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Markov Renewal Theory for Stationary (m+1)-Block Factors: Convergence Rate Results*

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This article continues work in [4] on random walks $(S_n)_{n\geq 0}$ whose increments X_n are $(m+1)$-block factors of the form $\varphi(Y_{n-m},...,Y_n)$ for i.i.d. random variables $Y_{-m}, Y_{-m+1}, ...$ taking values in an arbitrary measurable space $(\mathcal{S}, \mathfrak{S})$. Defining $M_n = (Y_{n-m},...,Y_n)$ for $n \geq 0$, which is a Harris ergodic Markov chain, the sequence $(M_n, S_n)_{n\geq 0}$ constitutes a Markov random walk with stationary drift $\mu = E_{F^{m+1}} X_1$ where F denotes the distribution of the Y_n's. Suppose $\mu > 0$, let $(\sigma_n)_{n\geq 0}$ be the sequence of strictly ascending ladder epochs associated with $(M_n, S_n)_{n\geq 0}$ and let $(M_{\sigma_n}, S_{\sigma_n})_{n\geq 0}$, $(M_{\sigma_n}, \sigma_n)_{n\geq 0}$ be the resulting Markov renewal processes whose common driving chain is again positive Harris recurrent. The Markov renewal measures associated with $(M_n, S_n)_{n\geq 0}$ and the former two sequences are denoted $U_\lambda, U_\lambda^>$ and $V_\lambda^>$, respectively, where λ is an arbitrary initial distribution for (M_0, S_0). Given the basic sequence $(M_n, S_n)_{n\geq 0}$ is spread-out or 1-arithmetic with shift function 0, we provide convergence rate results for each of $U_\lambda, U_\lambda^>$ and $V_\lambda^>$ under natural moment conditions. Proofs are based on a suitable reduction to standard renewal theory by finding an appropriate imbedded regeneration scheme and coupling. Considerable work is further spent on necessary moment results.

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1. Introduction

Let $m \in I\!N$. A stochastic sequence $(X_n)_{n\geq 0}$ is called *m-dependent* if $X_0, ..., X_n$ and $X_{n+m+1}, X_{n+m+2}, ...$ are independent for all $n \in I\!N$. Our concern is a special class of such sequences, called *stationary* $(m+1)$*-block factors*, given by

$$X_n \;=\; \varphi(Y_{n-m}, ..., Y_n), \quad n \geq 0, \tag{1.1}$$

where $\varphi : \mathcal{S}^{m+1} \to I\!R$ is a measurable function and $Y_{-m}, Y_{-m+1}, ...$ are i.i.d. random variables on a probability space $(\Omega, \mathfrak{A}, P)$ taking values in a measurable space $(\mathcal{S}, \mathfrak{S})$. We denote by F the common distribution of the Y_n's and assume that $\mathfrak{S}$ is countably generated. Let $S_n = \sum_{k=0}^n X_k$, $n \geq 0$, be the random walk associated with $(X_n)_{n\geq 0}$ and suppose $\mu \stackrel{\rm def}{=} EX_1 > 0$. Many interesting properties of $(S_n)_{n\geq 0}$ including renewal theory were derived by Janson [11], [12]. A number of these results have been improved in [4] by analyzing $(S_n)_{n\geq 0}$ within the framework of Markov renewal theory. For this purpose observe that

$$M_n \;\stackrel{\rm def}{=}\; (Y_{n-m}, ..., Y_n), \quad n \geq 0, \tag{1.2}$$

constitutes a positive Harris chain with stationary distribution F^{m+1}, the $(m+1)$-fold product of F, and $(M_n, S_n)_{n\geq 0}$ a Markov random walk, respectively a Marov renewal process if all X_n's are positive. We call $(M_n, S_n)_{n\geq 0}$ hereafter a (φ, F)*-m-dependent Markov random walk*, abbreviated as (φ, F)*-mdMRW*. For the definition of its lattice-span d, a notoriously important characteristic in renewal theory, see [4], Section 3.

Let us briefly summarize some notation from [4] which is kept throughout unless stated otherwise. Suppose a canonical model with probability measures $P_{x,y}$, $(x, y) \in \mathcal{S}^{m+1} \times I\!R$, such that $P_{x,y}(M_0 = x, S_0 = y) = 1$. For every distribution λ on $\mathcal{S}^{m+1} \times I\!R$ put $P_\lambda \stackrel{\rm def}{=} \int_{\mathcal{S}^{m+1}\times I\!R} P_{x,y}\,\lambda(dx, dy)$. If λ is a distribution on $\mathcal{S}^{m+1}$ only then $P_\lambda \stackrel{\rm def}{=} P_{\lambda\otimes\delta_0}$. In the stationary case $\lambda = F^{m+1}$ we simply write P instead of $P_{F^{m+1}}$. As usual, the corresponding expectation operators are denoted by $E_{x,y}, E_\lambda$ and E. Let $\lambda\!\!\lambda_0$ be Lebesgue measure on $I\!R$ and $\lambda\!\!\lambda_1$ counting measure on $\mathbb{Z}$. Finally, given a measure ζ on $\mathcal{S}^{m+1} \times I\!R$ [resp. $I\!R$], put $\zeta^+ \stackrel{\rm def}{=} \zeta(\cdot \cap \mathcal{S}^{m+1} \times (0, \infty))$ [resp. $\stackrel{\rm def}{=} \zeta(\cdot \cap (0, \infty))$].

The strictly ascending ladder epochs of $(S_n)_{n\geq 0}$ are given by $\sigma_0 = 0$ and

$$\sigma_n \;\stackrel{\rm def}{=}\; \inf\{k > \sigma_{n-1} : S_k > S_{\sigma_{n-1}}\}$$

for $n \geq 1$. Put $M_n^> \stackrel{\rm def}{=} M_{\sigma_n}$ and $S_n^> \stackrel{\rm def}{=} S_{\sigma_n}$. As pointed out in [4], $(M_n^>, \sigma_n)_{n\geq 0}$ and $(M_n^>, S_n^>)_{n\geq 0}$ are both MRP's, the first 1-arithmetic with shift function 0, the latter with the same lattice-span (and shift function) as $(M_n, S_n)_{n\geq 0}$. The driving chain $(M_n^>)_{n\geq 0}$ is also positive Harris recurrent with a unique stationary distribution ξ^*. Moreover, $\mu \in (0, \infty)$ implies $\mu^> \stackrel{\rm def}{=} E_{\xi^*} S_{\sigma_1} = \mu E_{\xi^*}\sigma_1 < \infty$. These conclusions do indeed follow from a more general result in [3]. The Markov renewal measures associated with $(M_n, S_n)_{n\geq 0}$, $(M_n^>, S_n^>)_{n\geq 0}$ and $(M_n^>, \sigma_n)_{n\geq 0}$ under P_λ are denoted by $U_\lambda, U_\lambda^>$ and $V_\lambda^>$, respectively, that is

$$U_\lambda \;\stackrel{\rm def}{=}\; \sum_{n\geq 0} P_\lambda^{(M_n, S_n)}, \quad U_\lambda^> \;\stackrel{\rm def}{=}\; \sum_{n\geq 0} P_\lambda^{(M_n^>, S_n^>)} \quad \text{and} \quad V_\lambda^> \;\stackrel{\rm def}{=}\; \sum_{n\geq 0} P_\lambda^{(M_n^>, \sigma_n)}. \tag{1.3}$$

Defining the stationary Markov delay distribution of $(M_n,S_n)_{n\ge0}$ and $(M_n^>,S_n^>)_{n\ge0}$

$$\nu^s(A\times B) \;=\; \frac{1}{\mu^>}\int_B P_{\xi^*}(M_1^>\in A, S_1^>\ge s)\;\lambda\!\!\lambda_d^+(ds),\quad A\in\mathfrak{S}^{m+1}, B\in\mathfrak{B},\tag{1.4}$$

one has

$$U_{\nu^s}^+ \;=\; \mu^{-1}F^{m+1}\otimes\lambda\!\!\lambda_d^+ \quad\text{and}\quad U_{\nu^s}^> \;=\; (\mu^>)^{-1}\xi^*\otimes\lambda\!\!\lambda_d^+.\tag{1.5}$$

ν^s is also the unique stationary distribution of the continuous-time Markov process of forward recurrence times $(M_{\tau(t)},S_{\tau(t)}-t)_{t\ge0}$ where $\tau(t)\overset{\text{def}}{=}\inf\{n\ge0: S_n>t\}$. Correspondingly, the stationary Markov delay distribution of $(M_n^>,\sigma_n)_{n\ge0}$ is

$$\phi^s(A\times\{k\}) \;\overset{\text{def}}{=}\; \vartheta^{-1}P_{\xi^*}(M_1^>\in A,\sigma_1\ge k),\quad A\in\mathfrak{S}^{m+1}, k\in I\!N,\tag{1.6}$$

$\vartheta\overset{\text{def}}{=}E_{\xi^*}\sigma_1$, and satisfies

$$V_*^> \;=\; \vartheta^{-1}\xi^*\otimes\lambda\!\!\lambda_1^+,\tag{1.7}$$

where $V_*^>(A\times\{n\})\overset{\text{def}}{=}\int_{\mathcal{S}^{m+1}}V_x^>(A\times\{n-k\})\,\phi^s(dx,dk)$.

