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Compact Ovoids in Quadrangles I: Geometric Constructions

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Abstract. This paper is about ovoids in infinite generalized quadrangles. Using the axiom of choice, Cameron showed that infinite quadrangles contain many ovoids. Therefore, we consider mainly closed ovoids in compact quadrangles.

After deriving some basic properties of compact ovoids, we consider ovoids which arise from full imbeddings. This leads to restrictions for the topological parameters (m, m'). For example, if there is a regular pair of lines or a full closed subquadrangle, then $m \leqslant m'$. The existence of full subquadrangles implies the nonexistence of ideal subquadrangles, so finite-dimensional quadrangles are either point-minimal or line-minimal. Another result is that (up to duality) such a quadrangle is spanned by the set of points on an ordinary quadrangle. This is useful for studying orbits of automorphism groups. Finally we prove general nonexistence results for ovoids in quadrangles with low-dimensional line pencils. As one consequence we show that the symplectic quadrangle over an algebraically closed field of characteristic 0 has no Zariski-closed ovoids or spreads.

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1. Background on Quadrangles and Regularity

We recall some terminology. An *incidence geometry* is a triple $(\mathcal{P}, \mathcal{L}, \mathcal{F})$, consisting of a set $\mathcal{P}$ of *points*, a set $\mathcal{L}$ of *lines*, and a set $\mathcal{F} \subset \mathcal{P} \times \mathcal{L}$ of *flags*. Two elements $x, y \in \mathcal{P} \cup \mathcal{L}$ are *incident* if $(x, y) \in \mathcal{F}$ or $(y, x) \in \mathcal{F}$. For $x \in \mathcal{P} \cup \mathcal{L}$ we let $D_1(x)$ denote the set of all elements which are incident with x. More generally, $D_k(x)$ denotes the set of all elements $y \in \mathcal{P} \cup \mathcal{L}$ whose graph-theoretic distance from x is k. We also put $x^\perp = \{x\} \cup D_2(x)$. If there is a (unique) line joining points $p, q \in \mathcal{P}$, then this line is denoted by pq.

The canonical maps $\mathcal{F} \to \mathcal{P}$ and $\mathcal{F} \to \mathcal{L}$ are denoted by $\mathrm{pr}_\mathcal{P}$ and $\mathrm{pr}_\mathcal{L}$, respectively. A *subgeometry* is a triple $(\mathcal{P}', \mathcal{L}', \mathcal{F}')$, where $\mathcal{P}' \subseteq \mathcal{P}$, $\mathcal{L}' \subseteq \mathcal{L}$, and $\mathcal{F}' = \mathcal{F} \cap (\mathcal{P}' \times \mathcal{L}')$. The *dual* of $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ is $(\mathcal{P}, \mathcal{L}, \mathcal{F})^{\mathrm{dual}} = (\mathcal{L}, \mathcal{P}, \mathcal{F}^{-1})$, where $\mathcal{F}^{-1} = \{(\ell, p) | (p, \ell) \in \mathcal{F}\}$.

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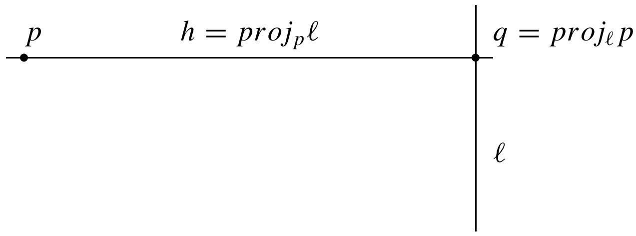
DEFINITION 1.1. We call $\mathfrak{Q} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ a *generalized quadrangle* (or quadrangle for short) if it satisfies the following two axioms:

(GQ_1) For each line $\ell \in \mathcal{L}$ the *point row* $D_1(\ell)$ has cardinality $|D_1(\ell)| \geqslant 3$ and dually, for each point $p \in \mathcal{P}$ the *line pencil* $D_1(p)$ has cardinality $|D_1(p)| \geqslant 3$.

(GQ_2) If (p, ℓ) is a nonincident point-line pair, then there exists a unique line $h = proj_p \ell$ passing through p which is confluent with ℓ in the unique point $q = proj_\ell p$.



Property (GQ_1) is also called *thickness*. If $\mathfrak{Q}$ satisfies (GQ_2) and the following weaker version of the first axiom,

(wGQ_1) For each line $\ell \in \mathcal{L}$ the point row $D_1(\ell)$ has cardinality $|D_1(\ell)| \geqslant 2$ and dually, for each point $p \in \mathcal{P}$ the line pencil $D_1(p)$ has cardinality $|D_1(p)| \geqslant 2$,

then $\mathfrak{Q}$ is called a *weak quadrangle*. Clearly, $\mathfrak{Q}$ is a (weak) quadrangle if and only if $\mathfrak{Q}^{\mathrm{dual}}$ is a (weak) quadrangle.

A (weak) *subquadrangle* $\mathfrak{Q}'$ of a generalized quadrangle $\mathfrak{Q}$ is a subgeometry which is a (weak) quadrangle. A subquadrangle is called *full* if for at least one line ℓ of $\mathfrak{Q}'$ we have $D_1'(\ell) = D_1(\ell)$ (it follows that this equality holds for all lines of $\mathfrak{Q}'$). Dually, one defines an *ideal* subquadrangle by requiring that $D_1'(p) = D_1(p)$ holds for some (and hence every) point of $\mathfrak{Q}'$.

The verification of the following result is straight-forward.

LEMMA 1.2. *Let $\mathfrak{Q}' \subseteq \mathfrak{Q}$ be a (weak) subquadrangle, and let $\ell \in \mathcal{L}$ be a line of $\mathfrak{Q}$. If $|D_1(\ell) \cap \mathcal{P}'| \geqslant 2$, then $\ell \in \mathcal{L}'$ (and dually); thus, if $\mathfrak{Q}'$ is a full (weak) subquadrangle, then either $D_1(\ell) \subset \mathcal{P}'$, or the intersection $D_1(\ell) \cap \mathcal{P}'$ contains 0 or 1 elements.* □

We need also the following result.

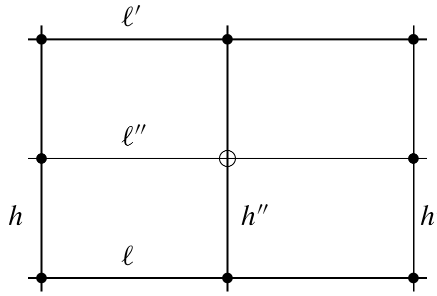
LEMMA 1.3. *Let $\mathfrak{Q}'$ be a full weak subquadrangle of $\mathfrak{Q}$. If $|D_1'(p)| = 2$ holds for one point $p \in \mathcal{P}'$, then every line pencil of $\mathfrak{Q}'$ has cardinality 2; thus $\mathfrak{Q}'$ is a grid.*

Proof. This follows from Tits' classification of proper weak quadrangles, cf. Van Maldeghem [33] 1.6.2, because the lines of $\mathfrak{Q}'$ are thick. □

Full and ideal subquadrangles play an important rôle in the theory of generalized quadrangles. For instance, they are used in the classification of Moufang quadrangles (in particular to classify the Moufang quadrangles of so-called type BC_2, see Tits [31] and Tits and Weiss [32] for more details). In the finite case, the existence of full or ideal subquadrangles puts severe restrictions on the parameters of the quadrangle. If a finite quadrangle of order (s, t) (i.e., there are $s + 1$ points on each point row and $t + 1$ lines in each line pencil) has a full subquadrangle of order (s', t'), then $s = s'$ and $t \geqslant st'$. Combined with the inequality $s^2 \geqslant t$, this also implies $t' \leqslant s$. See Payne and Thas [24] for proofs. We will show that the inequality $t \geqslant st'$ has an analogue in the compact connected case, but the inequality $t' \leqslant s$ cannot have an analogue.

Full subquadrangles are also used in the finite case to construct ovoids. The same construction works in the general case, and we will apply this to some compact connected quadrangles. As a further application, we will show that certain ovoids that we construct have a 2-transitive automorphism group within the quadrangle they live in.

DEFINITION 1.4. Let ℓ and ℓ' be two opposite lines of a quadrangle $\mathfrak{Q}$. Let $\{\ell, \ell'\}^\perp$ be the set of lines of $\mathfrak{Q}$ confluent with both ℓ and ℓ'. Let $h, h' \in \{\ell, \ell'\}^\perp$ be distinct lines. If the set $\{h, h'\}^\perp$ is independent of h and h', then we say that (ℓ, ℓ') is a *regular pair of lines*. In other words, (ℓ, ℓ') is regular if the lines h'' and ℓ'' in the picture below are always confluent.



In this case the set of lines $\{\ell, \ell'\}^\perp \cup \{h, h'\}^\perp$, for $h, h' \in \{\ell, \ell'\}^\perp$ is the line set of a full weak subquadrangle $\mathfrak{Q}'$ whose line pencils have exactly two elements. This is also called a *grid*.

