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WS 2012/13

1. Übung zur Vorlesung Räume nichtpositiver Krümmung

Please hand in your solutions on the morning of October 15 2012 before the lecture.

Aufgabe 1.1 (Continuity of the distance)

(2 marks) If A is a nonempty subset of a metric space X, we put $d(x, A) = \inf\{d(x, a) \mid a \in A\}$. Show that the map

$$x \mapsto d(x, A)$$

is continuous. What is the zero-set of this function?

(1 mark) Every closed set in a metric space is a G_δ-set, that is, an intersection of countably many open sets.

Aufgabe 1.2 (New metrics from old ones)

(4 marks) Suppose that $0 < \varepsilon \leq 1$. Prove that $|a + b|^\varepsilon \leq |a|^\varepsilon + |b|^\varepsilon$ holds for all $a, b \in \mathbb{R}$.

Conclude that if (X, d) is a metric space, then (X, d^ε) is also a metric space, with

$$d^\varepsilon(u, v) = d(u, v)^\varepsilon.$$

What can you say about the completeness of (X, d^ε) if (X, d) is complete?

Aufgabe 1.3 (Convex functions and the $\|-\|_p$-norm)

A function $f : J \to \mathbb{R}$, defined on some interval $J \subseteq \mathbb{R}$, is called *convex* if

$$f(sx + (1 - s)y) \leq sf(x) + (1 - s)f(y)$$

holds for all $x, y \in J$ and $s \in [0, 1]$.

(1 mark) A convex function is locally Lipschitz continuous and in particular continuous.

(2 marks) Suppose that f is two times continuously differentiable and that $f'' \geq 0$. Show that f is convex. Conclude that $f(x) = x^p$ is convex on $\mathbb{R}_{\geq 0}$, for all $p \geq 1$.

(2 marks) Prove Minkowski's inequality: for all $a, b \in \mathbb{R}^n$ and $p \geq 1$ we have

$$\left(\sum_{j=1}^n |a_j + b_j|^p\right)^{\frac{1}{p}} \leq \left(\sum_{j=1}^n |a_j|^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^n |b_j|^p\right)^{\frac{1}{p}}$$

(Hint: put $x = \frac{a_j}{\|a\|_p}$, $y = \frac{b_j}{\|b\|_p}$, and $s = \frac{\|a\|_p}{\|a\|_p + \|b\|_p}$.)

(1 mark) Show that $\mathbb{R}^n$ with the $\|-\|_p$-norm

$$\|x\|_p = \left(\sum_{j=1}^n |x_j|^p\right)^{\frac{1}{p}}$$

is a normed vextor space. Is it a Banach space?

Aufgabe 1.4 (Some Banach spaces)

(1 mark) Let X be a set and let $L_\infty(X, \mathbb{R})$ denote the vector space of all bounded real functions on X. Let $\|f\|_\infty = \sup\{|f(x)| \mid x \in X\}$. Show that $(L_\infty(X, \mathbb{R}), \|-\|_\infty)$ is a Banach space.

(3 marks) Let X be a topological space (or a metric space) and let $C_b(X, \mathbb{R})$ denote the vector space of all continuous bounded real functions on X. Show that $(C_b(X, \mathbb{R}), \|-\|_\infty)$ is a Banach space.