1-Laplacian on metric random walk spaces

Wojciech Górny
University of Vienna, University of Warsaw
wojciech.gorny@univie.ac.at

Abstract

We study the nonlocal 1-Laplacian operator in a quite general framework of metric random walk spaces. These include as special cases locally finite graphs and Euclidean spaces with nonlocality given by a nonnegative radial kernel. We provide some geometric conditions on the space and the random walk under which the problem admits a solution and study its properties. This poster is based on a joint work with Jose M. Mazón.

Introduction

The starting point of this presentation is the least gradient problem ([1,6])

$$\min \left\{ \int_\Omega |Du| : \quad u \in BV(\Omega), \quad u|_{\partial\Omega} = h \right\}. \tag{1}$$

Depending on the regularity of h and geometry of Ω, the boundary condition can be understood in several different ways. The approach we will focus is the relaxed formulation: assume that Ω is an open bounded set, $\partial\Omega$ is Lipschitz and $h \in L^1(\partial\Omega)$. The relaxed energy functional associated to problem (1) is the functional $\Phi_h : L^{\frac{N}{N-1}}(\Omega) \to (-\infty, +\infty]$:

$$\Phi_h(u) = \begin{cases} \int_\Omega |Du| + \int_{\partial\Omega} |u - h| \, d\mathcal{H}^{N-1} & \text{if } u \in BV(\Omega), \\ +\infty & \text{if } u \in L^{\frac{N}{N-1}}(\Omega) \setminus BV(\Omega). \end{cases} \tag{2}$$

The Euler-Lagrange equation corresponding to minimisation of Φ_h is the Dirichlet problem for the 1-Laplacian operator (see [4])

$$\begin{cases} -\mathrm{div}\left(\frac{Du}{|Du|} \right) = 0 \,, & \text{in } \Omega \,, \\ u = h \,, & \text{on } \partial\Omega. \end{cases} \tag{3}$$

It admits a solution $u \in BV(\Omega)$ for every $h \in L^1(\partial\Omega)$. Moreover, all the solutions can be characterised using a single vector field $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^N)$.

Here, we present a nonlocal counterpart of the above results developed in [2] together with Jose M. Mazón. This is done in a unified framework of metric random walk spaces, which include as special cases the nonlocality defined via a symmetric nonnegative kernel on $\mathbb{R}^N$, ε-step random walks on Ahlfors regular spaces, and the case of locally finite weighted graphs. We study a nonlocal counterpart of the 1-Laplacian and give a characterisation of solutions to the Dirichlet problem.

Metric random walk spaces

A *metric random walk space* (X, d, m) is a Polish metric space (X, d) equipped with a random walk m, i.e. a family of probability measures m_x (for $x \in X$) with finite first moment such that the measures m_x depend in a Borel measurable way on the point $x \in X$.

A Radon measure ν on X is *invariant* for the random walk m if for any $u \in L^1(X, \nu)$

$$\int_X u(x) \, d\nu(x) = \int_X \left(\int_X u(y) \, dm_x(y) \right) \, d\nu(x).$$

The measure ν is said to be *reversible* if a more detailed balance condition holds:

$$dm_x(y) \, d\nu(x) = dm_y(x) \, d\nu(y).$$

Examples of invariant and reversible random walks include $\mathbb{R}^N$ equipped with a radially symmetric kernel $J : \mathbb{R}^N \to [0, \infty)$ and locally finite weighted graphs.

We also define the space $BV_m(X)$ associated to the random walk $m = (m_x)$

$$BV_m(X) := \left\{ u : X \to \mathbb{R} \ \ \nu\text{-measurable} \ : \ \frac{1}{2} \int_X \int_X |u(y) - u(x)| \, dm_x(y) \, d\nu(x) < \infty \right\}.$$

We have $L^1(X, \nu) \subset BV_m(X)$. The quantity on the right is called the m-total variation and is denoted by $TV_m(u)$. It is convex, continuous with respect to convergence in $L^1(X, \nu)$ and lower semicontinuous with respect to weak convergence in $L^1(X, \nu)$.