The line ℓ is called *regular* if every such pair (ℓ, ℓ') is regular. Regular points and pairs of points are defined dually. A regular point p is called *projective* if $D_2(p) \cap D_2(q) \cap D_2(r) \neq \emptyset$ holds for all $q, r \in D_4(p)$. If $q \in D_4(p)$, then $p^q = p^\perp \cap q^\perp$ is also called a *trace*. The *derived geometry* at p is the incidence geometry

$$\mathfrak{Q}(p) = (p^\perp, \{p^\perp \cap q^\perp | q \in \mathcal{P}\backslash\{p\}\}, \subseteq).$$

It is easy to see that $\mathfrak{Q}(p)$ is a linear space if and only if p is regular, and a projective plane if and only if p is projective. Projective lines are defined dually.

DEFINITION 1.5. An *ovoid* in a generalized quadrangle $\mathfrak{Q} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ is a set $\mathcal{O} \subseteq \mathcal{P}$ of points with the following property:

(Ov) Every line $\ell \in \mathcal{L}$ meets $\mathcal{O}$ in a unique point $o(\ell) \in \mathcal{O}$.

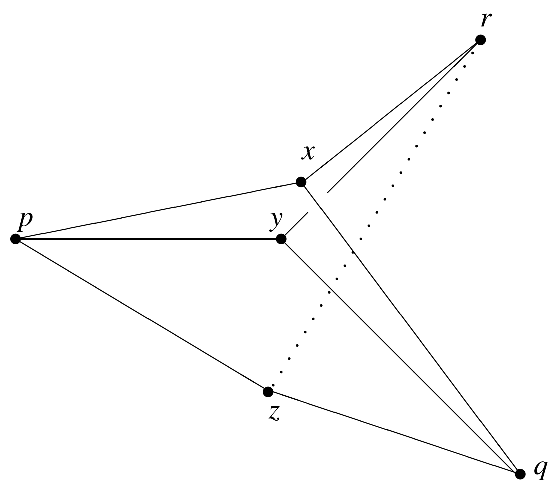
A *spread* $\mathcal{S}$ in $\mathfrak{Q}$ is an ovoid in $\mathfrak{Q}^{\text{dual}}$; thus, it is a set $\mathcal{S} \subseteq \mathcal{L}$ of lines with the property that every point $p \in \mathcal{P}$ lies on a unique line $s(p) \in \mathcal{S}$.

DEFINITION 1.6. A *geometric hyperplane* $\mathcal{H}$ of a generalized quadrangle $\mathfrak{Q}$ is a proper subset of the point set with the property that each point row of $\mathfrak{Q}$ which is not contained in $\mathcal{H}$ meets $\mathcal{H}$ in a unique point. If $\mathcal{H}$ does not contain any point row, then clearly $\mathcal{H}$ is an ovoid. If $\mathcal{H}$ contains a point row, but not the set of points on the lines of an ordinary quadrangle, then it is easy to see that $\mathcal{H}$ coincides with $p^\perp$, for some point p. In all other cases, $\mathcal{H}$ is the point set of a full subquadrangle of $\mathfrak{Q}$. If $\mathcal{H}$ coincides with $p^\perp$ for some point p, then we call $\mathcal{H}$ *trivial*, and p is a *deep point* of $\mathcal{H}$.

LEMMA 1.7. *Let $\mathfrak{Q}$ be a generalized quadrangle and let p be any point of $\mathfrak{Q}$.*

(i) *If all elements of $D_2(p)$ are regular, then also p is regular.*
(ii) *If p is projective and an element $q \in D_4(p)$ is regular, then q is projective.*
(iii) *If all elements of $D_2(p)$ are regular, and if p is projective, then all points of $\mathfrak{Q}$ are projective and $\mathfrak{Q}$ is isomorphic to the symplectic quadrangle over some commutative field K.*

Proof. (i) If $x, y \in D_2(p) \cap D_2(q) \cap D_2(r)$, for points $q, r \in D_4(p)$, with $x \neq y$, then, by regularity of y, every point z collinear with both p and q is also collinear with r (since x is collinear with p, q, r). Hence $D_2(p) \cap D_2(q) = D_2(p) \cap D_2(r)$.

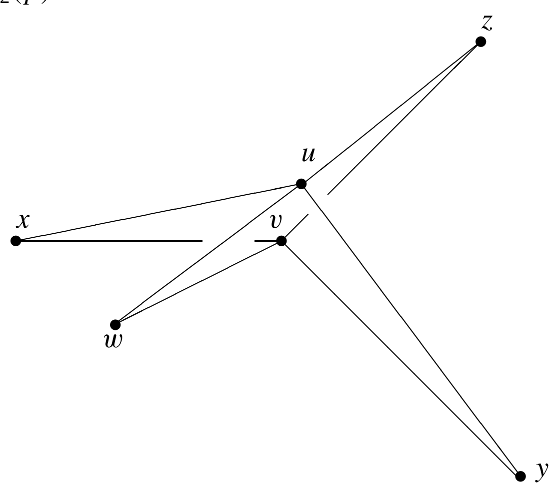


(ii) This follows immediately from the proof of Theorem 6.2.1 in Van Maldeghem [33].

(iii) Suppose x, y, z are three pairwise opposite points with $\{u, v\} \subseteq D_2(x) \cap D_2(y) \cap D_2(z)$, $u \neq v$, and $x \in D_4(p)$. We want to show that x is regular. If v

and/or u is collinear with p, then it is regular, and as in the proof of (i), we conclude that $D_2(x) \cap D_2(y) = D_2(x) \cap D_2(z)$.

Suppose now that u and v are opposite p. Since p is projective, there is at least one point $w \in D_2(p)$ collinear with both u and v.



Hence $\{u, v\} \subseteq D_2(w) \cap D_2(x)$, and by the regularity of w we conclude $D_2(x) \cap D_2(y) = D_2(x) \cap D_2(w) = D_2(x) \cap D_2(z)$. Hence x is regular. Consequently all points of $\mathfrak{Q}$ are regular and at least one point is projective. The result now follows directly from Theorem 6.2.1 in [33]. $\square$

We need one more definition.

DEFINITION 1.8. A generalized quadrangle is called *point-antiregular* if

$$|p_1^{\perp} \cap p_2^{\perp} \cap p_3^{\perp}| \in \{0, 2\}$$

holds for all triples of pairwise noncollinear points (p_1, p_2, p_3). Line-antiregular quadrangles are defined dually.

2. Compact Quadrangles

DEFINITION 2.1. A quadrangle $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ is called a *compact quadrangle* if $\mathcal{P}$ and $\mathcal{L}$ are compact Hausdorff spaces, and if the maps $(p, \ell) \mapsto proj_p \ell$ and $(p, \ell) \mapsto proj_\ell p$ defined in 1.1 are continuous on $(\mathcal{P} \times \mathcal{L}) \setminus \mathcal{F}$. This is equivalent with $\mathcal{F}$ being closed in the product space $\mathcal{P} \times \mathcal{L}$, see Grundhöfer and Van Maldeghem [10] 2.1(a). If one of the spaces $\mathcal{P}$, $\mathcal{L}$ or $\mathcal{F}$ is not connected, then all three spaces are totally disconnected, zero-dimensional, and in fact either finite or homeomorphic to the Cantor set, cf. Grundhöfer and Van Maldeghem [10] p. 466, Kramer [17] 2.5.6. Typical examples for this are finite quadrangles, p-adic classical quadrangles, and inverse limits of finite quadrangles, cf. [10]. Here, standard homotopy or homology theory is of little use, hence we disregard the zero-dimensional compact quadrangles. For technical reasons we will assume that the topological dimension of the quadrangle (that is, the covering dimension of $\mathcal{P}$, $\mathcal{L}$, or $\mathcal{F}$) is finite (cf. Grundhöfer and Knarr [8] Section 4). This is very likely a condition which is automatically satisfied – for example the presence of a sufficiently large group of topological automorphisms guarantees finite dimension (e.g., an open orbit in the point space suffices by Szenthe's result, cf. Grundhöfer, Knarr and Kramer [9] 2.2). Let (p, ℓ) be a flag of $\mathfrak{Q}$. The dimensions

$$m = \dim D_1(\ell), \qquad m' = \dim D_1(p),$$

are independent of the flag (p, ℓ), because any two point rows or line pencils are homeomorphic via projectivities. The numbers m, m' are positive if and only if $\mathfrak{Q}$ is finite-dimensional and connected, cf. [8] 3.3; we call such a quadrangle an (m, m')-quadrangle, and we call (m, m') the *topological parameters* of $\mathfrak{Q}$. They play a similar rôle as the *order* (s, t) of a finite quadrangle (cf. Payne and Thas [24]); for example, $\dim \mathcal{P} = m + m' + m$, $\dim \mathcal{L} = m' + m + m'$ and $\dim \mathcal{F} = 2(m + m')$, cf. [8] 4.3.

