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WS 2012/13

9. Übung zur Vorlesung Räume nichtpositiver Krümmung

Please hand in your solutions on the morning of December 10 2012 before the lecture.

Aufgabe 9.1 (Finitely generated abelian groups)

Let A be a finitely generated abelian group, and let $B \subseteq A$ be a subgroup.

(a) (1 mark) Show that the torsion group $T(A) = \{a \in A \mid a \text{ has finite order}\}$ is a subgroup. (Where do you need that A is abelian? Do you need that A is finitely generated?)

(b) (1 mark) Show that $A/T(A)$ has no torsion. (Where do you need that A is abelian? Do you need that A is finitely generated?)

(c) (2 marks) Show that B is finitely generated.

(d) (3 marks) Show that

$$\mathrm{rk}_{\mathbb{Q}} A = \mathrm{rk}_{\mathbb{Q}} B + \mathrm{rk}_{\mathbb{Q}}(A/B).$$

Hint: Show that $\mathrm{Hom}(A, \mathbb{Q})$ *is a vector space over* $\mathbb{Q}$ *of dimension* $\mathrm{rk}_{\mathbb{Q}} A$.

Aufgabe 9.2 (Isometric actions on $\mathbb{R}$-trees)

(a) (2 marks) Every isometry g of an $\mathbb{R}$-tree is semisimple.
Hint: consider the midpoint of $\{x, g(x)\}$.

We call a point x in an $\mathbb{R}$-tree a branch point if there are three distinct nonconstant geodesics starting at x having pairwise only the point x in common. We call an $\mathbb{R}$-tree a $\mathbb{Z}$-tree if the distance between any two branch points is a natural number.

(b) (3 marks) Suppose that $(\mathbb{Q}, +)$ acts isometrically on a $\mathbb{Z}$-tree having a nonempty set of branch points.. Show that every $t \in \mathbb{Q}$ is elliptic.

Aufgabe 9.3 (Euclidean space)

(a) (1 mark) Let $u, u', v \in \mathbb{R}^2$ be nonzero vectors, with $\|u\|_2 = \|u'\|_2$. Show that the following are equivalent.
(1) $\|u - v\|_2 < \|u' - v\|_2$.
(2) $\measuredangle_0(u, v) < \measuredangle_0(u', v)$.
Hint: what is the relation between the inner product and the cosine?

(b) (2 marks) Let A be a group and let $\tau : A \longrightarrow (\mathbb{R}^m, +)$ be a homomorphism. Suppose that there is a compact subset $K \subseteq \mathbb{R}^m$ such that $\bigcup_{a \in A}(\tau(a) + K) = \mathbb{R}^m$. Show that $\tau(A)$ generates $\mathbb{R}^m$ as a vector space.

(c) (2 marks) Assume that A and τ are as in part (b). Then A acts isometrically as a group of translations on $\mathbb{R}^m$, via $a : x \mapsto x + \tau(a)$. Suppose that the isometry $g \in \mathrm{Iso}(\mathbb{R}^m)$ commutes with this action, i.e. that $g(x + \tau(a)) = g(x) + \tau(a)$ holds for all $a \in A$. Show that g is a translation.