Notions of solutions

Formulation of the Dirichlet problem for the m-total variation faces an additional difficulty which does not appear in the Euclidean case. Since the range of interaction is positive, the boundary datum ψ needs to be defined on a set larger than the topological boundary of Ω. Given $\Omega \subset X$ a ν-measurable set, we define its m-boundary as

$$\partial_m \Omega := \{ x \in X \setminus \Omega \ : \ m_x(\Omega) > 0 \}.$$

We set $\Omega_m := \Omega \cup \partial_m \Omega$. For $\psi \in L^1(\partial_m \Omega, \nu)$ and $u \in BV_m(\Omega)$, we define the function

$$u_\psi(x) := \begin{cases} u(x) & \text{if } \ x \in \Omega \\ \psi(x) & \text{if } \ x \in \partial_m \Omega. \end{cases}$$

Consider the relaxed energy functional $\mathcal{J}_\psi : L^1(\Omega, \nu) \to [0, +\infty[$ given by

$$\mathcal{J}_\psi(u) := TV_m(u_\psi) = \frac{1}{2} \int_{\Omega_m} \int_{\Omega_m} |u_\psi(y) - u_\psi(x)| \, dm_x(y) \, d\nu(x). \tag{4}$$

For arbitrary $\psi \in L^\infty(\partial_m \Omega, \nu)$ we easily prove existence of minimisers of $\mathcal{J}_\psi$ using lower semicontinuity of the m-total variation with respect to weak convergence in $L^1(X, \nu)$.

Our goal is to provide an Euler-Lagrange characterisation of minimisers of $\mathcal{J}_\psi$. This is done using the m-1-Laplacian operator Δ_1^m, first introduced in [5] and formally defined as

$$\Delta_1^m u(x) = \int_{\Omega_m} \frac{u_\psi(y) - u(x)}{|u_\psi(y) - u(x)|} \, dm_x(y) \quad \text{for } x \in \Omega.$$

Now, we are ready to introduce a notion of solutions to the *nonlocal* 1-*Laplace problem* with Dirichlet boundary condition ψ

$$\begin{cases} -\Delta_1^m u(x) = 0, \ x \in \Omega, \\ u(x) = \psi(x), \quad x \in \partial_m \Omega. \end{cases} \tag{5}$$

Definition 1. Let $\psi \in L^1(\partial_m \Omega, \nu)$. We say that $u \in BV_m(\Omega)$ is a solution to (5) if there exists $g \in L^\infty(\Omega_m \times \Omega_m, \nu \otimes \nu)$ with $\|g\|_\infty \leq 1$ verifying

$$g(x, y) = -g(y, x) \quad \text{for } (\nu \otimes dm_x)\text{-a.e } \ (x, y) \ \text{ in } \Omega_m \times \Omega_m, \tag{6}$$

$$g(x, y) \in \mathrm{sign}(u_\psi(y) - u_\psi(x)) \quad \text{for } (\nu \otimes dm_x)\text{-a.e } \ (x, y) \ \text{ in } \Omega_m \times \Omega_m, \tag{7}$$

and

$$-\int_{\Omega_m} g(x, y) \, dm_x(y) = 0 \quad \text{for } \nu\text{-a.e } \ x \in \Omega. \tag{8}$$

Existence of solutions

We start by formulating a key geometric assumption required for existence of solutions in the sense of Definition 1.

Definition 2. Let $p \geq 1$. We say that Ω satisfies a *nonlocal p-Poincaré Inequality* if there exists $\lambda > 0$ such that

$$\lambda \int_\Omega |u(x)|^p \, d\nu(x) \leq \int_\Omega \int_{\Omega_m} |u_\psi(y) - u(x)|^p \, dm_x(y) \, d\nu(x) + \int_{\partial_m \Omega} |\psi(y)|^p \, d\nu(y)$$

for all $u \in L^p(\Omega, \nu)$ and $\psi \in L^p(\partial_m \Omega, \nu)$.