We call a (weak) subquadrangle $\mathfrak{Q}' \subseteq \mathfrak{Q}$ compact if $\mathcal{P}'$, $\mathcal{L}'$, $\mathcal{F}'$ are compact (or, equivalently, closed); this implies that the parameters of $\mathfrak{Q}'$ are also finite (possibly zero – this is an open problem); however, there exist no thick compact $(m, 0)$-quadrangles for $m > 0$, i.e., positive dimension of the point rows implies positive dimension of the line pencils (and dually, of course), see [8] 3.3. Therefore, the following definition is unambiguous: *for* $m, m' > 0$, *an* $(m, 0)$*-subquadrangle* is a full weak closed subquadrangle where every point is incident with precisely two lines and dually, a weak $(0, m')$-subquadrangle is an ideal weak closed subquadrangle with point rows of cardinality 2.

FIBRE BUNDLES IN COMPACT QUADRANGLES 2.2. In a compact quadrangle, the map $\mathrm{pr}_{\mathcal{P}}\colon \mathcal{F} \to \mathcal{P}$ is a fibre bundle with a line pencil $D_1(p)$ as a typical fibre,

$$\begin{tikzcd} \{p\} \times D_1(p) \arrow[r, hook] & \mathcal{F} \arrow[d] \\ & \mathcal{P}, \end{tikzcd}$$

and similarly, $\mathrm{pr}_{\mathcal{L}}\colon \mathcal{F} \to \mathcal{L}$ is a fibre bundle with a point row $D_1(\ell)$ as a typical fibre. This is important, since we will see that the existence of ovoids and spreads is closely related to the existence of sections of these maps (a section of a continuous map $f\colon X \to Y$ is a continuous right inverse $r\colon Y \to X$, i.e. $fr = \mathrm{id}_Y$).

DEFINITION 2.3. For our purposes, a *generalized n-manifold* is a locally compact metrizable space X which is an *n-cm$_R$* (n-dimensional cohomology manifold) for every principal ideal domain R (Bredon's book [2] provides a comprehensive introduction to cm_Rs). If the covering dimension of X is finite, then $\dim X = n$, cf. Löwen [21] 4.1, [2] IV.8. Generalized manifolds have the following nice property: a product $X \times Y$ is a generalized manifold if and only if X and Y are generalized manifolds (and in that case the dimensions sum up in the right way), cf. [2] V.16.11. Moreover, they satisfy domain invariance: if $X \subseteq Y$, and if both X and Y are generalized n-manifolds, then X is open in Y, cf. [2] V.16.19. If $n \leqslant 2$, then an n-$cm_{\mathbb{Z}}$ is in fact an n-manifold, cf. [2] V.16.32.

The following results are proved in Grundhöfer and Knarr [8] 4.2. Let $\mathfrak{Q}$ be an (m, m')-quadrangle, and $\ell \in \mathcal{L}$ be a line. If $m > 0$, then $D_1(\ell)$ is an ANR (*absolute neighborhood retract*, cf. Hu [12]) and a generalized m-manifold. Moreover, $D_1(\ell)$ is homotopy equivalent to an m-sphere $\mathbb{S}^m$, and (by the result on low dimensional $cm_{\mathbb{Z}}$s mentioned above) even homeomorphic to $\mathbb{S}^m$, provided that $m \leqslant 2$. For every $p \in D_1(\ell)$, the 'affine line' $D_1(\ell)\backslash\{p\}$ is locally and globally contractible. A similar statement holds for the line pencils. By the coordinazation, the spaces $\mathcal{P}$, $\mathcal{L}$ and $\mathcal{F}$ are also ANRs and generalized manifolds, of dimensions $2m + m'$, $2m' + m$ and $2(m + m')$, respectively.

If $\mathfrak{Q}$ is point-antiregular, then there is a geometric bijection between punctured point rows and punctured line pencils, cf. Schroth [27] 2.1. Thus, in a compact point-antiregular quadrangle the point rows and line pencils are homeomorphic and the topological parameters are equal.

LEMMA 2.4. *If $\mathfrak{Q}$ is a compact connected inite-dimensional point-antiregular quadrangle, then $m = m'$.* □

The following results are also due to Schroth.

THEOREM 2.5. *Let $\mathfrak{Q}$ be a compact connected finite-dimensional quadrangle with a regular point p. Then the following are equivalent,* cf. [28] Cor. 2.

(i) The topological parameters are equal, $m = m'$.
(ii) The point p is projective.
(iii) The derived geometry $\mathfrak{Q}(p)$ is (in a natural way) a compact projective plane.□

THEOREM 2.6. *Let $\mathfrak{Q}$ be a compact connected finite-dimensional quadrangle. If $m = m' \in \{1, 2\}$, then $\mathfrak{Q}$ or $\mathfrak{Q}^{\mathrm{dual}}$ is point-antiregular,* cf. [29] 2.15. □

The precise result is as follows: the $\mathbb{F}_2$-cohomology rings of $\mathcal{P}$ and $\mathcal{L}$ are not isomorphic. Let x_m, x_{2m} denote the (unique) generators of $\mathrm{H}^{\mathrm{k}}(\mathcal{P}; \mathbb{F}_2) \cong \mathbb{F}_2$, for $k = m, 2m$ respectively, and define the *mod 2 twisting number* $t_{\mathcal{P}}$ by $x_m^2 = t_{\mathcal{P}} \cdot x_{2m}$. Then $\mathfrak{Q}$ is point-antiregular if and only if $t_{\mathcal{P}} = 0$, and in this case $t_{\mathcal{L}} = 1$.

Knarr shows that $m = m'$ implies that $m = 1, 2, 4$, see Knarr [15] p. 610, Kramer [17] 3.3.6. The following theorem follows from recent results of Stolz.

THEOREM 2.7. *There exists no compact connected finite-dimensional quadrangle with $m = m' = 4$.* □

The proof uses the fact that $\mathcal{P}$ is the *S-dual* of $\mathcal{L}$ (see e.g. Husemoller [13] Chapter 16.3) (this follows from Knarr's topological Veronese imbedding) and resulting integrality properties of the K-theory of $\mathcal{P}$ and $\mathcal{L}$.

COROLLARY 2.8. *Every compact connected finite-dimensional quadrangle with equal parameters is (up to duality) point-antiregular.* □

Stolz proved his result for Dupin hypersurfaces in spheres, which are closely related to compact quadrangles. We are at present not sure if his proof requires that the point rows and line pencils are locally Euclidean. As the result is yet unpublished, we will not rely on 2.7 and 2.8.

3. Topological Properties of Compact Ovoids

By Cameron's result [4], ovoids and spreads abound in infinite quadrangles. Hence we are going to consider only closed ovoids and spreads in compact quadrangles. By duality, it suffices to consider either spreads or ovoids.

Given a set $X \subseteq \mathcal{P}$ of points, put $\mathcal{F}_X = \mathcal{F} \cap (X \times \mathcal{L})$. Then the following diagram is a topologically useful reformulation of 1.5 (Ov):

$$\begin{tikzcd} \mathcal{F} \arrow[d] & \mathcal{F}_{\mathcal{O}} \arrow[l, hook'] \arrow[r, "\mathrm{pr}_{\mathcal{L}}|_{\mathcal{F}_{\mathcal{O}}}"] \arrow[d] & \mathcal{L} \arrow[dl, dotted, "o"] \\ \mathcal{P} & \mathcal{O} \arrow[l, hook'] & \end{tikzcd}$$

the set $\mathcal{O} \subseteq \mathcal{P}$ is an ovoid if and only if $\mathrm{pr}_{\mathcal{L}}|_{\mathcal{F}_{\mathcal{O}}}: \mathcal{F}_{\mathcal{O}} \to \mathcal{L}$ is bijective; in that case $\ell \mapsto o(\ell) = \mathrm{pr}_{\mathcal{P}}(\mathrm{pr}_{\mathcal{L}}|_{\mathcal{F}_{\mathcal{O}}})^{-1}(\ell)$ is the dotted map.

If $\mathfrak{Q}$ is compact and $\mathcal{O}$ is closed (and hence compact Hausdorff), then o is continuous, and conversely, if o is continuous, then $\mathcal{O} = o(\mathcal{L})$ is compact. Note that the map $\ell \mapsto (o(\ell), \ell)$ is a section of the bundle map $\mathrm{pr}_{\mathcal{L}}: \mathcal{F} \to \mathcal{L}$. In particular, the existence of an ovoid implies the existence of a section of $\mathrm{pr}_{\mathcal{L}}$. Also, (o, id) maps $\mathcal{L}$ homeomorphically onto $\mathcal{F}_{\mathcal{O}}$. Thus o, being the composite $o = \mathrm{pr}_{\mathcal{P}} \circ (o, \mathrm{id})$ is a locally trivial bundle

$$\begin{tikzcd} D_1(p) \arrow[r, hook] & \mathcal{L} \arrow[d, "o"] \\ & \mathcal{O} \end{tikzcd}$$

over $\mathcal{O}$, with line pencils as fibres. Similar remarks hold, by duality, for compact spreads.

