The first two chapters from the book: Elements of Asymptotic Geometry

Sergei Buyalo

Viktor Schroeder

Steklov Institute for Mathematics at St. Petersburg
E-mail address: sbuyalo@pdmi.ras.ru

Institut für Mathematik, Universität Zürich
E-mail address: vschroed@math.unizh.ch

Chapter 1

Hyperbolic geodesic spaces

Here we recall basic notions related to metric spaces, define hyperbolic geodesic metric spaces and prove the fundamental theorem about the stability of geodesics in hyperbolic spaces.

1.1. Geodesic metric spaces

A *metric* on a set X is a function $d : X \times X \to \mathbb{R}$ which

(1) is positive: $d(x, x') \geq 0$ for every x, $x' \in X$ and $d(x, x') = 0$ if and only if $x = x'$;

(2) is symmetric: $d(x, x') = d(x', x)$ for every x, $x' \in X$;

(3) satisfies the triangle inequality: $d(x, x'') \leq d(x, x') + d(x', x'')$ for every x, x', $x'' \in X$.

Given a metric d, the value $d(x, x')$ is called *distance* between the points x, x'. We often use the notation $|xx'|$ for the distance between x, x' in a given metric space X, and λX for the metric space obtained from X by multiplying all distances by the factor $\lambda > 0$.

A map $f : X \to Y$ between metric spaces is said to be *isometric* if it preserves the distances, i.e. $|f(x)f(x')| = |xx'|$ for each x, $x \in X$. Clearly, every isometric map is injective.

A *geodesic* in a metric space X is any isometric map $\gamma : I \to X$, where $I \subset \mathbb{R}$ is an interval (open, closed or half-open, finite or infinite). The image $\gamma(I)$ of such a map is also called a geodesic. A metric space X is said to be *geodesic* if any two points in X can be connected by a geodesic. We use the notation xx' for a geodesic in X between x, x', calling it a *segment* (even in the case when there are possibly several such segments).

Remark. In many theories where the local geometry plays an essential role as e.g. in Riemannian geometry, a geodesic means a curve $\gamma : I \to X$ which is only locally isometric, while on large scales the length of a segment might be larger than the distance between its end points. However, we always consider geodesics in the sense of the definition above.

1.2. Hyperbolic geodesic spaces

A *triangle* xyz in a geodesic space X is the union of segments xy, yz, zx, called the *sides*, connecting pairwise its *vertices* x, y, $z \in X$. More generally an *n-gon* $x_1 \dots x_n$ in X is the union of segments $x_1x_2, \dots, x_nx_1$.

The property of a geodesic space to be hyperbolic is defined in terms of triangles and the Gromov product, which is a useful notion in many circumstances.

1.2.1. Gromov product. Let X be a metric space. Fix a base point $o \in X$ and for x, $x' \in X$ put $(x|x')_o = \frac{1}{2}(|xo| + |x'o| - |xx'|)$. The number $(x|x')_o$ is nonnegative by the triangle inequality, and it is called the *Gromov product* of x, x' w.r.t. o. Geometrically, the product can be interpreted as follows.

Lemma 1.2.1. *Let X be a geodesic space and xyz a triangle in X. There is a unique collection of points $u \in yz$, $v \in xz$, $w \in xy$ such that $|xv| = |xw|$, $|yu| = |yw|$, $|zv| = |zu|$.*

Proof. The equation system

$$\begin{array}{ccccccc} a & + & b & & & = & |xy| \\ a & & & + & c & = & |xz| \\ & & b & + & c & = & |yz| \end{array}$$

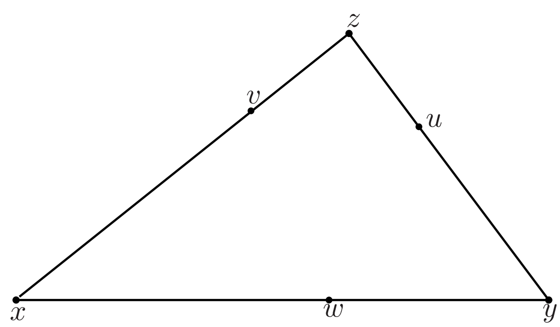

Figure 1.1: Gromov product and equiradial points

has a unique solution and a, b, c are nonnegative by the triangle inequality. Then, the points u, v, w are uniquely determined by the conditions $|xv| = a$, $|yw| = b$, $|zu| = c$. $\square$

The points $u \in yz$, $v \in xz$, $w \in xy$ are called *equiradial* points. Note that
$$a = \frac{1}{2}(|xy| + |xz| - |yz|) = (y|z)_x$$
and similarly $b = (x|z)_y$, $c = (x|y)_z$.

For example, if a triangle $xyz \subset X$ is a *tripod*, i.e. the union $wx \cup wy \cup wz$ with only one common point $w \in X$, then $(y|z)_x = |xw|$.

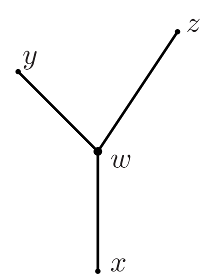

Figure 1.2: Tripod

Definition 1.2.2. A geodesic metric space is called *δ-hyperbolic*, $\delta \geq 0$, if for any triangle $xyz \subset X$ the following holds: If $y' \in xy$, $z' \in xz$ are points with $|xy'| = |xz'| \leq (y|z)_x$, then $|y'z'| \leq \delta$.

Roughly speaking, in a δ-hyperbolic geodesic space X two sides xy and xz of any triangle xyz coming out of the common vertex x run together within the distance δ up to the moment $(y|z)_x$ and after that they start to diverge with almost maximal possible speed. This point of view becomes effective at distances large compared to δ.

The space is *(Gromov) hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$. The constant δ is called a *hyperbolicity constant* for X. Clearly, in a δ-hyperbolic space any side of any triangle lies in the δ-neighborhood of the two other sides. This is the case $k = 1$ of the following Lemma.

Lemma 1.2.3. *Let $x_1 \dots x_n$ be an n-gon with $n \leq 2^k + 1$ for some $k \in N$, then every side is contained in the $k\delta$-neighborhood of the union of the other sides.*

Proof. We show that a point $x \in x_n x_1$ has distance $\leq k\delta$ from $x_1 x_2 \cup \dots \cup x_{n-1} x_n$. Choose the midpoint x_m with $m = [n/2] + 1$ where $[\ ]$ is the integer part and consider the triangle $x_1 x_m x_n$. By δ-hyperbolicity there exists $y \in x_1 x_m \cup x_m x_n$ with $|xy| \leq \delta$. In the case $y \in x_1 x_m$ (resp. $y \in x_m x_n$) the induction hypothesis for the polygon $x_1 \dots x_m$ (resp. $x_m \dots x_n$) implies that y has distance $\leq (k-1)\delta$ from $x_1 x_2 \cup \dots \cup x_{m-1} x_m$ (resp. $x_m x_{m+1} \cup \dots \cup x_{n-1} x_n$). The claim follows. $\square$

Exercise 1.2.4. Show that if any side of any triangle in a geodesic space X lies in the δ-neighborhood of the union of the two other sides for some fixed $\delta \geq 0$, then X is hyperbolic (Rips' definition of hyperbolicity). Estimate the hyperbolicity constant for X.