Markov renewal theorems for each of $U_\lambda, U_\lambda^>$ and $V_\lambda^>$ as well as a number of interesting consequences for various relevant quantities associated with $(M_n,S_n)_{n\ge0}$ and the other sequences introduced above are provided in [4]. The present paper continues the work by dealing with convergence rate results in the Markov renewal theorem. Polynomial as well as exponential rates under suitable moment conditions are established. Results of this type are already hard to derive for ordinary random walks, see e.g. [14], but are even harder to obtain for Markov random walks, at least when the driving chain has continuous state space as in the situation considered here, see however [2] for another special case and [9] for some recent progress in a more general setting based upon an analytic approach. In contrast to [9], our methods are purely probabilistic using regeneration and coupling. Although the class of (φ,F)-mdMRW is a very special one within the general class of Markov random walks with Harris recurrent driving chain, let us point out that each such general process contains a subsequence of the former type when sampling at a sequence of regeneration epochs. This fact in combination with the results of this article may eventually lead to corresponding rate results in the general setting. A major remaining obstacle is to convert suitable moment conditions on certain occupation measures arising from such an approach into verifiable moment conditions on the increments of the given Markov random walk itself. One can even say that this is the main problem whenever trying to prove rate results in renewal theory by regenerative arguments. We refer to a future publication.

Let us also point to some weakly related work on stochastic recursive sequences and the renovation method introduced by Borovkov for proving stability theorems in queueing, see [5], and also [6], [7]. The connection is roughly described by the fact that the considered renovative processes have an $(m+1)$-block structure on certain recurrent events which provides a regeneration scheme for these processes. Finally, we mention a recent article by Csenki [8] where some renewal theoretic results are proved for certain (φ,F)-mdMRW without utilizing the Markov renewal structure.

Our results are stated in Section 2 followed by the construction of a regeneration scheme (Section 3) that furnishes the use of known rate results for ordinary renewal measures and a further coupling which must be employed to prove the results for $U_\lambda^>$ and $V_\lambda^>$. Section 4 provides necessary moment results. The proofs of the main results can be found in Sections 5 and 6. Finally, a few facts from classical renewal theory are collected in a short Appendix.

2. Results

Let us further define for $\alpha > 0$

$$\begin{aligned} C_\lambda(\alpha) &\stackrel{\text{def}}{=} \sup_{n\ge0} E_\lambda|X_n|^\alpha &= \max_{0\le n\le m+1} E_\lambda|X_n|^\alpha, \\ C_\lambda^\pm(\alpha) &\stackrel{\text{def}}{=} \sup_{n\ge0} E_\lambda(X_n^\pm)^\alpha &= \max_{0\le n\le m+1} E_\lambda(X_n^\pm)^\alpha, \\ M_\lambda(\alpha) &\stackrel{\text{def}}{=} \sup_{n\ge0} E_\lambda e^{\alpha|X_n|} &= \max_{0\le n\le m+1} E_\lambda e^{\alpha|X_n|}, \\ M_\lambda^\pm(\alpha) &\stackrel{\text{def}}{=} \sup_{n\ge0} E_\lambda e^{\alpha X_n^\pm} &= \max_{0\le n\le m+1} E_\lambda e^{\alpha X_n^\pm}. \end{aligned}$$

In analogy to ordinary renewal theory, our convergence rate results below are given for (φ, F)-mdMRW's $(M_n, S_n)_{n\ge0}$ which are either 1-arithmetic with shift function 0 or *spread-out*. The latter means that there is an F^{m+1}-positive set $\mathbb{C}$ such that for each $x \in \mathbb{C}$ there exists $n(x) \in \mathbb{N}$ such that $P_x((M_{n(x)}, S_{n(x)}) \in \cdot) = \boldsymbol{P}^{*(n(x))}(x, \cdot)$ has an absolutely continuous component with respect to $F^{m+1} \otimes \lambda\!\!\!\lambda_0$. Here $\boldsymbol{P}$ denotes the transition kernel of $(M_n, X_n)_{n\ge0}$ and $\boldsymbol{P}^{*(n)}$ its n-fold convolution. We also call $\boldsymbol{P}$ spread-out under the previous condition. Note that F^{m+1} is the unique invariant distribution and thus a maximal irreducibility measure for the Harris chain $(M_n)_{n\ge0}$. One can easily show that if $(M_n, S_n)_{n\ge0}$ is spread-out the same holds true for the ladder height subsequence $(M_n^>, S_n^>)_{n\ge0}$. As in [4] we make the following

> Standing assumption: Whenever in the 1-arithmetic case, initial distributions λ are such that $P_\lambda(X_n \in \mathbb{Z}) = 1$ for all $n \ge 1$.

In order to state our results more efficiently, let $\mathcal{H}^\alpha$ be the space of functions $g : [0,\infty) \to \mathbb{R}$ satisfying $\int_0^\infty t^{\alpha-1} g(t)\,dt < \infty$ and $\lim_{t\to\infty} t^\alpha g(t) = 0$, where $\alpha \ge 1$. Let further $\mathcal{E}$ be the space of functions $g : [0,\infty) \to \mathbb{R}$ satisfying $\int_0^\infty e^{\theta t} g(t)\,dt < \infty$ and $\lim_{t\to\infty} e^{\theta t} g(t) = 0$ for some $\theta > 0$. If V and W denote arbitrary signed measures on $\mathcal{S}^{m+1} \times \mathbb{R}$ and $\mathbb{R}$, respectively, then put

$$V_{|B} \stackrel{\text{def}}{=} V(\cdot \cap (\mathcal{S}^{m+1} \times B)) \quad \text{and} \quad W_{|B} \stackrel{\text{def}}{=} W(\cdot \cap B)$$

for measurable subsets B of $\mathbb{R}$.

Theorem 2.1. *Let $(M_n, S_n)_{n\ge0}$ be a (φ, F)-mdMRW which is either 1-arithmetic with shift function 0 ($d = 1$) or spread-out ($d = 0$). Let further $\mu \in (0,\infty)$, $\alpha \ge 1$ and λ, λ' be distributions on $\mathcal{S}^{m+1} \times \mathbb{R}$.*

(a) If $C_\nu^+(\alpha) < \infty$ for $\nu \in \{\lambda, \lambda'\}$, then $\|(U_\lambda - U_{\lambda'})_{|t+I}\| \in \mathcal{H}^\alpha$ for every finite interval I.

(b) *If* $C^+_\lambda(\alpha) < \infty$ *and* $E(X^+_1)^{\alpha+1} < \infty$, *then* $\|(U_\lambda - \mu^{-1}F^{m+1} \otimes \lambda\!\!\lambda_d)_{|t+I}\| \in \mathcal{H}^\alpha$ *for every finite interval* I.

(c) *If* $C^+_\nu(\alpha+1) < \infty$ *for* $\nu \in \{\lambda, \lambda'\}$, *then* $\|(U_\lambda - U_{\lambda'})_{|[t,\infty)}\| \in \mathcal{H}^\alpha$.

(d) *If* $C^+_\lambda(\alpha+1) < \infty$ *and* $E(X^+_1)^{\alpha+2} < \infty$, *then* $\|(U_\lambda - \mu^{-1}F^{m+1} \otimes \lambda\!\!\lambda_d)_{|[t,\infty)}\| \in \mathcal{H}^\alpha$.

(e) *If* $C^+_\lambda(1) < \infty$ *and* $E(X^+_1)^2 < \infty$, *then*

$$\|U^+_\lambda - \mu^{-1}F^{m+1} \otimes \lambda\!\!\lambda^+_d\| \ < \ \infty. \tag{2.1}$$

The next theorem covers the case when t tends to $-\infty$.

THEOREM 2.2. *Let* $(M_n, S_n)_{n\ge 0}$ *as well as* μ, α *and* λ *be as in Theorem 2.1.*

(a) *If* $C^-_\lambda(\alpha) < \infty$ *and* $E(X^-_1)^{\alpha+1} < \infty$, *then* $U_{\lambda|-t+I} \in \mathcal{H}^\alpha$ *for every finite interval* I.