COROLLARY 3.1. *If $\mathfrak{Q}$ is a compact connected quadrangle with parameters (m, m'), and if $\mathcal{O} \subseteq \mathcal{P}$ is a compact ovoid, then $\mathcal{O}$ is a generalized $(m + m')$-manifold.*

Proof. Since the bundle $o: \mathcal{L} \to \mathcal{O}$ is locally trivial, and since the fibre and the total space of this bundle are generalized m'- and $(2m' + m)$-manifolds, respectively, the claim follows from the fact that factors of a generalized manifold are again generalized manifolds (of the right dimension), see Bredon [2]. □

The fact that $\mathcal{O}$ is a generalized $(m + m')$-manifold can also be seen in a more geometric way. Fix a point $p \in \mathcal{O}$, a line $\ell \in D_1(p)$, put $D = D_1(\ell)\backslash\{p\}$ and $U = \mathcal{O}\backslash\{p\}$. Then $\mathcal{O} = \overline{U}$ is the closure and the one-point compactification of the open subset $U \subseteq \mathcal{O}$. Now consider the map $U \to \mathcal{L}$, $q \mapsto proj_q \ell$. The image of U under this map is precisely the set E of all lines that meet ℓ in some point different from p, and an inverse $E \to \mathcal{O}$ is given by $h \mapsto o(h)$. It is easy to see that E is homeomorphic to $D \times (D_1(p)\backslash\{\ell\})$. Therefore $U \cong E$ is a contractible generalized $(m + m')$-manifold.

COROLLARY 3.2. *If $\mathcal{O} \subseteq \mathcal{P}$ is a compact ovoid in a finite-dimensional compact connected quadrangle $\mathfrak{Q}$ with parameters (m, m'), then $\mathcal{O}$ is homotopy equivalent to an $(m + m')$-sphere.*

Proof. The set $\mathcal{O}$ is a generalized $(m + m')$-manifold with the property that the complement of every point is contractible, hence the claim follows from Löwen [21] 6.2. □

4. Ovoids in Subquadrangles

THEOREM 4.1. *Let $\mathfrak{Q}' = (\mathcal{P}', \mathcal{L}', \mathcal{F}')$ be a full subquadrangle of the quadrangle $\mathfrak{Q}$, and let $p \in \mathcal{P}\backslash\mathcal{P}'$ be a point of $\mathfrak{Q}$ not in $\mathfrak{Q}'$. Then the set of points $\mathcal{O}_p = p^\perp \cap \mathcal{P}'$ is an ovoid in $\mathfrak{Q}'$. If ℓ is a line of $\mathfrak{Q}$ not incident with any point of $\mathfrak{Q}'$, then the set $\{\mathcal{O}_q | (q, \ell) \in \mathcal{F}\}$ is a partition of $\mathcal{P}'$ into ovoids.*

If $\mathfrak{Q}$ is compact, then $\mathcal{O}_p$ is closed in $\mathcal{P}$. If $\mathfrak{Q}$ is a compact connected (m, m')-quadrangle, and $\mathfrak{Q}'$ is a closed (m, m'')-subquadrangle, then $m' \geqslant m + m''$. Moreover, $m' = m + m''$ if and only if $\mathcal{P}'$ is a geometric hyperplane of $\mathfrak{Q}$.

Proof. First, note that $D_1(p) \cap \mathcal{L}' = \emptyset$ since $\mathfrak{Q}'$ is full. Let ℓ' be any line of $\mathfrak{Q}'$. Then there is a unique point $p' = proj_{\ell'} p$ incident with ℓ' and collinear with p. The point p' belongs to $\mathfrak{Q}'$ because $\mathfrak{Q}'$ is a full subquadrangle of $\mathfrak{Q}$. This shows that $\mathcal{O}_p$ is an ovoid of $\mathfrak{Q}'$.

Suppose that $D_1(\ell) \cap \mathcal{P}' = \emptyset$, and let p' be any point of $\mathfrak{Q}'$. Then there exists a unique point $q = proj_\ell p'$ on ℓ collinear with p'. Hence $p' \in \mathcal{O}_q$, and any two such ovoids are disjoint. This shows that $\{\mathcal{O}_q | q \in D_1(\ell)\}$ partitions $\mathcal{P}$ into ovoids.

Now suppose that $\mathfrak{Q}$ is compact. Since $p^\perp$ is closed, the intersection $\mathcal{O}_p = p^\perp \cap \mathcal{P}'$ is closed in $\mathcal{P}'$. If $\mathcal{P}'$ is closed in $\mathcal{P}$, then $\mathcal{O}$ is compact. Consider the continuous injection $f: \mathcal{O}_p \to D_1(p)$, $p' \mapsto pp'$. By compactness, $\mathcal{O}_p$ and $f(\mathcal{O}_p)$ are homeomorphic. If $\mathfrak{Q}$ is finite-dimensional with parameters (m, m''), then $\dim \mathcal{O}_p = \dim f(\mathcal{O}_p) = m + m''$. But $f(\mathcal{O}_p) \subseteq D_1(p)$, whence $\dim \mathcal{O}_p \leqslant \dim D_1(p) = m'$, cf. Kramer [17] 3.1.1.

If $\mathcal{P}'$ is a geometric hyperplane of $\mathfrak{Q}$, then every line through p must meet $\mathcal{P}'$ in a point of $\mathcal{O}_p$, hence $f: \mathcal{O}_p \to D_1(p)$ is a bijection, hence $m + m'' = m'$. Conversely, if $m + m'' = m'$, then $f(\mathcal{O}_p) \subseteq D_1(p)$ is a closed subset of the full dimension; by domain invariance, $f(\mathcal{O}_p) = D_1(p)$, cf. Bredon [2] V.16.19. This holds for every $p \in \mathcal{P} \setminus \mathcal{P}'$, hence $\mathfrak{Q}'$ is a geometric hyperplane of $\mathfrak{Q}$. The theorem is proved. □

The proof of the following result is straight-forward.

LEMMA 4.2. *Let $\mathfrak{Q}' = (\mathcal{P}', \mathcal{L}', \mathcal{F}')$ be a full weak and nonthick subquadrangle in a compact quadrangle $\mathfrak{Q} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ (so $\mathfrak{Q}'$ is a grid by* 1.3). *Then $\mathfrak{Q}'$ is closed.*

Let $p \in \mathcal{P}'$ and let $h, \ell \in D_1'(p)$ be the two unique lines through p. The point space $\mathcal{P}'$ is homeomorphic to $D_1(h) \times D_1(\ell)$ and the line space is homeomorphic to the disjoint union $D_1(h) \sqcup D_1(\ell)$.

Proof. Let $(p_0, \ell_0, p_1, \ell_1, p_2, \ell_2, p_3, \ell_3)$ be an ordinary quadrangle in $\mathfrak{Q}'$. Then $\mathcal{L}' = (\ell_0^\perp \cap \ell_2^\perp) \cup (\ell_1^\perp \cap \ell_3^\perp)$ is closed, and $\mathcal{P}'$ is the set of all points which are incident with this compact set of lines, so $\mathcal{P}'$ is also compact. □

A closed ovoid of such an $(m, 0)$-subquadrangle $\mathfrak{Q}'$ is readily seen to be a generalized m-manifold homeomorphic to $D_1(\ell)$ imbedded 'diagonally' into $D_1(\ell) \times D_1(h)$. The proof of Theorem 4.1 also holds in this case and we have $m \leqslant m'$.

LEMMA 4.3. *If $\mathfrak{Q}$ is a compact connected (m, m')-quadrangle with a regular pair of lines, then $m \leqslant m'$.* □

This was also proved by Schroth [28] Lemma 4 under the stronger assumption that ℓ is regular. We mention some applications.

DEFINITION 4.4. Let $\mathfrak{Q}$ be a compact connected finite-dimensional quadrangle. We say that $\mathfrak{Q}$ is *line-minimal* if $\mathfrak{Q}$ does not contain any proper full compact subquadrangles. Dually, one defines *point-minimality*.

THEOREM 4.5. *Let $\mathfrak{Q}$ be a compact connected finite-dimensional quadrangle with parameters (m, m'). If $m \leqslant m'$, then $\mathfrak{Q}$ is line-minimal, and if $m \geqslant m'$ then $\mathfrak{Q}$ is point-minimal. If $m = m'$, then $\mathfrak{Q}$ has no full or ideal closed subquadrangles.*

Proof. We may assume that $m' \leqslant m$. If $\mathfrak{Q}$ would have a proper compact full subquadrangle $\mathfrak{Q}'$, then the topological parameters (m, m'') of $\mathfrak{Q}'$ would satisfy $m + m'' \leqslant m'$, hence $m < m'$, a contradiction. □

Let X be a set of points of $\mathfrak{Q}$. Then we say that X *spans* $\mathfrak{Q}$ if no proper subgeometry of $\mathfrak{Q}$ which is closed under joining points, intersecting lines and projecting points to lines or lines to points contains X. Whenever X contains an ordinary quadrangle, this is equivalent with saying that no proper weak subquadrangle of $\mathfrak{Q}$ contains X.