Example 1.2.5. A *metric tree* is a geodesic space in which every triangle is a tripod (possibly degenerate). Clearly, every metric tree is a 0-hyperbolic space.

1.3. Stability of geodesics

In this section we show that geodesics in hyperbolic spaces are stable. This means that if we enlarge the class of geodesics to the larger class of quasi-geodesics, then still each quasi-geodesic stays in uniformly bounded distance to a geodesic. To make this concept precise we need the concept of quasi-isometric maps.

1.3.1. Quasi-isometric maps. The notion of a quasi-isometric map is a rough version of a bilipschitz map; recall that a map $f : X \to Y$ between metric spaces is *bilipschitz*, if

$$\frac{1}{a}|xx'| \leq |f(x)f(x')| \leq a|xx'|$$

for some $a \geq 1$ and all x, $x' \in X$ (in this definition, we do not require that $f(X) = Y$).

A subset $A \subset Y$ in a metric space Y is called a *net*, if the distances of all points $y \in Y$ to A are uniformly bounded.

A map $f : X \to Y$ between metric spaces is said to be *quasi-isometric,* if there are $a \geq 1$, $b \geq 0$, such that

$$\frac{1}{a}|xx'| - b \leq |f(x)f(x')| \leq a|xx'| + b$$

for all x, $x' \in X$. In other words, a map is quasi-isometric if it is bilipschitz on large scales.

If, in addition, the image $f(X)$ is a net in Y, then f is called a *quasi-isometry*, and the spaces X and Y are called *quasi-isometric*. We also say that f is (a, b)-quasi-isometric, and call a, b the *quasi-isometricity constants*.

A *quasi-geodesic* in X is a quasi-isometric map $\gamma : I \to X$ where $I \subset \mathbb{R}$ is an interval.

For general metric spaces a quasi-geodesic can be far from a geodesic. Consider for example in the Euclidean plane the spiral $\gamma : (0, \infty) \to \mathbb{R}^2$, $\gamma(t) = t(\cos(\ln t), \sin(\ln t))$.

Since $|\gamma(t)| = t$ and $|\gamma'(t)| = \sqrt{2}$ for all t, we easily see

$$\frac{1}{\sqrt{2}}|\gamma(t)\gamma(s)| \leq |t - s| \leq |\gamma(t)\gamma(s)|$$

which implies that γ is a quasi-geodesic. This curve is in no way close to any geodesic. In hyperbolic geodesic spaces the situation is completely different. We will show that in a geodesic hyperbolic space every quasi-geodesic will stay in uniform bounded distance to a honest geodesic.

To start our argument we first show that, roughly speaking, in order to avoid a ball in a hyperbolic space one needs to go an exponentially long path.

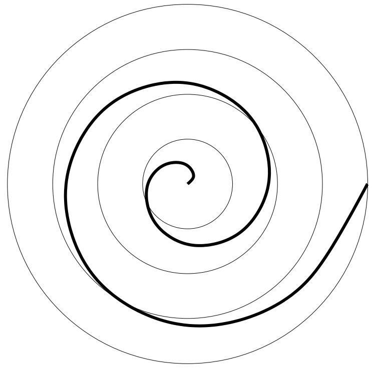

Figure 1.3: The spiral γ on logarithmic scale

We use the notation $B_r(x)$ for the open ball of radius r centered at x in a metric space X, $B_r(x) = \{x' \in X : |xx'| < r\}$. Furthermore, $\overline{B}_r(x)$ is the closed ball $\{x' \in X : |xx'| \leq r\}$.

By an a-path, $a > 0$, in a metric space we mean a finite or infinite sequence of points $\{x_i\}$ with $|x_i x_{i+1}| \leq a$ for each i.

Lemma 1.3.1. *Assume that an a-path $f : \{1, \dots, N\} \to X$ in a geodesic δ-hyperbolic space, $\delta > 0$, lies outside of the ball $B_r(x)$ centered at some point $x \in f(1)f(N)$. Then*

$$N \geq c \cdot 2^{r/\delta}$$

for some constant $c > 0$ depending only on a and δ.

Proof. Let k be the smallest integer with $N \leq 2^k + 1$ (then $N \geq 2^{k-1}$). By Lemma 1.2.3 there exists a point $y \in f(j)f(j+1)$ for some $j \in \{1, \dots, N-1\}$ such that $|xy| \leq k\delta$. Note that $|xy| \geq r - a/2$, and hence $k \geq r/\delta - a/(2\delta)$. Hence $N \geq 2^{k-1} \geq c \cdot 2^{r/\delta}$ with $c = 2^{-(a/2\delta+1)}$. □

We are now able to proof the stability of quasigeodesics.

Theorem 1.3.2 (Stability of geodesics)**.** *Let X be a δ-hyperbolic geodesic space and $a \geq 1$, $b \geq 0$. There exists $H = H(a, b, \delta) > 0$ such that for every $N \in \mathbb{N}$ the image of every (a, b)-quasi-isometric map $f : \{1, \dots, N\} \to X$, $\mathrm{im}(f)$, lies in the H-neighborhood of any geodesic $c : [0, l] \to X$ with $c(0) = f(1)$, $c(l) = f(N)$, and vice versa, c lies in the H-neighborhood of $\mathrm{im}(f)$.*

Proof. We first show that c lies in the h-neighborhood of $\mathrm{im}(f)$, where $h = h(a, b, \delta) > 0$ depends only on a, b and δ. Note that f is an $(a+b)$-path in X. Choose h maximal with the property that $\mathrm{im}(f)$ lies outside the ball $B_h(x)$ for some $x \in c$.

Take $y \in c(0)x$, $y' \in xc(l)$ with $|yx| = |xy'| = 2h$ (if the distance between x and one of the ends of c is less than $2h$, we take as y or y' the corresponding end). There are i, $i' \in A = \{1, \dots, N\}$ with $|f(i)y|$, $|f(i')y'| \le h$ and the segments $yf(i)$, $y'f(i')$ lie outside the ball $B_h(x)$. By taking appropriate points on these segments together with $f(i), \dots, f(i')$, we find an $(a+b)$-path between y and y' outside $B_h(x)$ which contains $K \le |i - i'| + 3 + \frac{2h}{a+b}$ points.

By quasi-isometricity of f, we have

$$|i - i'| \le a(|f(i)f(i')| + b) \le 6ah + ab.$$

On the other hand, $K \ge c \cdot 2^{h/\delta}$ by Lemma 1.3.1 where $c = c(a, b, \delta)$. These estimates together give an effective upper bound $h(a, b, \delta)$ for the radius h.

