Prof. Dr. L. Kramer
Dr. Rupert McCallum
Dr. Daniel Skodlerack
Antoine Beljean

WS 2012/13

3. Übung zur Vorlesung Räume nichtpositiver Krümmung

Please hand in your solutions on the morning of October 29 2012 before the lecture.

Aufgabe 3.1 (Simplicial complexes)

(1 mark) The n-simplex K is, by definition, the set of all subsets of $\{0, \dots, n\}$. Show that $|K|$ is homeomorphic to the closed n-ball and that $|K^{(n-1)}|$ is homeomorphic to the $n-1$-sphere $\mathbb{S}^{n-1}$.

(1 mark) If Δ is a simplicial complex and $a \in \Delta$ is an $n+1$-simplex, then $|\Delta^{(n)}| \cap |a| \cong \mathbb{S}^n$.

(2 marks) If Δ is a simplicial complex, then a subset $U \subseteq |\Delta| \times [0,1]$ is open (in the product topology, where $|\Delta|$ carries the weak topology) if and only if $U \cap (|a| \times [0,1])$ is open in $|a| \times [0,1]$, for all $a \in \Delta$.

(1 mark) If Δ is a simplicial complex and $a \in \Delta$ a nonempty simplex, then its star $|st(a)| = |\{b \in \Delta \mid a \cup b \in \Delta\}|$ is contractible.

Aufgabe 3.2 (Contractible and n-connected spaces)

(2 marks) A contractible space is n-connected, for all $n \in \mathbb{N}$.

(1 mark) A space is 0-connected if and only if it is path-connected.

(2 marks) If you know what the fundamental group $\pi_1(X, p)$ of a topological space is, show that a 0-connected space is 1-connected if and only if $\pi_1(X, p) = 1$, for some $p \in X$.

(1 mark) The Hilbert space $L_2(\mathbb{N})$ consists of all square summable sequences of real numbers, $\sum_{i \in \mathbb{N}} r_i^2 < \infty$, with norm $\|(r_i)_{i \in \mathbb{N}}\|_2 = \sqrt{\sum_{i \in \mathbb{N}} r_i^2}$. Show that the unit sphere

$$\{x \in L_2(\mathbb{N}) \mid \|x\|_2 = 1\}$$

is contractible.

(*) Is the unit sphere in $\mathbb{R}^n$ contractible, for $n \geq 1$?

Aufgabe 3.3 (Paracompact spaces and partitions of unity)

(2 marks) A closed subspace of a paracompact space is paracompact.

(1 mark) Let X be a compact Hausdorff space. Suppose that every point $x \in X$ has a neighborhood V_x that admits a continuous embedding $\beta_x : V_x \to \mathbb{R}^{n_x}$, for some $n_x \in \mathbb{N}$. Show that X can be continuously embedded into $\mathbb{R}^m$, for some m. (A map $f : P \longrightarrow Q$ is called an embedding if it is continuous, injective, and if f is a homeomorphism between P and $f(P)$.)

(3 marks) Prove (without using the theorem that metric spaces are paracompact) that the Euclidean space $\mathbb{R}^n$ is paracompact. (Hint: Consider compact subspaces of $\mathbb{R}^n$.)