THEOREM 4.6. *Up to duality, every compact connected finite-dimensional generalized quadrangle $\mathfrak{Q}$ is spanned by the set of points on the lines of some ordinary quadrangle Q. Moreover, if in the dual of $\mathfrak{Q}$ no ordinary quadrangle Q exists such that the set of points on Q spans $\mathfrak{Q}^{\text{dual}}$, then $\mathfrak{Q}$ is spanned by the set of points on every ordinary quadrangle of $\mathfrak{Q}$.*

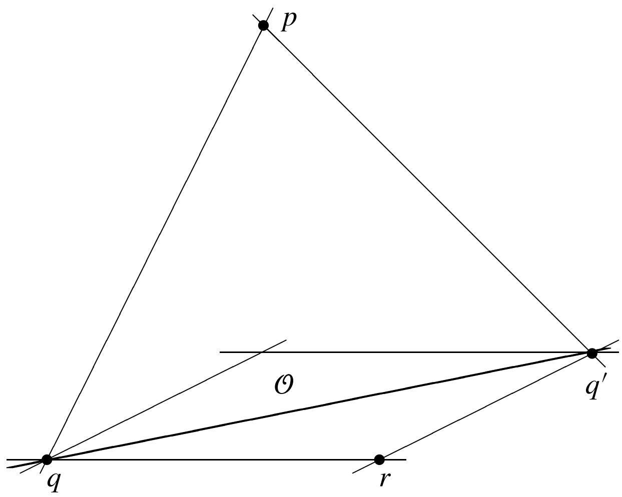
Proof. Let (m, m') be the topological parameters of $\mathfrak{Q}$. Up to duality, we may again assume that $m \geqslant m'$. If $m > m'$, then any compact weak subquadrangle of $\mathfrak{Q}$ containing the set X of points on some ordinary quadrangle has parameters (m, m'') with $m + m'' \leqslant m'$, hence $m \leqslant m'$, a contradiction. Hence we may assume that $m = m'$ and $m'' = 0$. We may suppose that in the dual, the set of points on any ordinary quadrangle is contained in a proper compact weak subquadrangle (which is a compact grid by the preceeding remarks), hence every point of $\mathfrak{Q}$ is regular. But then, by Schroth [28], $\mathfrak{Q}$ is a symplectic quadrangle over $\mathbb{R}$ or $\mathbb{C}$ and the result follows. $\square$

More generally, the question whether the set of points on an apartment of a spherical building spans the building has been considered by various authors, see Cooperstein and Shult [5], Blok and Brouwer [1], and Ronan and Smith [25]. Usually, in these papers, the rank 2 case is left out or treated very incompletely. The previous theorem gives a complete answer for finite-dimensional compact connected quadrangles. For Moufang quadrangles, see Van Maldeghem [33]. Another application is contained in the following results.

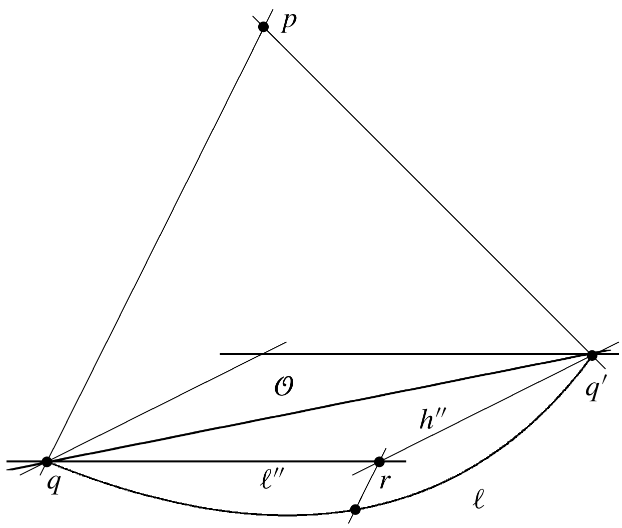
LEMMA 4.7. *Let $\mathfrak{Q}$ be a compact connected finite-dimensional quadrangle. Suppose that p is a regular point and (ℓ, ℓ') is a regular pair of lines. Then p is incident with some element of $\{\ell, \ell'\}^{\perp}$.*

Proof. Suppose that p is not incident with any element of $\mathcal{S}_1 = \{\ell, \ell'\}^{\perp}$. Put $\mathcal{S}_2 = \{h, h'\}^{\perp}$, where $h, h' \in \mathcal{S}_1$, $h \neq h'$. The sets $\mathcal{S}_1$ and $\mathcal{S}_2$ form the two sets of lines of a grid, i.e., a full weak subquadrangle $\mathfrak{Q}'$ with two lines per point. By Lemma 4.3, the topological parameters of $\mathfrak{Q}$ are equal, say (m, m). This means by 2.5 that $\mathfrak{Q}(p)$ is a compact projective plane. Let $\mathcal{P}'$ be the set of points on the lines of $\mathcal{S}_1$ (or equivalently, of $\mathcal{S}_2$). The set of points $\mathcal{O} = p^{\perp} \cap \mathcal{P}'$ is a closed ovoid in $\mathfrak{Q}'$ which is clearly a generalized m-manifold. Since $D_1(p)$ is also a generalized m-manifold, every line through p carries a point of $\mathcal{O}$ by domain invariance. Hence $\mathcal{O}$ can be seen as a closed curve in the projective plane $\mathfrak{Q}(p)$. We want to show that $\mathcal{O}$ is an oval. Let ℓ be a line of $\mathfrak{Q}(p)$ and suppose that ℓ meets $\mathcal{O}$ in at least two points q, q'. Clearly, this line does not meet p. There is a point r of $\mathcal{P}'$ collinear with both q and q', hence by the regularity of p, we have $\ell = p^{\perp} \cap r^{\perp}$. Clearly $\mathcal{O}$

does not contain any other point of $r^\perp$ (otherwise there arises a triangle in $\mathfrak{Q}'$), so ℓ meets $\mathcal{O}$ in exactly two points q, q'.



Suppose now that ℓ is any line of $\mathfrak{Q}(p)$ not through p but containing at least one point q of $\mathcal{O}$. Let ℓ'' be a line of $\mathfrak{Q}'$ through q and let r be the projection of a point on ℓ different from q onto ℓ''. Since p is regular, $\ell = r^\perp \cap p^\perp$. Let h'' the line of $\mathfrak{Q}'$ through r different from ℓ''. Then h'' meets ℓ in some point q' different from q; thus ℓ meets $\mathcal{O}$ in two points.



Hence $\mathcal{O}$ is a closed oval in $\mathfrak{Q}(p)$, and all the tangent lines go through p. This is in contradiction with the fact that the tangent lines constitute an oval in the dual projective plane, cf. Salzmann *et al.* [26] 55.17. The lemma follows. $\square$

The quadrangle $\mathfrak{Q}$ in Lemma 4.7 above has equal parameters $m = m'$. If one is willing to use Stolz' result 2.7, then $\mathfrak{Q}$ is point- or line-antiregular by 2.8; thus, it cannot have a regular point and a regular pair of lines. In other words, a quadrangle with these properties does not exist. In any case, we now have:

THEOREM 4.8. *Let $\mathfrak{Q}$ be a compact connected finite-dimensional quadrangle with topological parameters (m, m') and with a regular point p. If $\mathfrak{Q}$ has a full weak subquadrangle $\mathfrak{Q}'$ with parameters (m, m''), then $m = m'$, (hence $\mathfrak{Q}'$ is a grid) and p is a point of $\mathfrak{Q}'$.*

Proof. The existence of $\mathfrak{Q}'$ forces $m' \geqslant m$. The existence of a regular point forces $m \geqslant m'$. Hence $m = m'$. But $m' \geqslant m + m''$, hence $m'' = 0$ and the result follows from the previous lemma and 1.3. □

Lemma 4.7 states that any projective point in a compact quadrangle $\mathfrak{Q}$ containing a full grid $\mathfrak{Q}'$ must be a point of $\mathfrak{Q}'$. The following result, which improves a result by Schroth [28] significantly, implies in particular that not all points of such a $\mathfrak{Q}'$ can be projective.

THEOREM 4.9. *Let $\mathfrak{Q}$ be a compact connected finite-dimensional quadrangle and let $\mathcal{H}$ be a closed geometric hyperplane of $\mathfrak{Q}$. Then the following are equivalent:*

(i) *$\mathfrak{Q}$ has topological parameters (m, m) and all points of $\mathcal{H}$ are regular;*

(ii) *all points of $\mathcal{H}$ are projective;*

(iii) *$\mathcal{H}$ is trivial, all its nondeep points are regular, and at least one of them is projective;*

(iv) *$\mathcal{H}$ is trivial, all its nondeep points are regular and its deep point is projective;*

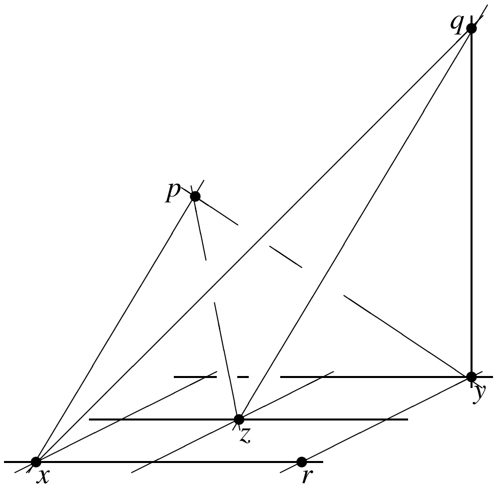
(v) *all points of $\mathfrak{Q}$ are projective;*

(vi) *$\mathfrak{Q}$ is isomorphic to the symplectic quadrangle over $\mathbb{R}$ or $\mathbb{C}$.*

Proof. By Lemma 1.7 and the main result of [28], it suffices to show that $\mathfrak{Q}$ cannot have a closed full subquadrangle all points of which are projective in $\mathfrak{Q}$. So suppose that $\mathfrak{Q}$ contains such a subquadrangle $\mathfrak{Q}'$. We know that $\mathfrak{Q}$ has then topological parameters (m, m), and hence by Theorem 4.8, $\mathfrak{Q}'$ is a grid. We could now finish the proof by quoting 2.8 which says that all lines of $\mathfrak{Q}$ are antiregular (because not all points are by the existence of regular points). This contradicts the existence of the grid $\mathfrak{Q}'$.