To complete the proof, consider a maximal sub-interval $\{j, \dots, j'\} \subset A$ such that $f(\{j, \dots, j'\})$ lies outside the h-neighborhood of c, $h = h(a, b, \delta)$. Since c is contained in the h-neighborhood of $\mathrm{im}(f)$, there are $i \in \{1, \dots, j\}$, $i' \in \{j', \dots, N\}$ and $z \in c$ so that $|zf(i)|$, $|zf(i')| \le h$. Then $|f(i)f(i')| \le 2h$, and $|i - i'| \le 2ah + ab$ by quasi-isometricity of f. Hence, $\mathrm{im}(f)$ is contained in the H-neighborhood of c, where $H = h + a(2ah + ab) + b$, $H = H(a, b, \delta)$. □

Exercise 1.3.3. Derive the following consequences of Theorem 1.3.2.

Corollary 1.3.4. *Let X be hyperbolic geodesic space. Then there is no quasi-isometric map $f : \mathbb{R}^2 \to X$. (Hint: Assuming that such a map exists, consider images of larger and larger equilateral triangles to obtain a contradiction using the stability of geodesics in X).*

Corollary 1.3.5. *If a geodesic space X is quasi-isometric to a hyperbolic geodesic space Y, then X is also hyperbolic. (Hint: take any triangle in X and compare it with its image in Y to conclude using stability of geodesics in Y that the triangle satisfies a δ-hyperbolicity condition).*

1.4. Additional and historical remarks

1.4.1. The real hyperbolic space H^n. The real hyperbolic space H^n is a simply connected, complete Riemannian manifold of dimension $n \geq 2$ having the constant sectional curvature -1. Various models of H^n are discussed in Appendix A. This is the basic example of Gromov hyperbolic spaces.

Exercise 1.4.1. Using the parallelism angle formula (see Appendix A, Lemma **??**), show that the space H^n is δ-hyperbolic with $\delta < \ln 3 = 1.0986\ldots$. Actually, $\delta = 2\ln\tau = 0.9624\ldots$ where τ is the golden ratio, $\tau^2 = \tau + 1$.

1.4.2. Gromov hyperbolic groups. An important class of hyperbolic spaces is the class of Gromov hyperbolic groups which are defined as follows.

Let G be a finitely generated group and $S \subset G$ a finite set generating G. We assume that S does not contain the unit element of G and it is symmetric, i.e. $g \in S$ if and only if $g^{-1} \in S$. The *Cayley graph* of (G, S) is a graph $\Gamma = \Gamma(G, S)$ with the vertex set G, and vertices g, $g' \in G$ are connected by an edge if and only if $g^{-1}g' \in S$. The Cayley graph Γ carries the path metric d_S for which every edge has length one. Such a metric when viewed on G is called a *word metric*. Clearly, Γ is a geodesic space.

A finitely generated group G is said to be *word hyperbolic* or *Gromov hyperbolic* if its Cayley graph $\Gamma(G, S)$ is a hyperbolic space for some generating system S.

Exercise 1.4.2. Show (using Corollary 1.3.5) that the property of a finitely generated group G to be hyperbolic is independent of the choice of a generating system S.

1.4.3. $\mathrm{CAT}(-1)$-spaces. Let xyz be a geodesic triangle in a geodesic metric space X. A *comparison triangle*

$$\widetilde{x}\,\widetilde{y}\,\widetilde{z} \subset \mathrm{H}^2$$

is a triangle with the same side-lengths. *Comparison points* on the sides are obtained by taking equal distances from the vertices.

A complete geodesic space X is a $\mathrm{CAT}(-1)$-*space* if for each triangle $xyz \subset X$ and each $u \in xy$, $v \in xz$, it holds that $|uv| \leq |\widetilde{u}\,\widetilde{v}|$, where $\widetilde{u} \in \widetilde{x}\,\widetilde{y}$, $\widetilde{v} \in \widetilde{x}\,\widetilde{z}$ are comparison points on the sides of $\widetilde{x}\,\widetilde{y}\,\widetilde{z} \subset \mathrm{H}^2$.

That is, any triangle in X is thinner than its comparison triangle in H^2. Thus by definition, every CAT(-1)-space is δ-hyperbolic with $\delta \le \delta_{\mathrm{H}^2}$.

The class of CAT(-1)-spaces is very large. Recall that a *Hadamard manifold* is a complete simply connected Riemannian manifold with nonpositive sectional curvatures. Every Hadamard manifold with sectional curvatures $K \le -1$ is a CAT(-1)-space. Furthermore, any metric tree is a CAT(κ)-space for each $\kappa < 0$, in particular, it is CAT(-1). The class of CAT(-1)-spaces also includes various hyperbolic buildings. One the other hand, there are compact nonpositively curved (in Alexandrov sense) 2-polyhedra with word hyperbolic fundamental group that admit no metric with CAT(-1) universal covering, see e.g. [**?**].

Taking comparison triangles in $\mathbb{R}^2$, one similarly obtains the important class of CAT(0) or *Hadamard spaces,* i.e. complete geodesic spaces with triangles thinner than the Euclidean comparison triangles. In any Hadamard space X, all points x, $x' \in X$ are connected by a unique geodesic segment.

1.4.4. The stability of geodesics was discovered in the twenties of the last century by M. Morse, [**?, ?**]. There are several approaches to its proof. The proof presented in sect. 1.3 is very close to the M. Gromov's proof, [**?**], see also [**?**].

Chapter 2

The boundary at infinity

We start this chapter with a discussion of further properties of the Gromov product with the aim of deriving the δ-inequality for hyperbolic geodesic spaces. This allows us to extend the notion of hyperbolicity to metric spaces which are not necessarily geodesic. An important point of this discussion is the Tetrahedron Lemma, which has various applications throughout the book.

Next, we define the boundary at infinity for any hyperbolic space and discuss various structures attached to it: Gromov product, quasi-metrics, visual metrics and topology. We also establish local self-similarity of the boundary at infinity of cocompact hyperbolic spaces.

2.1. δ-inequality and hyperbolic spaces

The Gromov product is monotone in the following sense.

Lemma 2.1.1. *Assume that $y' \in xy$ and $z' \in xz$ in a geodesic space X. Then $(y'|z')_x \le (y|z)_x$.*

Proof. Since

$$\begin{aligned} |xz| &= |xz'| + |z'z|, \\ |y'z| &\le |y'z'| + |z'z|, \end{aligned}$$

we have $|xz| - |y'z| \geq |xz'| - |y'z'|$. Thus $(y'|z')_x \leq (y'|z)_x$. Similarly $(y'|z)_x \leq (y|z)_x$. □

Proposition 2.1.2. *If a geodesic space X is δ-hyperbolic, then*

$$(x|y)_o \geq \min\{(x|z)_o, (z|y)_o\} - \delta$$

for any base point $o \in X$ and any x, y, $z \in X$.

Proof. Put $t_0 = \min\{(x|z)_o, (y|z)_o\}$ and assume that $x' \in ox$, $y' \in oy$ and $z' \in oz$ satisfy $|ox'| = |oy'| = |oz'| = t_0$. Then $|x'z'|$, $|y'z'| \leq \delta$, thus $|x'y'| \leq 2\delta$. On the other hand, by Lemma 2.1.1,

$$(x|y)_o \geq (x'|y')_o = t_0 - \frac{1}{2}|x'y'| \geq t_0 - \delta.$$

□

The inequality from Proposition 2.1.2 is called δ-*inequality.* This inequality is characteristic for the property of a space to be hyperbolic.