(b) *If* $C^-_\lambda(\alpha+1) < \infty$ *and* $E(X^-_1)^{\alpha+2} < \infty$, *then* $U_{\lambda|(-\infty,-t]} \in \mathcal{H}^\alpha$.

(c) *If* $C^-_\lambda(1) < \infty$ *and* $E(X^-_1)^2 < \infty$, *then*

$$\|U^-_\lambda\| \ = \ U_\lambda(\mathcal{S}^{m+1} \times (-\infty, 0]) \ < \ \infty. \tag{2.2}$$

Turning to exponential rates, we will prove

THEOREM 2.3. *Let* $(M_n, S_n)_{n\ge 0}$ *as well as* μ *and* λ, λ' *be as in Theorem 2.1.*

(a) *If* $M^+_\nu(\alpha) < \infty$ *for* $\nu \in \{\lambda, \lambda'\}$ *and some* $\alpha > 0$, *then* $\|(U_\lambda - U_{\lambda'})_{|[t,\infty)}\| \in \mathcal{E}$.

(b) *If* $M^+_\lambda(\alpha) < \infty$ *for some* $\alpha > 0$, *then* $\|(U_\lambda - \mu^{-1}F^{m+1} \otimes \lambda\!\!\lambda_d)_{|[t,\infty)}\| \in \mathcal{E}$.

(c) *If* $M^-_\lambda(\alpha) < \infty$ *for some* $\alpha > 0$, *then* $U_\lambda(\mathcal{S}^{m+1} \times (-\infty, -t]) \in \mathcal{E}$.

The counterpart of Theorem 2.1 for $(M^>_n, S^>_n)_{n\ge 0}$ is stated next.

THEOREM 2.4. *Let the situation of Theorem 2.1 be given and* $\alpha \ge 1$.

(a) *If* $C_\nu(\alpha) < \infty$ *for* $\nu \in \{\lambda, \lambda'\}$, *then* $\|(U^>_\lambda - U^>_{\lambda'})_{|t+I}\| \in \mathcal{H}^\alpha$ *for every finite interval* I.

(b) *If* $C_\lambda(\alpha) < \infty$ *and* $E(X^+_1)^{\alpha+1} < \infty$, *then* $\|(U^>_\lambda - (\mu^>)^{-1}\xi^* \otimes \lambda\!\!\lambda_d)_{|t+I}\| \in \mathcal{H}^\alpha$ *for every finite interval* I.

(c) *If* $C_\nu(\alpha+1) < \infty$ *for* $\nu \in \{\lambda, \lambda'\}$, *then* $\|(U^>_\lambda - U^>_{\lambda'})_{|[t,\infty)}\| \in \mathcal{H}^\alpha$.

(d) *If* $C_\lambda(\alpha+1) < \infty$ *and* $E(X^+_1)^{\alpha+2} < \infty$, *then* $\|(U^>_\lambda - (\mu^>)^{-1}\xi^* \otimes \lambda\!\!\lambda_d)_{|[t,\infty)}\| \in \mathcal{H}^\alpha$.

(e) *If* $C_\lambda(1) < \infty$ *and* $E(X^+_1)^2 < \infty$, *then*

$$\|U^>_\lambda - (\mu^>)^{-1}\xi^* \otimes \lambda\!\!\lambda^+_d\| \ < \ \infty. \tag{2.3}$$

The *two-sided* moment assumptions in Theorem 2.4 may be surprising because, in view of corresponding results in classic renewal theory, $E_\lambda(S^>_1)^\beta < \infty$ for suitable $\beta > 0$ seems to be the type of required condition which in turn follows from $C^+_\lambda(\beta) < \infty$, as can be easily verified with the help of Theorem 2.3 in [11]. The reason is that our method of proof uses a coupling construction which draws on the regeneration lemmata for the special class of (φ, F)-mdMRW given in Section 3. But since the ladder height process $(M^>_n, S^>_n)_{n\ge 0}$ is not of this type in general (see e.g. [11], Example 3.1), the construction must be for the original Markov random walk $(M_n,S_n)_{n\ge0}$ and may thus lack the optimal coupling rate. Roughly speaking, when a coupling of two versions of the original process occurs it generally takes an extra amount of time ψ, say, until the imbedded ladder height processes couple. We refer to the beginning of Section 6 for a more detailed explanation. The behavior of ψ, however, is tied to the degree of negative excursions of the two original processes before they couple. As a consequence, the existence of a moment of order $\beta>0$ for ψ is controlled by a moment condition of type $C_\lambda^-(\beta)<\infty$, see Proposition 6.3 and its proof. For the same reason, two-sided moment assumptions occur in the next theorem which is the counterpart of Theorem 2.3.

Theorem 2.5. *Let the situation of Theorem 2.1 be given.*

(a) If $M_\nu(\alpha)<\infty$ for $\nu\in\{\lambda,\lambda'\}$ and some $\alpha>0$, then $\|(U_\lambda^> - U_{\lambda'}^>)_{|[t,\infty)}\|\in\mathcal{E}$.

(b) If $M_\lambda(\alpha)<\infty$ for some $\alpha>0$, then $\|(U_\lambda^> - (\mu^>)^{-1}\xi^*\otimes\lambda\!\!\lambda_d)_{|[t,\infty)}\|\in\mathcal{E}$.

Our final convergence rate results deal with $V_\lambda^>$, the Markov renewal measure associated with the ladder epoch sequence $(M_n^>,\sigma_n)_{n\ge0}$. which is always 1-arithmetic with shift function 0 (in fact regardless of the lattice-type of $(M_n,S_n)_{n\ge0}$, Theorem 2.1 in [4]).

Theorem 2.6. *Let the situation of Theorem 2.1 be given and $\alpha\ge1$.*

(a) If $C_\nu^-(\alpha)<\infty$ for $\nu\in\{\lambda,\lambda'\}$, then $\|(V_\lambda^> - V_{\lambda'}^>)_{|t+I}\|\in\mathcal{H}^\alpha$ for every finite interval I.

(b) If $C_\lambda^-(\alpha)<\infty$ and $E(X_1^-)^{\alpha+1}<\infty$, then $\|(V_\lambda^> - \vartheta^{-1}\xi^*\otimes\lambda\!\!\lambda_1)_{|t+I}\|\in\mathcal{H}^\alpha$ for every finite interval I.

(c) If $C_\nu^-(\alpha+1)<\infty$ for $\nu\in\{\lambda,\lambda'\}$, then $\|(V_\lambda^> - V_{\lambda'}^>)_{|[t,\infty)}\|\in\mathcal{H}^\alpha$.

(d) If $C_\lambda^-(\alpha+1)<\infty$ and $E(X_1^-)^{\alpha+2}<\infty$, then $\|(V_\lambda^> - \vartheta^{-1}\xi^*\otimes\lambda\!\!\lambda_1)_{|[t,\infty)}\|\in\mathcal{H}^\alpha$.

(e) If $C_\lambda^-(1)<\infty$ and $E(X_1^-)^2<\infty$, then

$$\|V_\lambda^> - \vartheta^{-1}\xi^*\otimes\lambda\!\!\lambda_1^+\| \ < \ \infty. \tag{2.4}$$

Theorem 2.7. *Let the situation of Theorem 2.1 be given.*

(a) If $M_\nu^-(\alpha)<\infty$ for $\nu\in\{\lambda,\lambda'\}$ and some $\alpha>0$, then $\|(V_\lambda^> - V_{\lambda'}^>)_{|[t,\infty)}\|\in\mathcal{E}$.

(b) If $M_\lambda^-(\alpha)<\infty$ for some $\alpha>0$, then $\|(V_\lambda^> - \vartheta^{-1}\xi^*\otimes\lambda\!\!\lambda_1)_{|[t,\infty)}\|\in\mathcal{E}$.

3. Regeneration

The key to the proof of our main results is the following regeneration lemma and its generalization (Lemma 3.2 below) which will enable us to re-construct the considered (φ,F)-mdMRW $(M_n,S_n)_{n\ge0}$ together with a sequence of regeneration epochs that divides it into independent cycles which are further stationary except for the first one. An assumption on existence or positivity of the stationary drift $\mu=EX_1$ is not needed and thus not imposed here. The type of regeneration established through the re-construction of $(M_n,S_n)_{n\ge0}$ is called wide-sense regeneration in the literature, see Thorisson's monograph [17] for details.

LEMMA 3.1. *Let $(M_n, S_n)_{n\geq 0}$ be a (φ, F)-mdMRW which is 1-arithmetic with shift function 0 or spread-out. Then there exist $n_0 \in I\!N$, F^{m+1}-positive sets $\mathbb{A}, \mathbb{B} \in \mathfrak{S}^{m+1}$ and $\beta > 0$ such that*

$$\boldsymbol{P}^{*(n_0)}(x, \cdot) \;\geq\; \beta F^{m+1}(\cdot|\mathbb{B}) \otimes \Gamma \tag{3.1}$$

for all $x \in \mathbb{A}$ where $\Gamma = \delta_L$ for some $L \in \mathbb{Z}$ in the 1-arithmetic case and $\Gamma = \lambda\!\!\lambda_0(\cdot|J)$ for some finite, $\lambda\!\!\lambda_0$-positive interval $J \subset I\!R$ in the spread-out case.