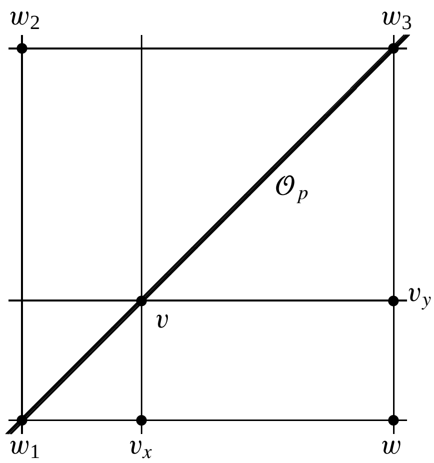
We now present an independent and rather geometric proof. Let $\mathcal{P}'$ be the point set of $\mathfrak{Q}'$ and let p be a point outside $\mathcal{P}'$.

(1) Let q be a point outside $\mathcal{P}'$ different from p. Then p and q induce closed ovals $\mathcal{O}_p = p^\perp \cap \mathcal{P}'$ and $\mathcal{O}_q = q^\perp \cap \mathcal{P}'$ in $\mathfrak{Q}'$. Suppose $\mathcal{O}_p$ and $\mathcal{O}_q$ meet in at least three different points x, y, z. Let r be one of the two points of $\mathfrak{Q}'$ collinear with both x and y.



By the regularity of x, we know that $\{r, p, q\} \subseteq D_2(x) \cap D_2(y) = D_2(x) \cap D_2(z)$, contradicting $r \notin D_2(z)$. Hence $|p^\perp \cap q^\perp \cap \mathcal{P}'| \leqslant 2$.

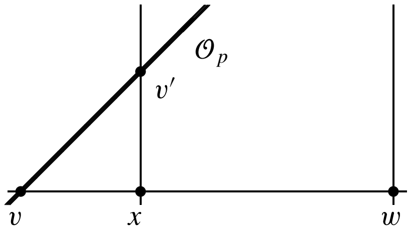
(2) Now consider a point w of $\mathfrak{Q}'$ off $\mathcal{O}_p$. Consider the set of lines $\mathcal{S} = \{v^\perp \cap w^\perp \mid v \in \mathcal{O}_p\}$ in the projective plane $\mathfrak{Q}(w)$ defined in $w^\perp$. We show that $\mathcal{S}$ is a dual closed oval. First we show that $\mathcal{S}$ is a closed curve in the line space of the compact projective plane $\mathfrak{Q}(w)$. Let w_1, w_3 denote the two unique points on $\mathcal{O}_p$ collinear with w. Then w, w_1, w_3 determine a unique ordinary quadrangle $(w, ww_1, w_1, w_1w_2, w_2, w_2w_3, w_3, w_3w)$ in $\mathfrak{Q}'$.



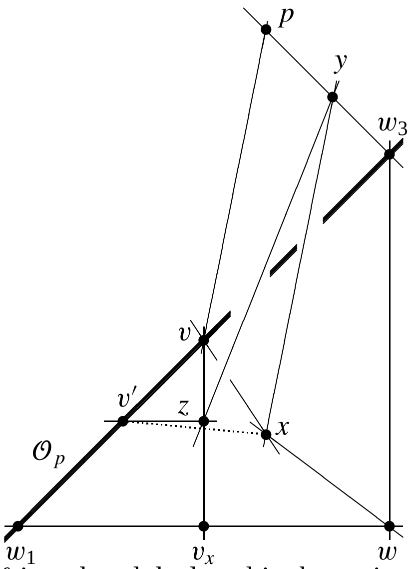
Let v_x denote the unique point on ww_1 which is equal to or collinear with $v \in \mathcal{O}_p$. We claim that v_x depends continuously on v. For $v \neq w_1$, we have $v_x = proj_{w_1w}v$, and for $v \neq w_3$, we have $v_x = proj_{ww_1}proj_{w_2w_3}v$; thus v_x depends continuously on v. Similarly, we define v_y. Then $v^\perp \cap w^\perp$ is the unique line through v_x, v_y in the projective plane $\mathfrak{Q}(w)$, hence $\mathcal{S} = \{v^\perp \cap w^\perp | v \in \mathcal{O}_p\}$ is compact and thus closed in the line space of $\mathfrak{Q}(w)$; also, every line through w meets $\mathcal{S}$ in a unique point.

(3) No point of $w^\perp$ can be collinear with three points of $\mathcal{O}_p$, because these three points are already collinear with p: such a line in $\mathfrak{Q}(w)$ would be of the form $q^\perp \cap w^\perp$, for $q \in D_4(w)$, and we showed in (1) that $|p^\perp \cap q^\perp \cap \mathcal{P}'| \leqslant 2$.

(4) Let w, w_1, w_2 as in (2), and let v be any point of $\mathcal{O}_p$.
If $v \in w^\perp$, then $v \in \{w_1, w_3\}$. Every point x of the line vw distinct from v itself lies on exactly two elements of $\mathcal{S}$, because every such point is collinear with exactly two points v, v' of $\mathcal{O}_p$.



Moreover, it is also clear that v itself is on a unique element of $\mathcal{S}$. Hence, if $v \in \{w_1, w_2\}$, then $v^\perp \cap w^\perp$ is, as an element of $\mathcal{S}$, on a unique ‘tangent point’ of $\mathcal{S}$.
Now let $v \in \mathcal{O} \backslash w^\perp$. The traces $w^\perp \cap p^\perp$ and $w^\perp \cap v^\perp$ intersect in a unique point x_0; in fact, p, v, x_0 are collinear. Thus x_0 is not collinear with any element in $\mathcal{O}$ different from v, i.e., x_0 lies on a unique element of $\mathcal{S}$.
Let now $x \in w^\perp \cap v^\perp$ be different from x_0. We have to show that x is collinear with exactly two points of $\mathcal{O}_p$. Put $y = proj_{pw_3} x$. Then $y \neq p, w_3$. Next, put $z = proj_{v_x v} y$. The point z is contained in $\mathcal{P}'$ and hence projective. Now $v, y \in z^\perp \cap x^\perp \cap p^\perp$, hence $z^\perp \cap x^\perp = z^\perp \cap p^\perp$. The point z is collinear with two distinct points $v, v' \in \mathcal{O}_p \subset p^\perp$, hence the same is true for x, i.e., $v' \in x^\perp$. By (3), x is collinear with at most two points of $\mathcal{O}_p$.



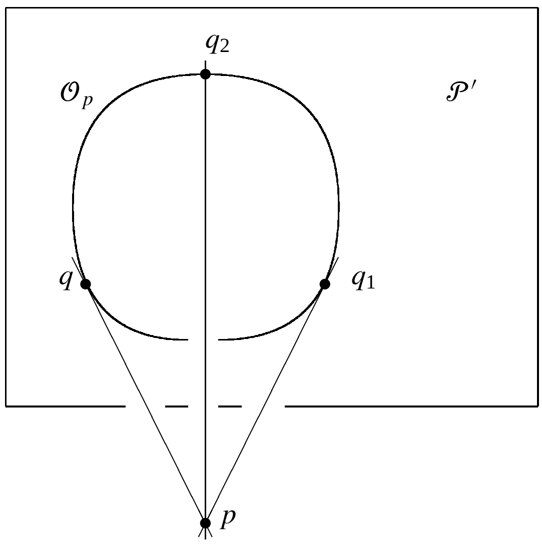
We have proved that $\mathcal{S}$ is a closed dual oval in the projective plane $\mathfrak{Q}(w)$.

(5) We can now finish the proof in two different ways. First, we can remark that by (4) all tangent points x_0 belong to the line $p^\perp \cap w^\perp$. Hence the dual of $\mathcal{S}$ is not a closed oval, a contradiction to Salzmann *et al.* [26], Proposition 55.17. Second, by [26], Proposition 55.14, the point rows in $\mathfrak{Q}$ have dimension 1 or 2. But then all lines of $\mathfrak{Q}$ are antiregular (by 2.6), contradicting the fact that $\mathfrak{Q}$ contains a full grid. □

We now consider the special case of a Moufang quadrangle. We can state a general theorem without referring to topology.