Proposition 2.1.3. *Assume that a geodesic space X satisfies the δ-inequality for every base point o and every x, y, $z \in X$. Then X is 4δ-hyperbolic.*

Proof. Assume that points $x' \in ox$, $y' \in oy$ of a triangle $oxy \subset X$ satisfy the condition $|ox'| = |oy'| = t \leq (x|y)_o$. It suffices to show that then $|x'y'| \leq 4\delta$. By the δ-inequality we have

$$\begin{aligned} (x'|y')_o &\geq \min\{(x'|y)_o, t\} - \delta \\ &\geq \min\{\min\{(x|y)_o, t\} - \delta, t\} - \delta \\ &= t - 2\delta, \end{aligned}$$

hence $|x'y'| = 2t - 2(x'|y')_o \leq 4\delta$. □

Finally, we show that the δ-inequality for some base point implies the 2δ-inequality for any other base point. The following terminology is useful. A δ-*triple* is a triple of real numbers a, b, c with the property that the two smallest of these numbers differ by at most δ. To rephrase the δ-inequality we can say that the numbers $(x|y)_o$, $(x|z)_o$, $(y|z)_o$ form a δ-triple.

It is also convenient to write $a \doteq b$ up to an error $\leq c$ or $a \doteq_c b$ instead of $|a - b| \leq c$.

The following important result, which has many applications in the sequel, is called Tetrahedron Lemma.

Lemma 2.1.4. *Let $d_{12}, d_{13}, d_{14}, d_{23}, d_{24}, d_{34}$ be six numbers, such that the four triples $A_1 = (d_{23}, d_{24}, d_{34})$, $A_2 = (d_{13}, d_{14}, d_{34})$, $A_3 = (d_{12}, d_{14}, d_{24})$ and $A_4 = (d_{12}, d_{13}, d_{23})$ are δ-triples. Then*

$$B = (d_{12} + d_{34}, d_{13} + d_{24}, d_{14} + d_{23})$$

is a 2δ-triple.

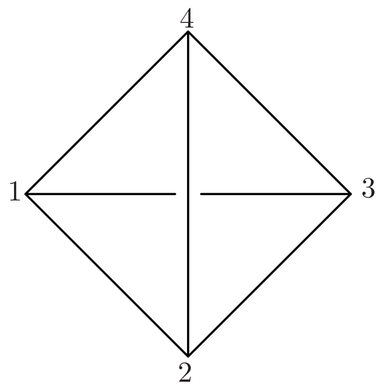

Figure 2.1: Tetrahedron Lemma

Proof. Without loss of generality, we can assume that d_{34} is maximal among the listed numbers. Then $d_{13} \doteq d_{14}$ up to an error $\leq \delta$ since A_2 is a δ-triple, and $d_{23} \doteq d_{24}$ up to an error $\leq \delta$ since A_1 is a δ-triple. Adding these approximate equalities, we obtain that $d_{13}+d_{24} \doteq d_{23}+d_{14}$ up to an error $\leq 2\delta$. Since d_{34} is maximal, this means, if we assume that B is not a 2δ-triple, that $d_{12} < \min\{d_{13}, d_{14}, d_{23}, d_{24}\} - 2\delta$. But this contradicts the fact that A_3 and A_4 are δ-triples. Thus B is a 2δ-triple. $\square$

Lemma 2.1.5. *Assume that a metric space X satisfies the δ-inequality for a base point o. Then for any other base point $x \in X$, the 2δ-inequality is fulfilled.*

Proof. Note that the expression

$$A = (t|y)_o + (x|z)_o - \min\{(t|z)_o + (y|x)_o, (x|t)_o + (y|z)_o\}$$

does not depend on the base point o. Choosing x as the base point, we see $A = (t|y)_x - \min\{(t|z)_x, (z|y)_x\}$. Thus, we have to prove $A \geq -2\delta$. From the δ-inequality for the base point o, it follows that the six numbers $(t|x)_o$, $(t|y)_o$, $(t|z)_o$, $(x|y)_o$, $(x|z)_o$, $(y|z)_o$ satisfy the condition of the Tetrahedron Lemma, which implies $A \geq -2\delta$. □

Now, we extend the notion of hyperbolicity to metric spaces which are not necessarily geodesic.

Definition 2.1.6. A metric space X is *(Gromov) hyperbolic* if it satisfies the δ-inequality

$$(x|y)_o \geq \min\{(x|z)_o, (z|y)_o\} - \delta$$

or, what is the same, the triple $((x|y)_o, (x|z)_o, (y|z)_o)$ is a δ-triple for some $\delta \geq 0$, every base point $o \in X$ and all x, y, $z \in X$.

For geodesic spaces this notion is equivalent to our initial definition by Propositions 2.1.2, 2.1.3. From now on, when speaking about a δ-hyperbolic space X we mean Definition 1.2.2 if X is geodesic, and Definition 2.1.6 otherwise. The same holds for hyperbolicity constants. This causes no ambiguity because of Proposition 2.1.2.

Remark 2.1.7. By Lemma 2.1.5, to prove that a space X is hyperbolic, it suffices to check that the δ-inequality holds for some $\delta \geq 0$, some base point $o \in X$ and all x, y, $z \in X$. We shall often use this remark.

2.2. The boundary at infinity of hyperbolic spaces

There are several possibilities to define the boundary at infinity of a hyperbolic space, ranging from the most geometric one, geodesic boundary, see sect. 2.4.2, to the most analytic one, called Higson corona, which is not discussed in this book. We choose the original Gromov definition, since it is well adapted to the basic property of hyperbolic geodesic spaces that quasi-isometric maps have a natural extension to boundary maps, and the definition appeals to the geometric intuition.

Let X be a hyperbolic space and $o \in X$ a base point. A sequence of points $\{x_i\} \subset X$ *converges to infinity,* if

$$\lim_{i,j\to\infty} (x_i|x_j)_o = \infty.$$

This property is independent of the choice of o since

$$|(x|x')_o - (x|x')_{o'}| \leq |oo'|$$

for any x, x', o, $o' \in X$. Two sequences $\{x_i\}$, $\{x'_i\}$ that converge to infinity are *equivalent* if

$$\lim_{i\to\infty} (x_i|x'_i)_o = \infty.$$

Using the δ-inequality, we easily see that this defines an equivalence relation for sequences in X converging to infinity. The *boundary at infinity* $\partial_\infty X$ of X is defined to be the set of equivalence classes of sequences converging to infinity.

Remark 2.2.1. If $\{x_i\}$ is a sequence converging to infinity and $\{x'_i\}$ a sequence equivalent to $\{x_i\}$ in the sense that $\lim(x_i|x'_i)_o = \infty$, then $\{x'_i\}$ converges to infinity itself. This easily follows from the δ-inequality.