PROOF. In the spread-out case the assertion follows directly from a more general result by Niemi [15] and Niemi and Nummelin [16], see their Minorization Lemma and Remark 4.2. We therefore restrict ourselves to the 1-arithmetic case and prove the slightly stronger result

$$P_x(M_{2m+2}, X_1, ..., X_{2m+2}) \in \cdot) \;\geq\; \beta F^{m+1}(\cdot|\mathbb{B}) \otimes \delta_l \tag{3.2}$$

for all $x \in \mathbb{A}$ and some $l = (l_1, ..., l_{2m+2}) \in \mathbb{Z}^{2m+2}$, thus $n_0 = 2m + 2$ and $L = \sum_{i=1}^{2m+2} l_i$.

For $a = (a_1, ..., a_{m+1})$, $b = (b_1, ..., b_{m+1}) \in \mathcal{S}^{m+1}$ put

$$\Phi(a, b) \;\stackrel{\text{def}}{=}\; (\varphi(a_2, ..., a_{m+1}, b_1), ..., \varphi(a_{m+1}, b_1, ..., b_m), \varphi(b_1, ..., b_{m+1}))$$

and

$$\Lambda(a, b) \;\stackrel{\text{def}}{=}\; ((a_2, ..., a_{m+1}, b_1), ..., (a_{m+1}, b_1, ..., b_m), (b_1, ..., b_{m+1})).$$

Hence $\Phi(M_n, M_{n+m+1}) = (X_{n+1}, ..., X_{n+m+1})$ and $\Lambda(M_n, M_{n+m+1}) = (M_{n+1}, ..., M_{n+m+1})$, in particular $M_{n+1}, ..., M_{n+m}, X_{n+1}, ..., X_{n+m}$ are fully determined by M_n, M_{n+m+1}. Since $P(X_n \in \mathbb{Z}$ for all $n \geq 1) = 1$ there exists $l \in \mathbb{Z}^{2m+2}$ such that $P((X_1, ..., X_{2m+2}) = l) > 0$. Define

$$\begin{aligned}
C &\stackrel{\text{def}}{=} \{(x, y) \in \mathcal{S}^{2m+2} : \Phi(x, y) = (l_1, ..., l_{m+1})\}, \\
D &\stackrel{\text{def}}{=} \{(y, z) \in \mathcal{S}^{2m+2} : \Phi(y, z) = (l_{m+2}, ..., l_{2m+2})\}, \\
E &\stackrel{\text{def}}{=} \{(x, y, z) \in \mathcal{S}^{3m+3} : \Phi(x, y) = (l_1, ..., l_{m+1}), \Phi(y, z) = (l_{m+2}, ..., l_{2m+2})\}.
\end{aligned}$$

Clearly, $E = (C \times \mathcal{S}^{m+1}) \cap (\mathcal{S}^{m+1} \times D)$, $F^{2m+2}(C) > 0, F^{2m+2}(D) > 0$ and $F^{3m+3}(E) > 0$.

Since $\mathfrak{S}^{m+1}$ is countably generated, there is an incrasing sequence of finite σ-fields $(\mathfrak{S}_n)_{n\geq 1}$ such that $\mathfrak{S}^{m+1} = \sigma(\cup_{n\geq 1}\mathfrak{S}_n)$ and each $\mathfrak{S}_n$ is generated by a finite partition of $\mathcal{S}^{m+1}$. For $x \in \mathcal{S}^{m+1}$ and $n \in I\!N$ denote by G_x^n the unique set containing x of the partition generating $\mathfrak{S}_n$. Put $G_{x,y}^n = G_x^n \times G_y^n$. From the Differentiation Theorem for measures we infer the existence of F^{2m+2}-null sets $N_1, N_2 \in \mathfrak{S}^{2m+2}$ such that

$$\lim_{n\to\infty} \frac{F^{2m+2}(C \cap G_{x,y}^n)}{F^{2m+2}(G_{x,y}^n)} \;=\; 1 \quad \text{for all } (x, y) \in C - N_1$$

and

$$\lim_{n\to\infty} \frac{F^{2m+2}(D \cap G_{y,z}^n)}{F^{2m+2}(G_{y,z}^n)} \;=\; 1 \quad \text{for all } (y, z) \in D - N_2.$$

Fix a triplet $(u, v, w) \in (C - N_1 \times \mathcal{S}^{m+1}) \cap (\mathcal{S}^{m+1} \times D - N_2)$ and an integer j such that

$$\frac{F^{2m+2}(C \cap G_{u,v}^j)}{F^{2m+2}(G_{u,v}^j)} \;\geq\; \frac{3}{4} \quad \text{and} \quad \frac{F^{2m+2}(D \cap G_{v,w}^j)}{F^{2m+2}(G_{v,w}^j)} \;\geq\; \frac{3}{4}. \tag{3.3}$$

Now put

$$\mathbb{A} \stackrel{\text{def}}{=} \{x \in G_u^j : F^{m+1}(G_v^j \cap \{y \in \mathcal{S}^{m+1} : (x,y) \in C\}) \geq (3/4)F^{m+1}(G_v^j)\},$$
$$\mathbb{B} \stackrel{\text{def}}{=} \{z \in G_w^j : F^{m+1}(G_v^j \cap \{y \in \mathcal{S}^{m+1} : (y,z) \in D\}) \geq (3/4)F^{m+1}(G_v^j)\},$$

which are both elements of $\mathfrak{S}^{m+1}$. Use (3.3) and Fubini's theorem to obtain

$$\begin{aligned} F^{2m+2}(C \cap G_{u,v}^j) &= \int_{G_u^j} F^{m+1}(G_v^j \cap \{y \in \mathcal{S}^{m+1} : (x,y) \in C\}) \; F^{m+1}(dx) \\ &\geq \frac{3}{4} F^{2m+2}(G_{u,v}^j) \; > \; 0 \end{aligned}$$

and analogously

$$\begin{aligned} F^{2m+2}(D \cap G_{v,w}^j) &= \int_{G_w^j} F^{m+1}(G_v^j \cap \{y \in \mathcal{S}^{m+1} : (y,z) \in D\}) \; F^{m+1}(dz) \\ &\geq \frac{3}{4} F^{2m+2}(G_{v,w}^j) \; > \; 0 \end{aligned}$$

and thereby $F^{m+1}(\mathbb{A}) > 0$ and $F^{m+1}(\mathbb{B}) > 0$. For $E_{x,z} \stackrel{\text{def}}{=} \{y \in \mathcal{S}^{m+1} : (x,y,z) \in E\}$, we finally conclude for all $(x,z) \in \mathbb{A} \times \mathbb{B}$

$$\begin{aligned} F^{m+1}(E_{x,z}) &\geq F^{m+1}(G_v^j \cap \{y \in \mathcal{S}^{m+1} : (x,y) \in C\} \cap \{y \in \mathcal{S}^{m+1} : (y,z) \in D\}) \\ &= F^{m+1}(G_v^j \cap \{y : (x,y) \in C\}) + F^{m+1}(G_v^j \cap \{y : (y,z) \in D\}) - F^{m+1}(G_v^j) \\ &\geq F^{m+1}(G_v^j)/2 \; > \; 0, \end{aligned}$$

thus proving (3.2) via

$$\begin{aligned} P_x((M_{2m+2} \in A, (X_1, ..., X_{2m+2}) = l) &= \int_{A \cap \mathbb{B}} F^{m+1}(E_{x,z}) \; F^{m+1}(dz) \\ &\geq \frac{1}{2} F^{m+1}(\mathbb{B}) F^{m+1}(G_v^j) F^{m+1}(A|\mathbb{B}) \end{aligned}$$

for all $x \in \mathbb{A}$ and $A \in \mathfrak{S}^{m+1}$ (thus $\beta = F^{m+1}(\mathbb{B})F^{m+1}(G_v^j)/2$). ♢

