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WS 2012/13

2. Übung zur Vorlesung Räume nichtpositiver Krümmung

Please hand in your solutions on the morning of October 22 2012 before the lecture.

Aufgabe 2.1 (Nested sequences of closed balls and completeness)

A nested sequence of closed balls in a metric space X is an infinite descending sequence

$$\bar{B}_{r_0}(x_0) \supseteq \bar{B}_{r_1}(x_1) \supseteq \bar{B}_{r_2}(x_2) \supseteq \cdots .$$

(2 mark) If every nested sequence of closed balls in X has a point in common, then X is complete. (A space with this property is called *spherically complete*.)

(2 marks) If X is complete, then every nested sequence of balls with $\lim_{i\in\mathbb{N}} r_i = 0$ has a point in common.

(*) Can you give an example of a complete metric space that is not spherically complete?

Aufgabe 2.2 (Families of subspaces)

(2 mark) Suppose that $(C_j)_{k\in J}$ is a locally finite family of subspaces of a topological space X. Prove that

$$\bigcup_{j\in J} \overline{C_j} = \overline{\bigcup_{j\in J} C_j}.$$

Is it true that the family $(\overline{C_j})_{k\in J}$ is still locally finite?

(1 mark) A Hausdorff space is compact if and only if every open covering has a finite refinement.

Recall that the interior of a subset $Y \subset X$ is

$$\operatorname{int}(Y) = X \setminus \overline{X \setminus Y} = \bigcup \{U \mid U \textit{ open in } X \textit{ and } U \subseteq Y\}.$$

(2 marks) If $(C_j)_{k\in J}$ is a finite closed covering, then $X = \bigcup_{j\in J} \overline{\operatorname{int}(C_j)} = X$.

(*) Can you prove the same if the closed covering is assumed to be locally finite? (It may be easier to assume that J is countably infinite.)

Aufgabe 2.3 (Isometries of compact metric spaces)

(4 marks) Let X be a compact metric space and let $f : X \longrightarrow X$ be an isometric embedding. Prove that f is surjective.

Hint: Consider an element of X which has positive distance from $f(X)$.

Is the same true for complete metric spaces? For locally compact metric spaces?

Aufgabe 2.4 (ANEs)

Recall that a metric space X is an ANE (absolute neighborhood extensor) if the following holds for every metric space Y: if $B \subseteq Y$ is closed and $f : B \longrightarrow X$ is continuous, then there exists an open neighborhood V of B and a continuous extension $F : V \longrightarrow X$ of f. If it is always *possible to put $V = Y$, then X is called an AE (absolute extensor). If you get stuck with the following problems, look up the first chapters of* S.-T. Hu, Theory of Retracts.

(3 mark) Every contractible ANE is an AE (look up the definition of 'contractible' if you don't remember it).

Hint: Use Urysohn's lemma.

(1 marks) Suppose that X is an AE. If $r : X \longrightarrow X$ is a continuous retraction ($r \circ r = r$) then $A = r(X)$ is an AE.

(2 marks) An open subspace of an ANE is an ANE.

Hint: Suppose that Y is an open subspace of an ANE X, V is an open neighbourhood of B and $F : V \longrightarrow X$ is a continuous extension of f. Consider $F^{-1}(Y)$.

A metric space X is called an ANR (absolute neighborhood retract) if the following hold for every metric space Z: if $f : X \longrightarrow Z$ is an isometric embedding and if $f(X) \subseteq Z$ is closed, then $f(X)$ there is an open neighborhood V of $f(X)$ and a retraction $r : V \longrightarrow f(X)$.

(1 mark) Every ANE is an ANR.