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WS 2012/13

4. Übung zur Vorlesung Räume nichtpositiver Krümmung

Please hand in your solutions on the morning of November 5 2012 before the lecture.

Aufgabe 4.1 (Monotonicity of the covering dimension)

(2 marks) Let X be a normal space and let $A \subseteq X$ be a closed subspace. Show that

$$\dim(A) \leq \dim(X).$$

Aufgabe 4.2 (CAT(0) spaces)

(1 mark) Let X be a CAT(0) space and let $p \in X$. Show that for every geodesic $\gamma : [a, b] \longrightarrow X$ the map $t \mapsto d(p, \gamma(t))$ is convex.

A normed vector space $(V, \|-\|)$ is called a pre-Hilbert space if there exists a symmetric positive definite bilinear form $b : V \times V \longrightarrow \mathbb{R}$ such that $b(v) = \|v\|^2$ holds for all $v \in V$.

(2 marks) A normed vector space is a pre-Hilbert space if and only if the parallelogram law

$$\|u - v\|^2 + \|u + v\|^2 = 2(\|u\|^2 + \|v\|^2)$$

holds for all $u, v \in V$.

(3 marks) Let $(V, \|-\|)$ be a normed vector space. Show that the metric $d(u, v) = \|u - v\|$ is CAT(0) if and only if V is a pre-Hilbert space.

A map $\gamma : [a, b] \longrightarrow X$ is called a local geodesic if for every point $p \in [a, b]$ there exists an $r_p > 0$ such that $d(\gamma(s), \gamma(t)) = |s - t|$ holds for all s, t with $|s - p|, |t - p| < r_p$.

(2 marks) Show that every local geodesic in a CAT(0) space is a geodesic.

Aufgabe 4.3 (Metric spaces are paracompact)

Suppose that X is a metric space and that we are given an open cover $\{C_\alpha \mid \alpha \in I\}$. By the well-ordering theorem there exists a well-ordering $<$ on the index set I; that is, a total ordering in which every nonempty subset has a least element. Choose such a well-ordering. For each positive integer n, define $D_{\alpha n}$, by induction on n, to be the union of all open balls $B_{2^{-n}}(x)$ such that:

(1) α is the least member of the index set I with respect to the ordering $<$ such that $x \in C_\alpha$,
(2) $x \notin D_{\beta j}$ if $j < n$,
(3) $B_{3 \cdot 2^{-n}}(x) \subseteq C_\alpha$.

(2 marks) Show that $\{D_{\alpha n}\}$ is a refinement of $\{C_\alpha\}$ which covers X.

(2 marks) Suppose that $x \in X$ and let α be the least element of the index set I, with respect to the chosen well-ordering $<$, such that $x \in D_{\alpha n}$ for some n. Choose a positive integer j such that $B_{2^{-j}}(x) \subseteq D_{\alpha n}$. Show that

(a) if $i \geq n + j$, $B_{2^{-n-j}}(x)$ does not meet $D_{\beta i}$ for any $\beta \in I$,
(b) if $i < n + j$, $B_{2^{-n-j}}(x)$ meets $D_{\beta i}$ for at most one $\beta \in I$.

(1 mark) Conclude that $D_{\alpha n}$ is a locally finite refinement of $\{C_\alpha\}$ which covers X, and hence that every metric space is paracompact.