For $c \geq 0$, we define the reduced (substochastic) kernel

$$\boldsymbol{P}_c(x, \cdot) \stackrel{\text{def}}{=} P_x((M_1, X_1) \in \cdot, |X_1| \leq c)$$

and note that

$$\boldsymbol{P}_c^{*(n)}(x, \cdot) \; = \; P_x((M_n, S_n) \in \cdot, |X_1| \leq c, ..., |X_n| \leq c)$$

for each $n \geq 0$. As a trivial consequence of (3.2), we have in the arithmetic case that with $n_0 = 2m+2$, $t_0 = \max(l_1, ..., l_{n_0})$ and $\Gamma = \delta_L$

$$\boldsymbol{P}_{t_0}^{*(n_0)}(x, dy, ds) \; \geq \; \beta F^{m+1}(dy|\mathbb{B}) \otimes \Gamma(ds) \tag{3.4}$$

for all $x \in \mathbb{A}$. For we need this be true also in the spread-out case given some t_0 sufficiently large, we next state the following generalization of Lemma 3.1:

LEMMA 3.2. *Let $(M_n,S_n)_{n\ge0}$ be a (φ,F)-mdMRW which is 1-arithmetic with shift function 0 or spread-out. Then there exist $n_0\in I\!N$, F^{m+1}-positive sets $\mathbb{A},\mathbb{B}\in\mathfrak{S}^{m+1}$, $\beta>0$ (in general different from those in Lemma 3.1) and $t_0>0$ such that (3.4) holds true for all $x\in\mathbb{A}$ where $\Gamma=\delta_L$ for some $L\in\mathbb{Z}$ in the 1-arithmetic case and $\Gamma=\lambda\!\!\lambda_0(\cdot|J)$ for some finite, $\lambda\!\!\lambda_0$-positive interval $J\subset I\!R$ in the spread-out case.*

PROOF. From the above we must only consider the spread-out case. But here the result follows again from Niemi [15] and Niemi and Nummelin [16] if we observe that, for sufficiently large t_0, the reduced kernel $\boldsymbol{P}_{t_0}$ is again spread-out and has the same irreducibility properties as $\boldsymbol{P}$ itself. Further details can thus be omitted. ◊

Observe that, upon setting $I_n\stackrel{\rm def}{=}\mathbf{1}_{\{|X_n|\le t_0\}}$ for $n\ge1$, (3.4) may be rewritten as

$$P_x((M_{n_0},S_{n_0},I_1,...,I_{n_0})\in\cdot)\ \ge\ \beta F^{m+1}(\cdot|\mathbb{B})\otimes\Gamma\otimes\delta_{(1,...,1)}\tag{3.4$'$}$$

for all $x\in\mathbb{A}$. Lemma 3.2 is now used for the re-construction of $(M_n,X_n)_{n\ge0}$ as follows: Let us stipulate without further notice that all occurring variables indexed by -1 are defined as 0. Let $(\eta_n)_{n\ge0}$ and $(\chi_n)_{n\ge0}$ be sequences of i.i.d. Bernoulli variables with parameter β ($\sim B(1,\beta)$), respectively i.i.d. geometric variables with parameter $\frac12$, each independent of all other occurring variables. Put $m_0\stackrel{\rm def}{=}n_0+m+1$ and

$$\upsilon_0\ \stackrel{\rm def}{=}\ \inf\{n\in\chi_0+m_0I\!N:M_{n-n_0}\in\mathbb{A}\}.$$

Hence $\kappa_0\stackrel{\rm def}{=}m_0^{-1}(\upsilon_0-\chi_0)-1$ has a geometric distribution with parameter $F^{m+1}(\mathbb{A})$ under every P_λ. Keep the segment $(M_k,X_k)_{0\le k\le\upsilon_0-n_0}$ unchanged. Re-generate $(M_{\upsilon_0},S_{\upsilon_0}-S_{\upsilon_0-n_0},$ $I_{\upsilon_0-n_0+1},...,I_{\upsilon_0})$ according to $F^{m+1}(\cdot|\mathbb{B})\otimes\Gamma\otimes\delta_{(1,...,1)}$, if $\eta_{\upsilon_0}=1$, and such that the overall distribution of that vector given $M_{\upsilon_0-n_0}$ remains the original one, otherwise. Finish this block by re-constructing $(M_k,X_k)_{\upsilon_0-n_0<k\le\upsilon_0}$ according to the prescribed conditional distribution under $(M_{\upsilon_0-n_0},M_{\upsilon_0},S_{\upsilon_0}-S_{\upsilon_0-n_0},I_{\upsilon_0-n_0+1},...,I_{\upsilon_0})$.

The next blocks are constructed similarly with υ_k, $k\ge1$, defined through

$$\upsilon_k\ \stackrel{\rm def}{=}\ \inf\{n\in\upsilon_{k-1}+\chi_k+m_0I\!N:M_{n-n_0}\in\mathbb{A}\}.$$

A regeneration occurs each time when $\eta_{\upsilon_k}=1$, more precisely at

$$T_k\ =\ \inf\{\upsilon_n>T_{k-1}:\eta_{\upsilon_n}=1\}$$

for $k\ge0$. The following assertions are valid *under every* P_λ and readily seen from the construction and given assumptions:

(R.1) The random vectors $(T_n-T_{n-1},M_{T_n},S_{T_n}-S_{T_{n-1}})$ are independent for $n\ge0$ and identically distributed for $n\ge1$ with the same distribution as (T_0,M_{T_0},S_{T_0}) under $P_{F^{m+1}(\cdot|\mathbb{B})}$. Moreover, M_{T_n} and S_{T_n} are independent for each $n\ge0$.

(R.2) $(S_{T_n})_{n\ge0}$ constitutes an ordinary delayed 1-arithmetic, respectively absolutely continuous random walk. In the arithmetic case the lattice-span assertion follows along similar lines as Lemma 3.3 in [1]. It is this property which makes use of the geometric variables χ_n.

(R.3) $(M_k, X_k)_{0\le k\le T_n-n_0}$, $S_{T_n}-S_{T_n-n_0}$ and $(M_{T_n+k}, X_{T_n+k+1})_{k\ge 0}$ are independent for every $n\ge 0$, the last sequence being distributed as $(M_k, X_{k+1})_{k\ge 0}$ under $P_{F^{m+1}(\cdot|\mathbb{B})}$.

(R.4) $\max_{1\le k\le n_0}|X_{T_n-n_0+k}|\le t_0$ for each $n\ge 0$.

(R.5) $\kappa_n \stackrel{\text{def}}{=} m_0^{-1}(\upsilon_n-\upsilon_{n-1}-\chi_n)-1$, $n\ge 0$, are i.i.d. geometric variables with parameter $F^{m+1}(\mathbb{A})$. They are further independent of $(\chi_n)_{n\ge 0}$.

(R.6) $T_0=\upsilon_\varrho$ where $\varrho=\inf\{n\ge 0:\eta_{\upsilon_n}=1\}$. Moreover, ϱ has a geometric distribution with parameter β and is independent of $(M_n, X_n, \chi_n, \upsilon_n)_{n\ge 0}$.

(R.5) and (R.6) show that T_0 is essentially a geometric sum of independent geometric variables. We determine its generating function in Lemma 3.5 at the end of the section. For the last assertion in (R.6), note that with $(\eta_n)_{n\ge 0}$ the subsequence $(\eta_{\upsilon_n})_{n\ge 0}$ is still independent of all other occurring random variables.

With the help of the previous construction we get the following key identity for the Markov renewal measure U_λ. Given a set $D\in\mathfrak{S}^{m+1}\otimes\mathfrak{B}$, $x\in\mathcal{S}^{m+1}$ and $z\in I\!R$, let $D_x\in\mathfrak{B}$ be the x-projection of D, i.e. $D_x=\{y\in I\!R:(x,y)\in D\}$, and $D-z\stackrel{\text{def}}{=}\{(v,w-z):(v,w)\in D\}$.