THEOREM 4.10. *Let $\mathfrak{Q}$ be a Moufang quadrangle and suppose that $\mathfrak{Q}'$ is a full subquadrangle of $\mathfrak{Q}$. Let G be the group of all automorphisms of $\mathfrak{Q}$ which belong to the little projective group of $\mathfrak{Q}$ and fix $\mathfrak{Q}'$ globally. Let p be a point of $\mathfrak{Q}$ not belonging to $\mathfrak{Q}'$. Then the stabilizer G_p induces on the ovoid $\mathcal{O}_p$ of $\mathfrak{Q}'$ a doubly transitive permutation group which is permutation isomorphic to a subgroup of the action on $D_1(p)$ of the stabilizer of G_p.*

Proof. Let q be any element of $\mathcal{O}_p$. We show that G_p acts transitively on $\mathcal{O}_p\backslash\{q\}$. Let $q_1, q_2 \in \mathcal{O}_p\backslash\{q\}$. Then both points q_1 and q_2 are opposite q in $\mathfrak{Q}$. Moreover, they have the same projection $proj_{pq}q_1 = proj_{pq}q_2 = p$ onto the line pq.



By the Moufang property, there exists a collineation $\theta = \theta_{q;q_1,q_2}$ of $\mathfrak{Q}$ fixing all lines through q, fixing all points on pq and mapping q_1 to q_2. Since θ fixes p, it stabilizes $\mathcal{O}_p$ whenever it stabilizes $\mathfrak{Q}'$. Now θ maps $\mathfrak{Q}'$ to some subquadrangle $\mathfrak{Q}''$. The set of lines of $\mathfrak{Q}''$ through q is precisely $D_1'(q)$, and all points on every line of $D_1'(q)$ belong to both $\mathfrak{Q}'$ and $\mathfrak{Q}''$. Moreover, $\mathfrak{Q}'$ and $\mathfrak{Q}''$ share an ordinary quadrangle through q and q_2: let $\ell, \ell' \in D_1'(q)$ be two different lines. Then $\theta(\ell) = \ell$ and $\theta(\ell') = \ell'$. Thus the points $\{q, proj_\ell q_2, q_2, proj_{\ell'} q_2\} \subset \mathcal{P}' \cap \mathcal{P}''$

determine an ordinary quadrangle. By Van Maldeghem [33] Corollary 1.8.5 we conclude $\mathfrak{Q}' = \mathfrak{Q}''$. The theorem is proved by considering the group generated by all $\theta_{q;q_1,q_2}$, for q, q_1, q_2 three different points of $\mathcal{O}_p$. □

COROLLARY 4.11. *The ovoids in classical Moufang quadrangles arising from hyperplanes admit a* 2*-transitive group.*

Proof. Let $\mathfrak{Q}$ be a classical Moufang quadrangle naturally imbedded in some d-dimensional projective space $\mathrm{PG}(V)$. One can easily check that, in all the finite-dimensional examples, if we find a hyperplane section which is an ovoid, then there exists a classical Moufang quadrangle $\mathfrak{Q}'$ containing $\mathfrak{Q}$ as a full subquadrangle and imbedded in $(d+1)$-dimensional projective space $\mathrm{PG}(V')$. If p is a point of $\mathfrak{Q}'$ not belonging to $\mathfrak{Q}$, then $p^{\perp}$ is contained in a hyperplane H of $\mathrm{PG}(V')$. But H meets $\mathrm{PG}(V)$ in a hyperplane of $\mathrm{PG}(V)$, hence the ovoid $\mathcal{O}_p$ is contained in a hyperplane of $\mathrm{PG}(V)$ and is now readily seen to be equal to a hyperplane section. □

EXAMPLES 4.12. We consider examples arising from imbeddings between the classical compact connected quadrangles, cf. Part II [18] Section 6 for the terminology (all possible imbeddings between the classical compact connected quadrangles have been determined recently by Wolfrom [34]). See Onishchik and Vinberg [23] for the Lie group terminology.

Let $Q_k(\mathbb{R})$ be the real orthogonal quadrangle arising from a nondegenerate quadratic form of Witt index 2 in $\mathbb{R}^{k+1}$, $k \geqslant 4$. Then $Q_k(\mathbb{R})$ has topological parameters $(1, k-3)$ and is a full subquadrangle of $Q_{k'}(\mathbb{R})$, for all $k' > k$. The corresponding ovoids are hyperplane sections and admit a 2-transitive orthogonal group $\mathrm{PSO}_{k-1,1}\mathbb{R}$.

Let $H_k(\mathbb{C})$ be the complex Hermitian quadrangle arising from a nondegenerate Hermitian form of Witt index 2 in $\mathbb{C}^{k+1}$, $k \geqslant 3$. Then $H_k(\mathbb{C})$ has topological parameters $(2, 2k-5)$ and is a full subquadrangle of $H_{k'}(\mathbb{C})$, for all $k' > k$. The corresponding ovoids are hyperplane sections and admit a 2-transitive unitary group $\mathrm{PSU}_{k-1,1}\mathbb{C}$.

Let $H_k(\mathbb{H})$ be the standard Hermitian quadrangle arising from a standard Hermitian form of Witt index 2 in $\mathbb{H}^{k+1}$, $k \geqslant 3$. Then $H_k(\mathbb{H})$ has topological parameters $(4, 4k-9)$ and is a full subquadrangle of $H_{k'}(\mathbb{H})$, for all $k' > k$. The corresponding ovoids are hyperplane sections and admit a 2-transitive quaternion unitary group $\mathrm{PU}_{k-1,1}\mathbb{H}$.

Let $H_k^{\alpha}(\mathbb{H})$ be the α-Hermitian quadrangle arising from a quaternion α-Hermitian form in $\mathbb{H}^{k+1}$, $k = 3, 4$. Then $H_3^{\alpha}(\mathbb{H})$ has topological parameters (4, 1), it is dual to $Q_7(\mathbb{R})$ and it is a full subquadrangle of $H_4^{\alpha}(\mathbb{H})$, which has topological parameters (4,5). The corresponding ovoid is also a hyperplane section and admits a 2-transitive quaternion α-unitary group $\mathrm{PU}_3^{\alpha}\mathbb{H} \cong \mathrm{PSU}_{3,1}\mathbb{C}$.

The orthogonal quadrangle $Q_9(\mathbb{R})$ is an ideal subquadrangle of the exceptional Moufang quadrangle $Q(E_6, \mathbb{R})$ of type E_6, and hence $Q_9(\mathbb{R})$ admits a 2-transitive spread. The corresponding group is $\mathrm{PSU}_{4,1}\mathbb{C}$. We conjecture that these spreads coincide with the J-spreads constructed in Part II [18] 7.3.

For a complete account on existence and nonexistence of spreads, ovoids and partitions into spreads and ovoids in the Moufang quadrangles, we refer to Part II [18] of the present paper. It follows e.g. from Part III [19] that $Q(E_6, \mathbb{R})$ is a full subquadrangle in the (non Moufang) Clifford quadrangle FKM(9, 32), and contains ovoids.

5. General Nonexistence Results

Using the fibre bundle interpretation of an ovoid, we can prove some general nonexistence results. If $\mathfrak{Q} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ is a (1, 1)-quadrangle, we can distinguish between $\mathfrak{Q}$ and its dual by the mod 2 twisting number $t_{\mathcal{P}}$ of the point space, cf. the remarks after 2.6.

THEOREM 5.1. *$\mathfrak{Q} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ be a compact connected $(1, 1)$-quadrangle with mod 2 twisting number $t_{\mathcal{P}} = 1$. Then the map $\mathcal{F} \to \mathcal{L}$ does not admit a section; in particular $\mathfrak{Q}$ has no closed ovoid $\mathcal{O} \subseteq \mathcal{P}$.*

Proof. The $\mathbb{F}_2$-cohomology of the map $\mathrm{pr}_{\mathcal{L}}\colon \mathcal{F} \to \mathcal{L}$ is given by the injection of graded $\mathbb{F}_2$-algebras

$$\mathbb{F}_2[x_1, y_1, y_2]/(x_1^4, y_1^2, x_1^2 + y_2 + x_1 y_1) \leftarrow \mathbb{F}_2[y_1, y_2]/(y_1^2, y_2^2),$$

see [17], Hebda [11], Münzner [22], Grundhöfer *et al.* [9] (the subscripts indicate the degrees of the homogeneous generators of the cohomology ring). If $r\colon \mathcal{L} \to \mathcal{F}$ is a section of $\mathrm{pr}_{\mathcal{L}}$, then the composite

$$\mathrm{H}^\bullet(\mathcal{L}; \mathbb{F}_2) \xleftarrow{r^\bullet} \mathrm{H}^\bullet(\mathcal{F}; \mathbb{F}_2) \xleftarrow{\mathrm{pr}_{\mathcal{L}}^\bullet} \mathrm{H}^\bullet(\mathcal{L}; \mathbb{F}_2)$$

is the identity. Since $\dim_{\mathbb{F}_2}\mathrm{H}^1(\mathcal{F}; \mathbb{F}_2) = 2$ and $\dim_{\mathbb{F}_2}\mathrm{H}^1(\mathcal{L}; \mathbb{F}_2) = 1$, the induced map in cohomology $r^\bullet$ has to kill some element $ay_1 + x_1 \in \mathrm{H}^1(\mathcal{L}; \mathbb{F}_2)$, where $a \in \mathbb{F}_2$. Since $r^\bullet$ is a ring homomorphism,

$$0 = r^\bullet(((1 + a)y_1 + x_1)(ay_1 + x_1)) = r^\bullet(y_1 x_1 + x_1^2) = r^\bullet(y_2) = y_2 \neq 0,$$

a contradiction. □

COROLLARY 5.2. *The real symplectic quadrangle $W(\mathbb{R})$ has no closed ovoids.* □

The dual of Theorem 5.1 is false: the real orthogonal (1, 1)-quadrangle $Q_4(\mathbb{R})$ has closed ovoids.

