We recall that a Hausdorff space X is called *paracompact* if every open covering of X has a locally finite refinement. Every metric space is paracompact.

Proposition 4.10. *Suppose that $f : X \longrightarrow Y$ is a continuous open surjective map, that X is Čech complete and that Y is paracompact. Then Y is Čech complete.*

Proof. We subdivide the proof into several steps.

Step 1. There exists a Tychonoff space Z containing X as a G_δ-subspace and a continuous closed proper map $F : Z \longrightarrow Y$ which extends f.

We consider the map $\beta f : \beta X \longrightarrow \beta Y$ between the Čech-Stone compactifications, and we put $Z = \beta f^{-1}(Y) \supseteq X$. We denote the restriction-corestriction of βf by

$$F : Z \longrightarrow Y.$$

If $B \subseteq Y$ is compact, then $F^{-1}(B) = \beta f^{-1}(B)$ is compact, hence F is proper. If $A \subseteq \beta f(X)$ is closed, then A is compact and $F(A \cap Z) = \beta f(A) \cap Y$ is closed in Y, hence F is closed.

Step 2. Suppose that $U \subseteq Z$ is open and that $F(U \cap X) = Y$. Then there exists an open set $U' \subseteq Z$ with $F(U' \cap X) = Y$ and with $\overline{U'} \subseteq U$.

Suppose that $U \subseteq Z$ is open and that $F(X \cap U) = Y$. We choose, for every $y \in Y$, an open set set $V_y \subseteq Z$ as follows. We choose an element $x \in X \cap U$ with $f(x) = y$, and then an open neighborhood $V_y \subseteq U$ of x with $\overline{V_y} \subseteq U$. This is possible because Z is regular. Since $V_y \cap X$ is open in X and since f is open, $U_y = f(V_y \cap X)$ is an open neighborhood of y. Since Y is paracompact, the open cover $\{U_y \mid y \in Y\}$ has a locally finite refinement $\mathcal{W}$. For every $W \in \mathcal{W}$ we choose $y(W) \in Y$ such that $W \subseteq U_{y(W)}$. We put $U_W = V_{y(W)} \cap F^{-1}(W)$ and we note that $\overline{U_W} \subseteq U$ is open. We also put $U' = \bigcup\{U_W \mid W \in W\}$. If $W \in \mathcal{W}$, then $W \subseteq f(V_{y(W)} \cap X)$ and thus $f(U_W \cap X) = W$. Hence $F(U' \cap X) = Y$. Suppose that $z \in Z$ is in the closure of U'. There exists an open neighborhood O of $f(z)$ such that set $\mathcal{W}_O = \{W \in \mathcal{W} \mid O \cap W \neq \varnothing\}$ is finite. Hence

$$z \in \overline{\bigcup\{U_W \mid W \in \mathcal{W}_O\}} = \bigcup\{\overline{U_W} \mid W \in \mathcal{W}_O\} \subseteq U.$$

Step 3. There exists a closed G_δ-set $C \subseteq X$ such that $f|_C : C \longrightarrow Y$ is surjective.

We write $X = \bigcup_{n \geq 1} U_n$, where $U_n \subseteq Z$ is open. we construct a sequence of open sets Z_n as follows We put $Z_0 = Z$. Given Z_{n-1}, we choose an open set Z_n such that $\overline{Z_n} \subseteq U_n \cap Z_n$, with $F(X \cap Z_N) = Y$. We put $C = \bigcap_{n \geq 0} Z_n = \bigcap_{n \geq 0} \overline{Z_n}$. Thus $C \subseteq X$ is a closed G_δ-set. The set $f^{-1}(y) \subseteq X$ is compact for every y, hence $\bigcap \overline{Z_n} \cap f^{-1}(y) \neq \varnothing$.

Step 4. The claim of the proposition is true.

The set $C \subseteq X$ is a G_δ-set in the Čech complete space X, and therefore Čech complete. Since $C \subseteq Z$ is closed, $F|_C = f|_C$ is continuous, proper, closed and surjective. Therefore Y is Čech complete. ☐