Lemma 3.3. *For all initial distributions λ on $\mathcal{S}^{m+1}\times I\!R$ and $D\in\mathfrak{S}^{m+1}\otimes\mathfrak{B}$*

$$\begin{aligned} U_\lambda(D) &= U_\lambda^{T_0}(D) + \int_{I\!R} U^{T_0}_{F^{m+1}(\cdot|\mathbb{B})}(D-y)\ \mathbb{U}_\lambda(dy), \\ &= U_\lambda^{T_0}(D) + \int_{\mathcal{S}^{m+1}\times I\!R} \mathbb{U}_\lambda(D_x-y)\ U^{T_0}_{F^{m+1}(\cdot|\mathbb{B})}(dx,dy) \end{aligned} \tag{3.5}$$

where $\mathbb{U}_\lambda=\sum_{n\ge 0}P_\lambda(S_{T_n}\in\cdot)$ equals the renewal measure of $(S_{T_n})_{n\ge 0}$ under P_λ and

$$U_\lambda^{T_0}(D) \stackrel{\text{def}}{=} E_\lambda\left(\sum_{n=0}^{T_0-1}\mathbf{1}_D(M_n,S_n)\right).$$

Proof. Using the strong Markov property, the independence of M_{T_n} and S_{T_n} and $M_{T_n}\sim F^{m+1}(\cdot|\mathbb{B})$ for all $n\ge 0$, we obtain under every P_λ

$$\begin{aligned} U_\lambda(D) &= U_\lambda^{T_0}(D) + \sum_{n\ge 0}E_\lambda\left(\sum_{k=T_n}^{T_{n+1}-1}\mathbf{1}_D(M_k,S_k)\right) \\ &= U_\lambda^{T_0}(D) + \sum_{n\ge 0}\int_{\mathcal{S}^{m+1}\times I\!R} U_x^{T_0}(D-y)\ P_\lambda^{(M_{T_n},S_{T_n})}(dx,dy) \\ &= U_\lambda^{T_0}(D) + \int_{I\!R} U^{T_0}_{F^{m+1}(\cdot|\mathbb{B})}(D-y)\ \mathbb{U}_\lambda(dy) \end{aligned}$$

that is the first identity of (3.5). If we write the final integral in previous line as

$$\int_{I\!R}\int_{\mathcal{S}^{m+1}\times I\!R}\mathbf{1}_D(x,y+z)\ U^{T_0}_{F^{m+1}(\cdot|\mathbb{B})}(dx,dz)\ \mathbb{U}_\lambda(dy)$$

and interchange the order of integration we also obtain the second equality in (3.5). ◊

Let $\mathbb{U}^*$ denote the renewal measure of $(S_{T_n})_{n\geq -1}$ under $P_{F^{m+1}(\cdot|\mathbb{B})}$, which is a zero-delayed random walk under that probability measure. With the help of (3.5) we get the following bound for $\sup_{t\in I\!R} U_\lambda(\mathcal{S}^{m+1}\times[t,t+a])$ independent of λ:

COROLLARY 3.4. *For all initial distributions λ on $\mathcal{S}^{m+1}\times I\!R$ and all $a>0$*

$$\sup_{t\in I\!R} U_\lambda(\mathcal{S}^{m+1}\times[t,t+a]) \;\leq\; ET_0\Big(1+\mathbb{U}^*[-a,a]\Big) \;<\; \infty. \tag{3.6}$$

PROOF. Clearly, $U_\lambda^{T_0}$ has total mass $E_\lambda T_0 = ET_0 < \infty$, finiteness and independence of λ following from (R.5) and (R.6), see Lemma 3.5 below. Moreover,

$$\sup_{t\in I\!R} \mathbb{U}_\lambda[t,t+a] \;\leq\; \mathbb{U}^*[-a,a]$$

is a well-known inequality from classical renewal theory. Combining these facts with (3.5) (second line) immediately gives the assertion. $\diamondsuit$

We close this section with an explicit computation of the generating function of T_0 showing in particular that T_0 has finite moments of exponential order. Let $g_\theta(s)=\frac{\theta}{1-(1-\theta)s}$ denote the generating function of a geometric distribution with parameter $\theta\in(0,1)$.

LEMMA 3.5. *The distribution of T_0 under P_λ is the same for every λ, its generating function given by*

$$Es^{T_0} \;=\; \frac{\beta}{1-(1-\beta)g_{1/2}(s)g_{F^{m+1}(\mathbb{A})}(s^{m_0})s^{m_0}} \tag{3.7}$$

and finite for all $s\in(0,s^*)$ for some $s^*>1$. Moreover,

$$ET_0 \;=\; E(\varrho+1)\,E(\chi_0+m_0\kappa_0+m_0) \;=\; \frac{1}{\beta}\left(2+\frac{m_0}{F^{m+1}(\mathbb{A})}\right). \tag{3.8}$$

PROOF. In view of (R.5) and (R.6) we have

$$T_0 \;=\; \upsilon_\varrho \;=\; \sum_{j=0}^{\varrho}(\chi_j+m_0\kappa_j+m_0)$$

with mutually independent geometric variables ϱ,χ_j,κ_j. This easily leads to the assertions of the lemma whence we omit further details. $\diamondsuit$

4. MOMENT RESULTS

Let $(M_n,S_n)_{n\geq 0}$ be any (φ,F)-mdMRW with finite, but not necessarily positive stationary drift $\mu=EX_1$. The following two propositions contain the moment results which are of essential importance when proving the main results in the next section.

PROPOSITION 4.1. *Let $\alpha>0$.*
(a) If $C_\lambda^\pm(\alpha)<\infty$ then $E_\lambda(S_{T_0}^\pm)^\alpha<\infty$.
(b) If $M_\lambda^\pm(\alpha)<\infty$ then $E_\lambda e^{\theta S_{T_0}^\pm}<\infty$ for some $\theta\in(0,\alpha]$.

Let the occupation measure $U_\lambda^{T_0}$ be defined as in Lemma 3.3.

PROPOSITION 4.2. *Let* $\alpha > 0$, $I_+ = (0,\infty)$ *and* $I_- = (-\infty, 0)$.
(a) If $C_\lambda^\pm(\alpha) < \infty$ *then* $\int_{\mathcal{S}^{m+1}\times I_\pm} |t|^\alpha\, U_\lambda^{T_0}(dx,dt) = E_\lambda(\sum_{n=0}^{T_0-1}(S_n^\pm)^\alpha) < \infty$.
(b) If $M_\lambda^\pm(\alpha) < \infty$ *then* $\int_{\mathcal{S}^{m+1}\times I_\pm} e^{\theta|t|}\, U_\lambda^{T_0}(dx,dt) = E_\lambda(\sum_{n=0}^{T_0-1} e^{\theta S_n^\pm}) < \infty$ *for some* $\theta \in (0,\alpha]$.

REMARK. Since $C_{F^{m+1}}^\pm(\alpha) = E(X_1^\pm)^\alpha \le C_\lambda^\pm(\alpha)$ for *every* λ, the conclusions of the previous propositions remain true when λ is replaced by F^{m+1} or any $\nu \le c\,F^{m+1}$ for some $c > 0$ (like $F^{m+1}(\cdot|\mathbb{B})$), the latter because $E_\nu(X_1^\pm)^\alpha \le c^{-1}E(X_1^\pm)^\alpha$.

The proofs are presented after some furnishing lemmata. We keep the notation of the previous section. Recall from the construction there that Γ is the distribution of $S_{\upsilon_0} - S_{\upsilon_0-n_0}$ given $\eta_{\upsilon_0} = 1$ under every P_λ. One can easily see that $n_0 \ge m+1$ in (3.1) which in turn implies

$$\inf_{(x,y)\in\mathbb{A}\times\mathbb{B}} P(S_{n_0} \in \cdot|M_0 = x, M_{n_0} = y) \;\ge\; \beta\Gamma(\cdot)/F^{m+1}(\mathbb{B})$$

Let $Y_1,...,Y_{n_0}$ and $Z_1,...,Z_{n_0}$ be generic random variables with

$$\mathcal{L}(Y_k) \;=\; P(X_{\upsilon_0-n_0+k} \in \cdot|\eta_{\upsilon_0} = 1) \quad\text{and}\quad \mathcal{L}(Z_k) \;=\; P(X_{\upsilon_0-n_0+k} \in \cdot|\eta_{\upsilon_0} = 0)$$

under each P_λ.

LEMMA 4.3. *There are finite constants* c_1, c_2 *such that* $Eg(Y_k) \le c_1 Eg(X_1)$ *and* $Eg(Z_k) \le c_2 Eg(X_1)$ *for all* $k = 1,...,n_0$ *and all measurable functions* $g : I\!R \to [0,\infty)$.

