three-dimensional Landau–de Gennes model". In: *Arch. Ration. Mech. Anal.* 223.2 (2017), pp. 591–676.

[4] E. C. Gartland. "Scalings and Limits of Landau-deGennes Models for Liquid Crystals: A Comment on Some Recent Analytical Papers". In: *Math. Modelling and Anal.* 23.3 (2018), pp. 414–432.

[5] J. Loudet, O. Mondain-Monval, and P. Poulin. "Line defect dynamics around a colloidal particle". In: *Eur. Phys. J. E* 7.3 (2002), pp. 205–208.

[6] H. Stark. "Director field configurations around a spherical particle in a nematic liquid crystal". In: *Eur. Phys. J. B* 10.2 (1999), pp. 311–321.