The nonlocal Poincaré inequality may be viewed as a result that describes how the metric random walk m_x and the topology of X are related. Roughly speaking, its validity means that the probability measures m_x are uniformly distributed around x (or equivalently that the measures m_x do not distinguish between directions). The nonlocal Poincaré inequality holds for a variety of settings, including radial kernels on Euclidean spaces, finite graphs, and ε-step random walks on Ahlfors regular spaces.

The nonlocal Poincaré inequality is a crucial tool in the proof of existence of solutions to (5), because they are obtained using p-Laplace type approximation. Moreover, without this assumptions existence of solutions may fail; there is a counterexample which is a locally finite weighted graph.

Theorem 1. *Suppose that $\psi \in L^\infty(\partial_m \Omega, \nu)$. If Ω satisfies a nonlocal p-Poincaré inequality for all $p > 1$, then there exists a solution to problem* (5).

Once we know that a solution exists, we can use the antisymmetric function g given by Definition 1 to prove a characterisation of minimisers of the functional $\mathcal{J}_\psi$.

Theorem 2. *Let $\psi \in L^\infty(\partial_m \Omega, \nu)$. If Ω satisfies a nonlocal p-Poincaré inequality for all $p > 1$, then $u \in L^1(\Omega)$ is a minimiser of the functional $\mathcal{J}_\psi$ given by the formula* (4) *if and only if it is a solution to problem* (5).

Moreover, as a corollary of the proof we get that all the solutions to these problems can be characterised using a single function g.

Median value property

Given a ν-measurable function $u : \Omega \to \mathbb{R}$, we decompose the space X as

$$E_+^x = \{ y \in X : u_\psi(y) > u_\psi(x) \}, \qquad E_-^x = \{ y \in X : u_\psi(y) < u_\psi(x) \},$$
$$E_0^x = \{ y \in X : u_\psi(y) = u_\psi(x) \}.$$

Definition 3. We say that a ν-measurable function $u : \Omega \to \mathbb{R}$ satisfies the *m-median value property* if the condition

$$m_x(E_+^x \cup E_0^x) \geq \frac{1}{2} \quad \text{and} \quad m_x(E_-^x \cup E_0^x) \geq \frac{1}{2}. \tag{9}$$

is satisfied for ν-a.e. $x \in \Omega$.

It turns out that solutions to (5) satisfy the nonlocal median value property. In the other direction, it is not necessarily the case; examples to the contrary exist even in Euclidean spaces (see [3]). Nonetheless, a partial converse still holds.

Theorem 3. *Suppose that $u \in BV_m(\Omega)$ is a solution to* (5)*. Then it satisfies the m-median value property. Moreover, if $u \in BV_m(\Omega)$ satisfies the m-median value property, then it satisfies Definition 1 except for antisymmetry of the function g.*

Funding

The work of the author has been partially supported by the National Science Centre, Poland, research project no. 2017/27/N/ST1/02418.

References

[1] E. Bombieri, E. de Giorgi, and E. Giusti, *Minimal cones and the Bernstein problem*, Invent. Math. **7** (1969), 243–268.

[2] W. Górny and J.M. Mazón, *Least gradient functions in metric random walk spaces*, ESAIM:COCV, to appear.

[3] J.M. Mazón, M. Perez-Llanos, J.D. Rossi, and J. Toledo, *A nonlocal 1-Laplacian problem and median values*, Publ. Mat. **60** (2016), 27–53.

[4] J.M. Mazón, J.D. Rossi, and S. Segura de Leon, *Functions of least gradient and 1-harmonic functions*, Indiana Univ. Math. J. **63** (2014), 1067–1084.

[5] J.M. Mazón, M. Solera, and J. Toledo, *The total variation flow in metric random walk spaces*, Calc. Var. PDE **59** (2020), 29.

[6] P. Sternberg, G. Williams, and W.P. Ziemer, *Existence, uniqueness, and regularity for functions of least gradient*, J. Reine Angew. Math. **430** (1992), 35–60.