PROOF. Our argument is based on the simple fact that, given $\nu \le c\lambda$ for a finite constant c, $E_\lambda Z < \infty$ for any random variable $Z \ge 0$ implies $E_\nu Z < \infty$. Recall $I_n = \mathbf{1}_{\{|X_n|\le t_0\}}$, put $J_n \stackrel{\text{def}}{=} \prod_{k=0}^{n_0-1} I_{n-k}$ and then

$$\begin{aligned}\mathbb{K}(x,\cdot) &\stackrel{\text{def}}{=} P((M_{\upsilon_0}, S_{\upsilon_0} - S_{\upsilon_0-n_0}, J_{\upsilon_0}) \in \cdot|\eta_{\upsilon_0} = 0, M_{\upsilon_0-n_0} = x)\\ &= (1-\beta)^{-1}(P_x((M_{n_0}, S_{n_0}, J_{n_0}) \in \cdot) - \beta F^{m+1}(\cdot|\mathbb{B}) \otimes \Gamma \otimes \delta_1)\end{aligned}$$

for $x \in \mathbb{A}$. Obviously,

$$\begin{aligned}F^{m+1}(\cdot|D) &\le F^{m+1}(D)^{-1}F^{m+1} \quad \text{for all } D \in \mathfrak{S}^{m+1}, F^{m+1}(D) > 0,\\ F^{m+1}(\cdot|\mathbb{B}) \otimes \Gamma \otimes \delta_1 &\le \beta^{-1}P_x((M_{n_0}, S_{n_0}, J_{n_0}) \in \cdot) \quad \text{for all } x \in \mathbb{A},\\ \mathbb{K}(x,\cdot) &\le (1-\beta)^{-1}P_x((M_{n_0}, S_{n_0}, J_{n_0}) \in \cdot) \quad \text{for all } x \in \mathbb{A},\end{aligned}$$

where (3.4') should be recalled. Since furthermore

$$\begin{aligned}\mathcal{L}((M_{\upsilon_0-n_0}, M_{\upsilon_0}, S_{\upsilon_0} - S_{\upsilon_0-n_0}, J_{\upsilon_0})|\eta_{\upsilon_0} = 1) &= F^{m+1}(\cdot|\mathbb{A}) \otimes F^{m+1}(\cdot|\mathbb{B}) \otimes \Gamma \otimes \delta_1,\\ \mathcal{L}((M_{\upsilon_0-n_0}, M_{\upsilon_0}, S_{\upsilon_0} - S_{\upsilon_0-n_0}, J_{\upsilon_0})|\eta_{\upsilon_0} = 0) &= \mathbb{K}(x, dy, dz)\, F^{m+1}(dx|\mathbb{A}),\end{aligned}$$

we now infer

$$\begin{aligned} Eg(Y_k) &= \int_{\mathbb{A}}\int_{\mathbb{B}}\int_{I\!R}\int_{\{0,1\}} E_x(g(X_k)|M_{n_0}=y, S_{n_0}=z, J_{n_0}=j) \\ &\qquad\qquad \times \Gamma(dz)\ F^{m+1}(dy|\mathbb{B})\ F^{m+1}(dx|\mathbb{A})\ \delta_1(dj) \\ &\le \left(\beta F^{m+1}(\mathbb{A})\right)^{-1} Eg(X_1) \ < \ \infty \end{aligned}$$

and similarly

$$\begin{aligned} Eg(Z_k) &= \int_{\mathbb{A}}\int_{\mathcal{S}^{m+1}\times I\!R\times\{0,1\}} E_x(g(X_k)|M_{n_0}=y, S_{n_0}=z, J_{n_0}=j) \\ &\qquad\qquad \times \mathbb{K}(x,dy,dz,dj)\ F^{m+1}(dx|\mathbb{A}) \\ &\le \left((1-\beta)F^{m+1}(\mathbb{A})\right)^{-1} Eg(X_1) \ < \ \infty. \end{aligned}$$

◊

Lemma 4.4. *Let $\alpha>0$. Then $C_\lambda(\alpha)<\infty$ implies $E_\lambda|S_{\upsilon_0}|^\alpha<\infty$.*

Proof. We only consider the case $\alpha\ge 1$. The modifications of the subsequent inequalities if $\alpha\in(0,1)$ are obvious. Put $\hat\chi_0 \stackrel{\text{def}}{=} \chi_0+m+1$. We start by noting

$$E_\lambda|S_{\upsilon_0}|^\alpha \ = \ \beta\, E_\lambda(|S_{\upsilon_0}|^\alpha|\eta_{\upsilon_0}=1) \ + \ (1-\beta)\, E_\lambda(|S_{\upsilon_0}|^\alpha|\eta_{\upsilon_0}=0)$$

and

$$\begin{aligned} |S_{\upsilon_0}| &\le |S_{\hat\chi_0}| \ + \ |S_{\upsilon_0}-S_{\upsilon_0-n_0}| \ + \ \sum_{j=1}^{\kappa_0} |S_{\hat\chi_0+jm_0}-S_{\hat\chi_0+(j-1)m_0}| \\ &= |S_{\hat\chi_0}| \ + \ |S_{\upsilon_0}-S_{\upsilon_0-n_0}| \ + \ \sum_{j\ge 1} |S_{\hat\chi_0+jm_0}-S_{\hat\chi_0+(j-1)m_0}|\mathbf{1}_{\{\kappa\ge j\}} \end{aligned}$$

As can be seen from the construction in the previous section, only the middle term depends on η_{υ_0} and only the first term depends on the initial distribution λ, whence

$$\begin{aligned} E_\lambda(|S_{\upsilon_0}|^\alpha|\eta_{\upsilon_0})^{1/\alpha} &\le (E_\lambda|S_{\hat\chi_0}|^\alpha)^{1/\alpha} \ + \ E(|S_{\upsilon_0}-S_{\upsilon_0-n_0}|^\alpha|\eta_{\upsilon_0})^{1/\alpha} \\ &\quad + \ \sum_{j\ge 1}(E|S_{\hat\chi_0+jm_0}-S_{\hat\chi_0+(j-1)m_0}|^\alpha\mathbf{1}_{\{\kappa_0\ge j\}})^{1/\alpha}. \end{aligned}$$

Use the independence of $\hat\chi_0$ and $(S_n)_{n\ge 0}$ to obtain

$$E_\lambda|S_{\hat\chi_0}|^\alpha \ = \ \sum_{n\ge 0} 2^{-n} E_\lambda|S_{n+m+1}|^\alpha \ \le \ C_\lambda(\alpha)\sum_{n\ge 0}2^{-n}(n+m+1)^\alpha \ < \ \infty.$$

Moreover, with the help of Lemma 4.3

$$\begin{aligned} E_\lambda(|S_{\upsilon_0}-S_{\upsilon_0-n_0}|^\alpha|\varepsilon_{\upsilon_0}=1) &= E|Y_1+...+Y_{n_0}|^\alpha \ < \ \infty \\ E_\lambda(|S_{\upsilon_0}-S_{\upsilon_0-n_0}|^\alpha|\varepsilon_{\upsilon_0}=0) &= E|Z_1+...+Z_{n_0}|^\alpha \ < \ \infty. \end{aligned}$$

Finally, recalling the definition of κ_0 and the fact that it has a geometric distribution with parameter $F^{m+1}(\mathbb{A})$, we infer

$$\begin{aligned} E_\lambda|S_{\hat{\chi}_0+jm_0} - S_{\hat{\chi}_0+(j-1)m_0}|^\alpha \mathbf{1}_{\{\kappa_0\ge j\}} &= P(\kappa_0 \ge j)E(|S_{m_0}|^\alpha | M_0 \in \mathbb{A}^c, M_{m_0} \in \mathbb{A}^c) \\ &= F^{m+1}(\mathbb{A}^c)^j \int_{\mathbb{A}^c}\int_{\mathbb{A}^c} E_x(|S_{m_0}|^\alpha | M_{m_0} = y)\ F^{m+1}(dy|\mathbb{A}^c)\ F^{m+1}(dx|\mathbb{A}^c) \\ &\le F^{m+1}(\mathbb{A}^c)^{j-2} E|S_{m_0}|^\alpha \ < \ \infty \end{aligned}$$

for each $j \ge 1$ and thereby

$$\sum_{j\ge1}(E|S_{\hat{\chi}_0+jm_0} - S_{\hat{\chi}_0+(j-1)m_0}|^\alpha \mathbf{1}_{\{\kappa_0\ge j\}})^{1/\alpha} \ \le \ (E|S_{m_0}|^\alpha)^{1/\alpha} \sum_{j\ge1} F^{m+1}(\mathbb{A}^c)^{(j-2)/\alpha} \ < \ \infty.$$

Putting all previous inequalities together the assertion obviously follows. ♢

Turning to exponential moments we need

LEMMA 4.5. *For all $\alpha > 0$ and $n \in \mathbb{N}_0$*

$$E_\lambda e^{\alpha|S_n|} \ \le \ \prod_{k=0}^n \Big(E_\lambda e^{\alpha(n+1)|X_k|}\Big)^{1/(n+1)} \ \le \ M_\lambda(\alpha(n+1)); \tag{4.1}$$

$$\begin{aligned} E_\lambda e^{\alpha|S_{n+m}|} &\le \Big(E_\lambda e^{2\alpha|S_m|}\Big)^{1/2}\Big(Ee^{2\alpha|S_n|}\Big)^{1/2} \\ &\le M_\lambda(2(m+1)\alpha)^{1/2}\Big(Ee^{2\alpha|S_n|}\Big)^{1/2}; \end{aligned} \tag{4.2}$$

$$Ee^{\alpha|S_n|} \ \le \ (m+1)\Big(Ee^{\alpha(m+1)|X_1|}\Big)^{(n+m+1)/(m+1)}; \tag{4.3}$$

$$E_\lambda e^{\alpha|S_{n+m}|} \ \le \ (m+1)M_\lambda(2(m+1)\alpha)^{1/2}\Big(Ee^{2\alpha(m+1)|X_1|}\Big)^{(n+m+1)/2(m+1)}. \tag{4.4}$$