Now, we introduce natural metric structures on the boundary at infinity of a Gromov hyperbolic space X. This is done in three steps. In a first step, we extend the Gromov product to the boundary at infinity. More precisely, we define for a base point $o \in X$ and points ξ, $\eta \in \partial_\infty X$ the product $(\xi|\eta)_o$. In a second step, we define the map $\rho : \partial_\infty X \times \partial_\infty X \to [0, \infty)$ by $\rho(\xi, \eta) = a^{-(\xi|\eta)_o}$, where $a > 1$ is some parameter. The map ρ turns out to be a quasi-metric. In a third step, we apply a standard procedure to obtain from ρ a metric for parameters $a > 1$, a small enough.

2.2.1. Gromov product on the boundary. Fix a base point $o \in X$. For points ξ, $\xi' \in \partial_\infty X$, we define their Gromov product by

$$(\xi|\xi')_o = \inf \liminf_{i\to\infty} (x_i|x'_i)_o,$$

where the infimum is taken over all sequences $\{x_i\} \in \xi$, $\{x'_i\} \in \xi'$. Note that $(\xi|\xi')_o$ takes values in $[0, \infty]$, that $(\xi|\xi')_o = \infty$ if and only if $\xi = \xi'$, and that $|(\xi|\xi')_o - (\xi|\xi')_{o'}| \leq |oo'|$ for any o, $o' \in X$. Furthermore, we obtain the following properties.

Lemma 2.2.2. *Let $o \in X$, let X satisfy the δ-inequality for o, and let ξ, ξ', $\xi'' \in \partial_\infty X$.*

(1) *For arbitrary sequences $\{y_i\} \in \xi$, $\{y'_i\} \in \xi'$, we have*

$$(\xi|\xi')_o \leq \liminf_{i\to\infty} (y_i|y'_i)_o \leq \limsup_{i\to\infty} (y_i|y'_i)_o \leq (\xi|\xi')_o + 2\delta$$

(2) $(\xi|\xi')_o$, $(\xi'|\xi'')_o$, $(\xi|\xi'')_o$ *is a δ-triple.*

Proof. (1) We only need to show that $\limsup_{i\to\infty}(y_i|y_i')_o \le (\xi|\xi')_o + 2\delta$. We can assume that $\xi \ne \xi'$. Applying the standard diagonal procedure, we find sequences $\{x_i\} \in \xi$, $\{x_i'\} \in \xi'$ with $\lim(x_i|x_i')_o = (\xi|\xi')_o$. Let $\{y_i\} \in \xi$, $\{y_i'\} \in \xi'$. For i sufficiently large, we have $(x_i|x_i')_o \doteq (y_i|x_i')_o$ up to an error $\le \delta$ since $(x_i|x_i')_o$, $(y_i|x_i')_o$, $(x_i|y_i)_o$ is a δ-triple, $(x_i|y_i)_o \to \infty$, and two other members are bounded due to the assumption $\xi \ne \xi'$. In the same way, we see $(y_i|x_i')_o \doteq (y_i|y_i')_o$ up to an error $\le \delta$ for i large enough. Thus, $(x_i|x_i')_o \doteq (y_i|y_i')_o$ up to an error $\le 2\delta$ which implies the claim.

(2) Without loss of generality, we have to show

$$(\xi|\xi'')_o \ge \min\{(\xi|\xi')_o, (\xi'|\xi'')_o\} - \delta.$$

Choose $\{x_i\} \in \xi$, $\{x_i'\} \in \xi'$, $\{x_i''\} \in \xi''$ such that $\lim(x_i|x_i'')_o = (\xi|\xi'')_o$. Then

$$(\xi|\xi'')_o \ge \limsup_{i\to\infty} \min\{(x_i|x_i')_o, (x_i'|x_i'')_o\} - \delta \ge \min\{(\xi|\xi')_o, (\xi'|\xi'')_o\} - \delta.$$

□

Similarly, the Gromov product

$$(x|\xi)_o = \inf \liminf_{i\to\infty} (x|x_i)_o$$

is defined for any $x \in X$, $\xi \in \partial_\infty X$, where the infimum is taken over all sequences $\{x_i\} \in \xi$, and the δ-inequality holds for any three points from $X \cup \partial_\infty X$.

2.2.2. Quasi-metric on the boundary. A *quasi-metric space* is a set Z with a function $\rho : Z \times Z \to \mathbb{R}$ which satisfies the conditions

(1) $\rho(z, z') \ge 0$ for every z, $z' \in Z$, and $\rho(z, z') = 0$ if and only if $z = z'$;
(2) $\rho(z, z') = \rho(z', z)$ for every z, $z' \in Z$;
(3) $\rho(z, z'') \le K \max\{\rho(z, z'), \rho(z', z'')\}$ for every z, z', $z'' \in Z$ and some fixed $K \ge 1$.

The function ρ is then called a *quasi-metric*, or more specifically, a K-quasi-metric. The property (3) is a generalized version of the *ultrametric* triangle inequality which is the case $K = 1$.

Remark 2.2.3. If (Z,d) is a metric space, then d is a K-quasi-metric for $K=2$. In general the p-th power d^p of the distance d is not a metric on Z for $p>1$. But d^p is still a 2^p-quasi-metric.

Coming back to the Gromov hyperbolic space X, we fix $a>1$ and consider the function $\rho : \partial_\infty X \times \partial_\infty X \to \mathbb{R}$, $\rho(\xi,\xi') = a^{-(\xi|\xi')_o}$. Then, ρ is a K-quasi-metric on $\partial_\infty X$ with $K=a^\delta$: the properties (1), (2) are obvious, and (3) immediately follows from Lemma 2.2.2(2).

Remark 2.2.4. The quasi-metric ρ defined on $\partial_\infty X$ depends on the base point $o\in X$ and the chosen parameter $a>1$. If we emphasize this dependence, we write $\rho_{o,a}$. Let o, $o'\in X$. Since $|(\xi|\xi')_o-(\xi|\xi')_{o'}|\le |oo'|$ we compute

$$c^{-1}\le \frac{\rho_{o,a}(\xi,\xi')}{\rho_{o',a}(\xi,\xi')}\le c$$

where $c=a^{|oo'|}$. If a, $a'>1$ are different parameters then we have

$$\rho_{o,a'}=\rho_{o,a}^{\alpha}$$

where $\alpha=\frac{\ln a'}{\ln a}$.

There is a standard procedure to construct a metric from a quasi-metric. Let (Z,ρ) be a quasi-metric space. We are interested in obtaining a metric on Z. Since the only problem is the triangle inequality, the following approach is natural. Define a map $d: Z\times Z\to\mathbb{R}$, $d(z,z')=\inf\sum_i\rho(z_i,z_{i+1})$, where the infimum is taken over all sequences $z=z_0,\dots,z_{k+1}=z'$ in Z. By definition, d is then symmetric and satisfies the triangle inequality. We call this construction of d the *chain construction*. The problem with the chain construction is that $d(z,z')$ could be 0 for different points z, z' and the axiom (1) would no longer be satisfied for (Z,d).

