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WS 2012/13

11. Übung zur Vorlesung Räume nichtpositiver Krümmung

Please hand in your solutions on the morning of January 7 2013 before the lecture.

Aufgabe 11.1 (Euclidean domains)

(a) (1 mark) Prove that $\mathbb{Z}$, $\mathbb{Z}[\sqrt{-1}]$, and $F[X]$ for any field F are Euclidean domains.

(b) (2 marks) Suppose that $D > 4$ is an integer. Prove that $\mathbb{Z}[\sqrt{-D}]$ is not a Euclidean domain.
Hint: First show that if δ is a Euclidean norm for $\mathbb{Z}[\sqrt{-D}]$ and $x \in \mathbb{Z}[\sqrt{-D}]$ has the value of $\delta(x)$ minimal for x not zero or a unit, then $x = \pm 2$ or ± 3. Now get a contradiction from showing that every $y \in \mathbb{Z}[\sqrt{-D}]$ must be congruent to ± 1 modulo x.

Aufgabe 11.2 (Infinite-dimensional hyperbolic space)

(a) (2 marks) Suppose that $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$ are both in the Hilbert space $\ell^2(\mathbb{R})$ of square summable real sequences. Define $\beta(x, y) := x_0y_0 - \sum_{n=1}^{\infty} x_n y_n$. Explain how this idea can be used to define an infinite-dimensional hyperbolic space H^∞ and prove that it is a complete CAT(0)-space.

(b) (2 marks) We previously did an example of an infinitely generated abelian group of elliptic isometries of the regular ternary tree without a common fixed point. Construct a similar example of an infinitely generated abelian group of elliptic isometries of H^∞ without a common fixed point.

Aufgabe 11.3 (The product of two trees)

(a) (2 marks) Consider the product of two regular ternary trees T_1, T_2. Prove that there is exactly one elliptic isometry with a fixed point (p, q) where p and q are both branch points which is not the product of two isometries of T_1 and T_2 respectively.

(b) (2 marks) Prove that every isometry of $T_1 \times T_2$ is semisimple.

Aufgabe 11.4 (The action of $\mathrm{SL}_2\mathbb{R}$ on the hyperbolic plane again)

(3 marks) Recall the action of $\mathrm{SL}_2\mathbb{R}$ by isometries on the hyperbolic plane described in Problem 7.1. Prove that the isometry corresponding to a matrix in $\mathrm{SL}_2\mathbb{R}$ is semisimple if and only if the matrix is diagonalizable. Show that the isometry is elliptic if the eigenvalues are complex numbers of absolute value 1 and hyperbolic if the eigenvalues are real numbers not of absolute value 1.