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WS 2012/13

7. Übung zur Vorlesung Räume nichtpositiver Krümmung

Please hand in your solutions on the morning of November 26 2012 before the lecture.

Aufgabe 7.1 (Isometries of the hyperbolic plane)

(a) (3 marks) Consider the group $\mathrm{SL}_2\mathbb{R}$ acting by $g(x) = gxg^T$ on the space of symmetric 2-by-2 matrices. Identify this space with $\mathbb{R}^3$ via the vector space isomorphism

$$\begin{pmatrix} x_0 - x_1 & x_2 \\ x_2 & x_0 + x_1 \end{pmatrix} \longmapsto (x_0, x_1, x_2).$$

Prove that the action of $\mathrm{SL}_2\mathbb{R}$ on $\mathbb{R}^3$ so defined preserves the Lorentzian form β. In this way we obtain an action of $\mathrm{SL}_2\mathbb{R}$ via isometries on H^2. Show that this action on H^2 is transitive.
Hint: symmetric matrices are diagonizable.

(b) (2 marks) Prove that the isometry $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ corresponds to a parabolic isometry.

Aufgabe 7.2 (Spherical geometry)

(a) (2 marks) Consider the sphere $\mathbb{S}^n$ with the induced metric from $\mathbb{E}^{n+1}$. Show that this is not a geodesic space.

(b) (3 marks) Given two elements $u, v \in \mathbb{S}^n$ viewed as vectors in $\mathbb{R}^{n+1}$, define $d(u, v)$ by

$$\cos(d(u, v)) = \langle u, v\rangle = \sum_j u_j v_j.$$

Show that this is a well-defined metric on $\mathbb{S}^n$, the so called *angular metric*.

(c) (2 marks) Show that the angular metric makes $\mathbb{S}^n$ into a complete geodesic space. Is it CAT(0)?

(d) (2 marks) Prove that $\mathrm{O}_{n+1}\mathbb{R}$ is the isometry group of the sphere $\mathbb{S}^n$ with respect to the angular metric.

(e) (1 mark) Show that every $g \in \mathrm{SO}_3\mathbb{R}$ has a fixed point on $\mathbb{S}^2$.

(f) (1 mark) Show that every isometry of a compact metric space is semisimple. In particular, every isometry of $\mathbb{S}^n$ is semisimple.

Aufgabe 7.3 (Hilbert space)

(a) (2 marks) Show that every isometry g of $\mathbb{R}^n$ with the euclidean metric can be written as

$$v \longmapsto av + t,$$

for some $t \in \mathbb{R}^n$ and $a \in \mathrm{O}_n\mathbb{R}$.

(b) (3 marks) In the Hilbert space $\ell_2(\mathbb{Z})$ consisting of all square-summable sequences $x = (x_j)_{j\in\mathbb{Z}}$, consider the shift operator $\sigma : (x_j)_{j\in\mathbb{Z}} \longmapsto (x_{j+1})_{j\in\mathbb{Z}}$. Put $t = (t_j)_{j\in\mathbb{Z}}$, with $t_0 = 1$ and $t_j = 0$ for $j \neq 0$. Show that the map

$$g : v \longmapsto \sigma(v) + t$$

is a parabolic isometry.