Lemma 2.2.5. *Let ρ be a K-quasi-metric on a set Z with $K\le 2$. Then, the chain construction applied to ρ yields a metric d with $\frac{1}{2K}\rho\le d\le\rho$.*

Proof. Clearly, d is nonnegative, symmetric, satisfies the triangle inequality and $d\le\rho$. We prove by induction over the length of sequences $\sigma=\{z=z_0,\dots,z_{k+1}=z'\}$, $|\sigma|=k+2$, that

(2.1)
$$\rho(z,z')\le\sum(\sigma):=K\left(\rho(z_0,z_1)+2\sum_1^{k-1}\rho(z_i,z_{i+1})+\rho(z_k,z_{k+1})\right).$$

For $|\sigma| = 3$, this follows from the triangle inequality (3) for ρ. Assume that (2.1) holds true for all sequences of length $|\sigma| \le k+1$, and suppose that $|\sigma| = k+2$.

Given $p \in \{1, \dots, k-1\}$, we let $\sigma'_p = \{z_0, \dots, z_{p+1}\}$, $\sigma''_p = \{z_p, \dots, z_{k+1}\}$, and note that $\sum(\sigma) = \sum(\sigma'_p) + \sum(\sigma''_p)$.

Because $\rho(z, z') \le K \max\{\rho(z, z_p), \rho(z_p, z')\}$, there is a maximal $p \in \{0, \dots, k\}$ with $\rho(z, z') \le K\rho(z_p, z')$. Then $\rho(z, z') \le K\rho(z, z_{p+1})$.

Assume now that $\rho(z, z') > \sum(\sigma)$. Then, in particular, $\rho(z, z') > K\rho(z, z_1)$ and $\rho(z, z') > K\rho(z_k, z')$. It follows that $p \in \{1, \dots, k-1\}$ and thus by the inductive assumption

$$\rho(z, z_{p+1}) + \rho(z_p, z') \le \sum(\sigma'_p) + \sum(\sigma''_p) = \sum(\sigma) < \rho(z, z').$$

On the other hand,

$$\rho(z, z') \le K \min\{\rho(z, z_{p+1}), \rho(z_p, z')\} \le \rho(z, z_{p+1}) + \rho(z_p, z')$$

because $K \le 2$. This is a contradiction. Now, it follows from (2.1) that $\rho \le 2Kd$. Hence, d is a metric as required. □

Proposition 2.2.6. *Let ρ be a K-quasi-metric on a set Z. Then, there exists $\varepsilon_0 > 0$ only depending on K, such that ρ^ε is bilipschitz equivalent to a metric for each $0 < \varepsilon \le \varepsilon_0$. More precisely, there exists a metric d_ε on Z such that*

$$\frac{1}{2K^\varepsilon}\rho^\varepsilon(z, z') \le d_\varepsilon(z, z') \le \rho^\varepsilon(z, z')$$

for all z, $z' \in Z$.

Proof. ρ^ε is a K^ε-quasi-metric for every $\varepsilon > 0$. If $K^\varepsilon \le 2$ then the chain construction applied to ρ^ε yields a required metric d_ε by Lemma 2.2.5. □

2.2.3. Visual metrics at infinity. We now apply this construction to the quasi-metric ρ on $\partial_\infty X$. A metric d on the boundary at infinity $\partial_\infty X$ of X is said to be *visual*, if there are $o \in X$, $a > 1$ and positive constants c_1, c_2, such that

$$c_1 a^{-(\xi|\xi')_o} \le d(\xi, \xi') \le c_2 a^{-(\xi|\xi')_o}$$

for all ξ, $\xi' \in \partial_\infty X$. In this case, we say that d is a *visual metric* w.r.t. the base point o and the parameter a. The inequalities above are called the *visual inequalities.*

Applying Proposition 2.2.6, we see:

Theorem 2.2.7. *Let X be a hyperbolic space. Then for any $o \in X$, there is $a_0 > 1$ such that for every $a \in (1, a_0]$ there exists a metric d on $\partial_\infty X$, which is visual w.r.t. o and a.* $\square$

Now, we consider what happens if the base point is changed.

Proposition 2.2.8. *Visual metrics d, d' on $\partial_\infty X$ w.r.t. the same parameter $a > 1$ and base points o, o' respectively are bilipschitz equivalent,*

$$c^{-1} \leq \frac{d'}{d} \leq c$$

for some constant $c \geq 1$.

Proof. This immediately follows from the visual inequalities for d, d' and from the fact that $|(\xi|\xi')_o - (\xi|\xi')_{o'}| \leq |oo'|$ for all ξ, $\xi' \in \partial_\infty X$ (see Remark 2.2.4). $\square$

Next, we consider the effect of the parameter change.

Proposition 2.2.9. *Visual metrics d, d' on $\partial_\infty X$ w.r.t. the same base point o and parameters a, $a' > 1$ respectively are Hölder equivalent, namely, there is a constant $c \geq 1$ such that*

$$\frac{1}{c} d^\alpha(\xi, \xi') \leq d'(\xi, \xi') \leq c d^\alpha(\xi, \xi')$$

for all ξ, $\xi' \in \partial_\infty X$, where $\alpha = \frac{\ln a'}{\ln a}$.

Proof. This immediately follows from the visual inequalities for the metrics d, d' and from the fact that $a' = a^\alpha$ (see Remark 2.2.4). $\square$

We define the topology on the boundary at infinity $\partial_\infty X$ for a hyperbolic space X as the metric topology for some visual metric on $\partial_\infty X$. It follows from Propositions 2.2.8 and 2.2.9 that this topology is independent of the choice of a visual metric.

Exercise 2.2.10. Let X be a hyperbolic space. Show that $\partial_\infty X$ is bounded and complete for any visual metric on $\partial_\infty X$.

2.3. Local self-similarity of the boundary

Hyperbolic groups and more general cobounded hyperbolic spaces have a remarkable and useful property: their boundary at infinity are locally self-similar.

A map $f : Z \to Z'$ between metric spaces is called *homothetic* with coefficient R, if

$$|f(z)f(z')| = R|zz'|$$

for all z, $z' \in Z$. Here we need a more flexible property.

Let $\lambda \geq 1$ and $R > 0$ be given. A map $f : Z \to Z'$ between metric spaces is λ-*quasi-homothetic with coefficient* R if for all z, $z' \in Z$, we have

$$R|zz'|/\lambda \leq |f(z)f(z')| \leq \lambda R|zz'|.$$

Note that f is also λ'-quasi-homothetic with coefficient R for every $\lambda' \geq \lambda$.

This property can be regarded as a perturbation of the property to be homothetic, and the coefficient λ describes the perturbation. We apply this notion usually to a family of quasi-homothetic maps with fixed λ when the coefficients R go to infinity.

A metric space Z is *locally similar* to a metric space Y, if there is $\lambda \geq 1$ such that for every sufficiently large $R > 1$ and every $A \subset Z$ with $\operatorname{diam} A \leq \frac{1}{R}$ there is a λ-quasi-homothetic map $f : A \to Y$ with coefficient R. If a metric space Z is locally similar to itself then we say that Z is *locally self-similar*.