THEOREM 5.3. *Let $\mathfrak{Q}$ be a compact connected (m, m)-quadrangle, for $m > 1$. Then the maps $\mathcal{F} \to \mathcal{L}$ and $\mathcal{F} \to \mathcal{P}$ admit no sections; in particular, $\mathfrak{Q}$ has neither closed ovoids nor closed spreads.*

Proof. We use the same method as in the previous proof; however, this time we use rational cohomology. The $\mathbb{Q}$-cohomology of the map $\mathrm{pr}_{\mathcal{L}}: \mathcal{F} \to \mathcal{L}$ is given by

$$\mathbb{Q}[x_m, x_{2m}, y_m, y_{2m}] \leftarrow \mathbb{Q}[y_m, y_m]$$

and we have the relations

$$x_m^2 = cx_{2m}, \qquad y_m^2 = dy_{2m}, \qquad x_{2m}^2 = y_{2m}^2 = 0, \qquad x_m y_m = x_{2m} + y_{2m}.$$

The structure constants c, d are given by $(c, d) \in \{(1, 2), (2, 1)\}$. Again, $\dim \mathrm{H}^{\mathrm{m}}(\mathcal{F}; \mathbb{Q}) = 2$ and $\dim \mathrm{H}^{\mathrm{m}}(\mathcal{L}; \mathbb{Q}) = 1$. The existence of a section r of $\mathrm{pr}_{\mathcal{L}}$ implies that $r^\bullet(ay_m + x_m) = 0$, for some $a \in \mathbb{Q}$. Hence

$$\begin{aligned} 0 &= r^\bullet((ay_m + x_m)(x_m - (c + a)y_m)) \\ &= -r^\bullet((da^2 + acd + c)y_m) = -(da^2 + 2a + c)y_m. \end{aligned}$$

But the polynomial $g(a) = da^2 + 2a + c$ has no roots in $\mathbb{Q}$, hence the right-hand side cannot be 0, a contradiction. □

COROLLARY 5.4. *The complex symplectic quadrangle $W(\mathbb{C})$ has neither closed spreads nor closed ovoids.* □

This holds in particular for Zariski-closed ovoids in $W(\mathbb{C})$.

COROLLARY 5.5. *An algebraic quadrangle over an algebraically closed field K of characteristic 0 has no Zariski-closed ovoids or spreads; in particular, $W(K)$ has neither Zariski-closed ovoids nor Zariski-closed spreads.*

Proof. By Kramer and Tent [20], such a quadrangle is K-isomorphic to $W(K)$ or to $Q_4(K)$. By similar model-theoretic transfer methods as in [20], one shows that the nonexistence of ovoids for the special case $K = \mathbb{C}$ implies the general result. □

COROLLARY 5.6. *A compact connected finite-dimensional quadrangle which has a projective point does not have closed ovoids.*

Proof. The parameters (m, m') of $\mathfrak{Q}$ are equal. If $m = 1$, then $p^\perp \cong \mathbb{R}\mathrm{P}^2$, cf. Kramer [17] 4.3.1, and thus $t_{\mathcal{P}} = 1$. The result follows for $m = 1$ from 5.1 and for $m > 1$ from 5.3. □

If the parameters (m, m') become large, then the existence problem becomes rather subtle, as we will see in Parts II, III [18], [19]. However, the nonexistence of spreads in $(1, m')$-quadrangles, for $m' > 1$ odd, can be settled in full generality.

The point rows of a $(1, m')$-quadrangle are homeomorphic to the circle $\mathbb{S}^1$; thus, $\mathrm{pr}_{\mathcal{L}}: \mathcal{F} \to \mathcal{L}$ is a circle bundle. We want to view this as an orthogonal sphere bundle $S(\xi)$ of a vector bundle ξ. Let ξ denote the associated open 2-disk bundle of this circle bundle, let TOP(2) denote the group of base-point preserving homeomorphisms of $\mathbb{R}^2$, and let $\mathrm{O}(2) \subseteq \mathrm{TOP}(2)$ denote the subgroup of all orthogonal transformations. The question is then whether the classifying map

$$\begin{tikzcd} & \mathrm{BO}(2) \arrow[d] \\ \mathcal{L} \arrow[r] \arrow[ur, dotted] & \mathrm{BTOP}(2) \end{tikzcd}$$

of the disk bundle ξ lifts to BO(2). By a result of Kneser the inclusion $\mathrm{O}(2) \subseteq \mathrm{TOP}(2)$ is a homotopy equivalence, cf. Kneser [16], Friberg [6], Kirby and Siebenmann [14] p. 253, therefore there exists no obstruction to this lifting problem. (The obstructions are certain elements of $\mathrm{H}^k(\mathcal{L}; \pi_k(\mathrm{TOP}(2)/\mathrm{O}(2)))$, for $k \geqslant 0$. Since $\mathrm{TOP}(2)/\mathrm{O}(2)$ is contractible, all obstructions vanish.)

The upshot is that we can think of $\mathcal{F} \to \mathcal{L}$ as the sphere bundle $S(\xi)$ of an orthogonal vector bundle ξ. If $m' > 1$, then $\mathcal{L}$ is simply connected, therefore we can choose an orientation for ξ, making ξ into a *complex* line bundle (thus we lift the classifying map from BO(2) into BU(1)). Complex line bundles over a paracompact space X are classified by their first Chern class $\mathrm{c}_1 \in \mathrm{H}^2(X)$ (to see this, note that BU(1) is an Eilenberg–MacLane space $K(\mathbb{Z}, 2)$, hence there is an equivalence of homotopy functors $\mathrm{H}^2(-) \cong [-; \mathrm{BU}(1)]$). The other piece of information that we need is the following: in a $(1, m')$-quadrangle with $m' \geqslant 2$, the generalized manifold $\mathcal{P}$ is orientable if and only if m' is even (this is proved in Kramer [17] 3.4.9, cf. also Grove and Halperin [7] 4.8).

PROPOSITION 5.7. *Let $\mathfrak{Q} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ be a finite-dimensional quadrangle with parameters $(1, m')$, for $m' \geqslant 3$ odd. Then $\mathfrak{Q}$ does not have closed spreads.*

Proof. Let $\mathcal{S} \subseteq \mathcal{L}$ be a closed spread, and consider the restriction $\mathcal{F}_\mathcal{S} \to \mathcal{S}$. If $m' \geqslant 2$, then $\mathrm{H}^2(\mathcal{S}) = \mathrm{H}^2(\mathbb{S}^{m'+1}) = 0$, hence $\xi|_\mathcal{S}$ is a trivial complex line bundle, and $\mathcal{P} \cong \mathcal{F}_\mathcal{S} = \mathbb{S}^1 \times \mathcal{S}$. Thus $\mathcal{P}$ is orientable and m' is even. □

These arguments do not go through for $m' = 1$, but this case is already covered by Theorem 5.1 above.

We will see that the point spaces of the compact Moufang quadrangles can often be partitioned in ovoids, cf. also 4.1. This property has strong consequences for the bundle $\mathrm{pr}_\mathcal{L}: \mathcal{F} \to \mathcal{L}$.

PROPOSITION 5.8. *Suppose that the point space of a compact quadrangle $\mathfrak{Q}$ can be partitioned into closed ovoids. Then the bundle $\mathcal{F} \to \mathcal{L}$ is topologically a trivial product bundle.*

Proof. Let $L = D_1(\ell)$ be a point row, and let $(\mathcal{O}_p)_{p \in L}$ be a family of ovoids which partitions $\mathcal{P}$, and such that $\mathcal{O}_p \cap L = \{p\}$. For each ovoid $\mathcal{O}_p$ let r_p denote the corresponding section $\mathcal{L} \to \mathcal{F}$. Then $R = \{r_p | p \in L\}$ is a subspace of the space $C(\mathcal{L}, \mathcal{F})$ of all continuous maps from $\mathcal{L}$ to $\mathcal{F}$, endowed with the compact-open topology. The evaluation map $r_p \mapsto r_p(\ell)$, $R \to \mathcal{F}$ is continuous and injective. Its image is the compact set $L \times \{\ell\} \subseteq \mathcal{F}$; therefore $r_p \mapsto r_p(\ell)$, $R \to L \times \{\ell\}$ is a homeomorphism. The map $(r_p, h) \mapsto r_p(h)$, $R \times \mathcal{L} \to \mathcal{F}$ is a continuous bijection, hence, by compactness of $\mathcal{F}$, also a homeomorphism. Combining these homeomorphisms we find that $L \times \mathcal{L} \cong \mathcal{F}$, as claimed. □

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