Example 2.3.1. The standard ternary Cantor set X is locally self-similar. One can take $\lambda = 3$ in this case. Indeed, given $R > 3$ and $A \subset X$ with $\operatorname{diam} A \leq 1/R$, there is $k \in \mathbb{N}$ with $3^k < R \leq 3^{k+1}$. Then $\operatorname{diam} A < 1/3^k$. Hence, A is contained in the k-th step interval which it intersects. This interval is 3^k-homothetic to $[0,1]$ and thus it is λ-quasi-homothetic to $[0,1]$ with coefficient R.

The basic example of locally self-similar spaces is the boundary at infinity of a hyperbolic group. We consider a more general situation. A metric space X is *cobounded* if there is a bounded subset $A \subset X$ such that the orbit of A under the isometry group of X covers X.

A metric space X is *proper*, if every closed ball $\overline{B}_r(x) \subset X$ is compact.

Theorem 2.3.2. *The boundary at infinity $\partial_\infty X$ of every cobounded, hyperbolic, proper, geodesic space X is locally self-similar with respect to any visual metric.*

For the proof we need the following

Lemma 2.3.3. *Let o, g, x', x'' be points of a metric space X such that the Gromov products $(x'|g)_o$, $(x''|g)_o \geq |og| - \sigma$ for some $\sigma \geq 0$. Then*

$$(x'|x'')_o \leq (x'|x'')_g + |og| \leq (x'|x'')_o + 2\sigma.$$

Proof. The left hand inequality immediately follows from the triangle inequality: since $|ox'| \leq |og| + |gx'|$ and $|ox''| \leq |og| + |gx''|$, we have $(x'|x'')_o \leq (x'|x'')_g + |og|$.

Next, we note that $(x'|o)_g = |og| - (x'|g)_o \leq \sigma$. This yields $|x'o| = |og| + |gx'| - 2(x'|o)_g \geq |og| + |gx'| - 2\sigma$ and similarly $|x''o| \geq |og| + |gx''| - 2\sigma$. Now, the right hand inequality follows. $\square$

Proof of Theorem 2.3.2. We can assume that the geodesic space X is δ-hyperbolic, $\delta \geq 0$, and that a visual metric d on $\partial_\infty X$ satisfies

$$c^{-1}a^{-(\xi|\xi')_o} \leq d(\xi,\xi') \leq ca^{-(\xi|\xi')_o}$$

for some base point $o \in X$, some constants $c \geq 1$, $a > 1$ and all ξ, $\xi' \in \partial_\infty X$. Note that then $\operatorname{diam} \partial_\infty X \leq c$.

There is $\rho > 0$ such that the orbit of the ball $B_\rho(o)$ under the isometry group of X covers X. Now, we put $\lambda = c^2 a^{\rho+4\delta}$. Fix $R > 1$ and consider $A \subset \partial_\infty X$ with $\operatorname{diam} A \leq 1/R$. For each ξ, $\xi' \in A$, we have

$$(\xi|\xi')_o \geq \log_a \frac{R}{c} \geq \log_a R.$$

We fix $\xi \in A$. Since X is proper, there is a geodesic ray $o\xi \subset X$ representing ξ (see Exercise 2.4.3). We take $g \in o\xi$ with $a^{|og|} = R$. Then using the δ-inequality, we obtain for every $\xi' \in A$

$$(\xi'|g)_o \geq \min\{(\xi'|\xi)_o, (\xi|g)_o\} - \delta = |og| - \delta$$

because $(\xi|g)_o = |og|$.

For arbitrary ξ', $\xi'' \in A$, consider sequences $\{x_i'\} \in \xi'$, $\{x_i''\} \in \xi''$ such that $(x_i'|x_i'')_g \to (\xi'|\xi'')_g$. We can assume without loss of generality that $(x_i'|g)_o$, $(x_i''|g)_o \geq |og| - \delta$ because possible errors in these estimates disappear while taking the limit, see below.

Applying Lemma 2.3.3 to the points o, g, x_i', $x_i'' \in X$ and $\sigma = \delta$, we obtain

$$(x_i'|x_i'')_o - |og| \le (x_i'|x_i'')_g \le (x_i'|x_i'')_o - |og| + 2\delta.$$

Passing to the limit, this yields

$$(\xi'|\xi'')_o - |og| \le (\xi'|\xi'')_g \le (\xi'|\xi'')_o - |og| + 4\delta.$$

There is an isometry $f : X \to X$ with $|of(g)| \le \rho$. Then

$$(\xi'|\xi'')_g - \rho \le (f(\xi')|f(\xi''))_o \le (\xi'|\xi'')_g + \rho$$

because the Gromov products with respect to different points differ one from another at most by the distance between the points. The last two double inequalities give

$$(\xi'|\xi'')_o - |og| - \rho \le (f(\xi')|f(\xi''))_o \le (\xi'|\xi'')_o - |og| + \rho + 4\delta,$$

and therefore,

$$c^{-2}a^{-(\rho+4\delta)}Rd(\xi',\xi'') \le d(f(\xi'),f(\xi'')) \le c^2a^\rho Rd(\xi',\xi'').$$

This shows that $f : A \to \partial_\infty X$ is λ-quasi-homothetic with coefficient R and hence $\partial_\infty X$ is locally self-similar. $\square$

We say that a metric space Z is *doubling* if there is a constant $N \in \mathbb{N}$ such that for every $r > 0$ every ball in Z of radius $2r$ can be covered by N balls of radius r.

If the property above holds for all sufficiently small $r > 0$ only, then we say that Z is *doubling at small scales*. Clearly, if a compact space is doubling at small scales then it is doubling.

Lemma 2.3.4. *Assume that a metric space Z is locally similar to a compact metric space Y. Then Z is doubling at small scales.*

Proof. There is $\lambda \ge 1$ such that for every sufficiently large $R > 1$ and every $A \subset Z$ with $\operatorname{diam} A \le 1/R$ there is a λ-quasi-homothetic map $f : A \to Y$ with coefficient R.

We fix a positive $\rho \le 1/(4\lambda)$. Since Y is compact, there is $N \in \mathbb{N}$ such that any subset $B \subset Y$ can be covered by at most N balls of radius ρ centered at points of B. Take $r > 0$ small enough so that $R = \lambda\rho/r$ satisfies the assumption above. Then for any ball $B_{2r} \subset Z$, we have

$$\operatorname{diam} B_{2r} \le 4r \le 1/R,$$

and thus there is a λ-quasi-homothetic map $f : B_{2r} \to Y$ with coefficient R. The image $f(B_{2r})$ is covered by at most N balls of radius ρ centered at points of $f(B_{2r})$. The preimage under f of every such a ball is contained in a ball of radius $\leq \lambda\rho/R = r$ centered at a point in B_{2r}. Hence, B_{2r} is covered by at most N balls of radius r, and Z is doubling at small scales. $\square$

Example 2.3.5. The space H^n, $n \geq 2$, is locally similar to a compact subspace, e.g. to any closed ball of radius 1. However, H^n is by no means doubling.

Corollary 2.3.6. *Assume that a hyperbolic space X satisfies the condition of Theorem 2.3.2, e.g. X is a hyperbolic group. Then $\partial_\infty X$ is doubling w.r.t. any visual metric.* $\square$

2.4. Additional and historical remarks

2.4.1. A quadruple condition for hyperbolicity. Given a quadruple $Q = (x, y, z, u)$ of points in a metric space X with fixed base point o, we form the triple $A = A(Q) = ((x|y)_o + (z|u)_o, (x|z)_o + (y|u)_o, (x|u)_o + (y|z)_o)$ as in the Tetrahedron Lemma and call it the *cross-difference triple* of Q. We define the *small cross-difference* of Q, $\mathrm{scd}(Q)$, as the distance between the two smaller entries of the cross-triple $A(Q)$.

Proposition 2.4.1. *The metric space X is δ-hyperbolic, $\delta \geq 0$, if and only if* $\mathrm{scd}(Q) \leq \delta$ *for every quadruple $Q \subset X$.*

Proof. The condition $\mathrm{scd}(Q) \leq \delta$ is a reformulation of the property of $A(Q)$ to be a δ-triple. Note that this property is independent of the choice of o and take as o any point of Q. $\square$

Explicitly written, the condition for $A(Q)$ to be a δ-triple for $Q = (x, y, z, u)$ is the inequality

$$|xz| + |yu| \leq \max\{|xy| + |zu|, |xu| + |yz|\} + 2\delta.$$

This formulation is more symmetric than the δ-inequality and has a geometric interpretation in the spirit of the Tetrahedron Lemma. Consider Q as an abstract tetrahedron. Adding the length of opposite edges of Q, we obtain three numbers which we can order as $a \leq b \leq c$. Then, the inequality says $c - b \leq 2\delta$.

2.4.2. Geodesic boundary. Two geodesic rays γ, $\gamma' : [a,\infty) \to X$ in a geodesic space X are called *asymptotic* if $|\gamma(t)\gamma'(t)| \le C < \infty$ for some constant C and all $t \ge a$. To be asymptotic is an equivalence relation on the set of the rays in X, and the set of classes of asymptotic rays is sometimes called the *geodesic boundary* of X, $\partial^g X$.

In a geodesic hyperbolic space, asymptotic rays are at a uniformly bounded distance from each other. Moreover, we have

Lemma 2.4.2. *Let X be a geodesic δ-hyperbolic space. Assume that for some constant $C > 0$, geodesic rays γ, γ' in X with common vertex o contain points $\gamma(t)$, $\gamma'(t')$ with $|\gamma(t)\gamma'(t')| \le C$ for arbitrarily large t, t'. Then $|\gamma(\tau)\gamma'(\tau)| \le \delta$ for all $\tau \ge 0$, in particular, the rays γ, γ' are asymptotic.*

Proof. We have

$$(\gamma(t)|\gamma'(t'))_o = \frac{1}{2}(t + t' - |\gamma(t)\gamma'(t)|) \ge \min\{t,t'\} - C/2.$$

Thus for $\tau \le \min\{t,t'\} - C/2$ we have $|\gamma(\tau)\gamma'(\tau)| \le \delta$ by δ-hyperbolicity. Since t, t' can be chosen arbitrarily large, this inequality holds for all $\tau \ge 0$. $\square$

If a geodesic space X is Gromov hyperbolic, then obviously $\partial^g X \subset \partial_\infty X$. In general, there is no reason for the geodesic boundary of a hyperbolic geodesic space to coincide with the boundary at infinity. However, there are several important cases when $\partial^g X = \partial_\infty X$.

Exercise 2.4.3. Show that if X is a proper hyperbolic geodesic space, then $\partial^g X = \partial_\infty X$.

Another important case when $\partial^g X = \partial_\infty X$ is described in Chapter **??**, see Proposition **??**.

2.4.3. Meaning of the function $\rho_{o,e}(\xi_1,\xi_2) = e^{-(\xi_1,\xi_2)_o}$ for H^n. For the unit ball model of the hyperbolic space H^n, $n \ge 2$ (see Appendix A, sect. **??** and **??**), the quasi-metric $\rho_{o,e} : S^{n-1} \times S^{n-1} \to \mathbb{R}$, $\rho_{o,e}(\xi_1,\xi_2) = e^{-(\xi_1,\xi_2)_o}$, where the unit sphere $S^{n-1} \subset \mathbb{R}^n$ is identified with $\partial_\infty \mathrm{H}^n$ and o is the center of the ball, has a clear geometric interpretation: This function coincides with half of the chordal metric,

$$e^{-(\xi_1,\xi_2)_o} = \frac{1}{2}|\xi_1 - \xi_2|$$

for every $\xi_1,\,\xi_2 \in S^{n-1}$. This immediately follows from the next lemma which also implies that the angle metric on S^{n-1} is a visual metric w.r.t. the center o and the parameter $a = e$.

Lemma 2.4.4. *For every $\xi_1,\,\xi_2 \in \partial_\infty \mathrm{H}^n = S^{n-1}$ we have*

$$e^{-(\xi_1,\xi_2)_o} = \sin(\theta/2),$$

where $\theta = \measuredangle_o(\xi_1, \xi_2)$.

Proof. For the geodesic rays $\gamma_i : [0,\infty) \to \mathrm{H}^n$ from o to ξ_i, $i = 1, 2$, we obviously have

$$e^{-(\xi_1|\xi_2)_o} = \lim_{t\to\infty} (e^{h_t} e^{-2t})^{1/2},$$

where $h_t = d(\gamma_1(t), \gamma_2(t))$ is the distance in H^n. From the hyperbolic law of cosine

$$\cosh(h_t) = \cosh^2(t) - \sinh^2(t)\cos\theta$$

and the trigonometric formula $1 - \cos\theta = 2\sin^2(\theta/2)$, we easily obtain

$$e^{h_t} \sim e^{2t}\sin^2(\theta/2)$$

as $t \to \infty$. Hence, the claim. □

2.4.4. The chain construction. Lemma 2.2.5 and the idea of its proof is due to A.H. Frink, [**?**]. It provides a better constant than contemporary simpler arguments, see e.g. [**?**, Ch. 14]. Moreover, the condition of that Lemma cannot be improved according to the following result:

Example 2.4.5 ([**?**])**.** For every $\varepsilon > 0$, there exists a $(2+\varepsilon)$-quasi-metric space (Z, ρ) such that the chain construction applied to ρ yields only a pseudo-metric d with $d(z, z') = 0$ for some distinct $z,\, z' \in Z$.

2.4.5. It is proven in [**?**] that the function $\rho_o(\xi_1, \xi_2) = e^{-(\xi_1|\xi_2)_0}$ is a metric on the boundary at infinity, $\xi_1,\,\xi_2 \in \partial_\infty X$, of any CAT$(-1)$-space X for every $o \in X$ (the only nontrivial point is to prove the triangle inequality). For further references, we call this metric the *Bourdon metric.* Bourdon metrics associated with different $o,\, o' \in X$ are conformal to each other, and any isometry of X induces a conformal transformation of $(\partial_\infty X, \rho_o)$ [**?**].

Local self-similarity of the boundary at infinity of hyperbolic groups and more general cocompact hyperbolic spaces is certainly well known to experts in the field. Explicitly, it is established in [?] from where basic results of section 2.3 are taken.