PROOF. W.l.o.g. suppose $X_n \ge 0$ for all $n \ge 0$. (4.1) and (4.2) follow by simple applications of Hölder's inequality so that we can turn immediately to (4.3). Write $n = j(m+1) + r$ with $0 \le r \le m$ and define

$$\begin{pmatrix} S_n(1) \\ \vdots \\ S_n(r) \\ S_n(r+1) \\ \vdots \\ S_n(m+1) \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} X_1 \\ \vdots \\ X_r \\ X_{r+1} \\ \vdots \\ X_{m+1} \end{pmatrix} + ... + \begin{pmatrix} X_{(j-1)(m+1)+1} \\ \vdots \\ X_{(j-1)(m+1)+r} \\ X_{(j-1)(m+1)+r+1} \\ \vdots \\ X_{j(m+1)} \end{pmatrix} + \begin{pmatrix} X_{j(m+1)+1} \\ \vdots \\ X_{j(m+1)+r} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Plainly, $S_n = S_n(1) + ... + S_n(m+1)$ and by m-dependence each $S_n(k)$ is a sum of $j+1$ (if

$k \le r$) or j (if $k > r$) i.i.d. random variables under $P = P_{F^{m+1}}$. Thus we conclude

$$\begin{aligned} Ee^{\alpha S_n} &= \int_0^\infty \alpha e^{\alpha t} P(S_n > t)\, dt \\ &\le \sum_{k=1}^{m+1} \int_0^\infty \alpha e^{\alpha t} P\Big(S_n(k) > \frac{t}{m+1}\Big)\, dt \\ &= \sum_{k=1}^{m+1} \int_0^\infty \alpha(m+1) e^{\alpha(m+1)s} P(S_n(k) > s)\, ds \\ &= \sum_{k=1}^{m+1} Ee^{\alpha(m+1)S_n(k)} \\ &= r\Big(Ee^{\alpha(m+1)X_1}\Big)^{j+1} + (m+1-r)\Big(Ee^{\alpha(m+1)X_1}\Big)^{j} \\ &\le (m+1)\Big(Ee^{\alpha(m+1)X_1}\Big)^{j+1} \;\le\; (m+1)\Big(Ee^{\alpha(m+1)|X_1|}\Big)^{(n+m+1)/(m+1)} \end{aligned}$$

which is (4.3). (4.4) is an obvious consequence of (4.2) and (4.3). $\diamondsuit$

We are now ready for the

PROOF OF PROPOSITION 4.1. Defining $S_n^{(\pm)} \stackrel{\text{def}}{=} \sum_{k=0}^n X_k^{\pm}$, we clearly have $S_n^{\pm} \le S_n^{(\pm)}$ for all $n \ge 0$. Moreover, $(M_n, S_n^{(+)})_{n\ge 0}$ and $(M_n, S_n^{(-)})_{n\ge 0}$ are (ψ, F)-mdMRW's for obvious choices of ψ with the same T_n's as regeneration epochs as $(M_n, S_n)_{n\ge 0}$ itself. It is therefore enough to prove the proposition for the case of nonnegative X_n's.

(a) Again we restrict ourselves to the case $\alpha \ge 1$, our assumption being $C_\lambda(\alpha) = C_\lambda^+(\alpha) < \infty$. Recall $n_0 \ge m+1$, put $\Lambda \stackrel{\text{def}}{=} (1-\beta)^{-1}(F^{m+1} - \beta F^{m+1}(\cdot|\mathbb{B}))$ and note that

$$\mathcal{L}(M_{\upsilon_n}|\eta_{\upsilon_n} = 0) \;=\; \mathcal{L}(M_{\upsilon_n}|\varrho = k) \;=\; \Lambda$$

for $k > n$ as well as

$$\begin{aligned} \mathcal{L}(S_{\upsilon_n} - S_{\upsilon_{n-1}}|\varrho = n) &= P_\Lambda(S_{\upsilon_0} \in \cdot|\eta_{\upsilon_0} = 1), \\ \mathcal{L}(S_{\upsilon_n} - S_{\upsilon_{n-1}}|\varrho > n) &= P_\Lambda(S_{\upsilon_0} \in \cdot|\eta_{\upsilon_0} = 0), \\ \mathcal{L}(S_{\upsilon_n} - S_{\upsilon_{n-1}}|\varrho \ge n) &= P_\Lambda(S_{\upsilon_0} \in \cdot) \end{aligned}$$

for all $n \ge 0$ and under each P_λ. Now use the latter fact to obtain

$$\begin{aligned} (E_\lambda S_{T_0}^\alpha)^{1/\alpha} &\le (E_\lambda S_{\upsilon_0}^\alpha)^{1/\alpha} + \sum_{n\ge 1} (E_\lambda (S_{\upsilon_{n+1}} - S_{\upsilon_n})^\alpha \mathbf{1}_{\{\varrho \ge n\}})^{1/\alpha} \\ &= (E_\lambda S_{\upsilon_0}^\alpha)^{1/\alpha} + \sum_{n\ge 1} (E_\Lambda (S_{\upsilon_0}^\alpha)^{1/\alpha} P(\varrho \ge n)^{1/\alpha} \\ &= (E_\lambda S_{\upsilon_0}^\alpha)^{1/\alpha} + (E_\Lambda (S_{\upsilon_0}^\alpha)^{1/\alpha} \sum_{n\ge 1} (1-\beta)^{n/\alpha} \end{aligned}$$

which is finite because $E_\nu S_{\upsilon_0}^\alpha < \infty$ for $\nu \in \{\lambda, \Lambda\}$ by Lemma 4.4. In case $\nu = \Lambda$ we mention that $E_\Lambda X_n^\alpha < \infty$ for all $n \ge 0$ follows from $EX_1^\alpha < \infty$ and $\Lambda \le (1-\beta)^{-1} F^{m+1}$.

(b) Note first that Lemma 3.5 yields

$$P_\lambda(T_0 \geq n) \ = \ P(T_0 \geq n) \ \leq \ C_1 \gamma_1^{-n} \tag{4.5}$$

for all $n \geq 1$, some $C_1 \in (0,\infty)$ and $\gamma_1 \in (1,\infty)$. Since $Ee^{aX_1} \downarrow 1$ for $a \downarrow 0$ and $M_\lambda(\alpha) < \infty$, we infer from (4.4) for sufficiently small positive θ that

$$E_\lambda e^{2\theta S_n} \ \leq \ C_2 \gamma_2^n \tag{4.6}$$

for all $n \geq 0$, some $C_2 \in (0,\infty)$ and $\gamma_2 < \gamma_1$. Hence by Hölder's inequality

$$\begin{aligned} E_\lambda e^{\theta S_{T_0}} \ &= \ \sum_{n\geq 1} E_\lambda e^{\theta S_n} \mathbf{1}_{\{T_0=n\}} \ \leq \ \sum_{n\geq 1} \left(E_\lambda e^{2\theta S_n}\right)^{1/2} P(T_0=n)^{1/2} \\ &\leq \ C_1 C_2 \sum_{n\geq 1} (\gamma_2/\gamma_1)^{n/2} \ < \ \infty. \end{aligned} \tag{4.7}$$

This completes the proof of Proposition 4.1. ◊

For the proof of Proposition 4.2(a), we need a further lemma. Let $\mathcal{G}$ be the σ-field generated by $(\chi_n, \eta_n, \upsilon_n)_{n\geq 1}$ and note that the T_n, κ_n are all $\mathcal{G}$-measurable.

LEMMA 4.6. *There is a finite constant C_0 such that*

$$P_\lambda(X_k \in \cdot|\mathcal{G}) \ \leq \ C_0 P_\lambda(X_k \in \cdot) \quad P_\lambda\text{-a.s.}$$

for all $k \in I\!N_0$ and initial distributions λ.

PROOF. The following listing shows that $P_\lambda(X_k \in \cdot|\mathcal{G})$, if not equal to $P_\lambda(X_k \in \cdot)$, can vary only within a set of finitely many distributions which are all bounded by some constant times $P_\lambda(X_k \in \cdot)$ as claimed. Note that the latter is the same as $P(X_1 \in \cdot)$ for all $k \geq m+1$ by m-dependence. It is convenient to put

$$\lambda_n \ \stackrel{\text{def}}{=} \ \begin{cases} \lambda, & \text{if } n=0 \\ F^{m+1}(\cdot|\mathbb{B}), & \text{if } n \geq 1, \eta_{\upsilon_n}=1 \\ \Lambda, & \text{if } n \geq 1, \eta_{\upsilon_n}=0 \end{cases}$$

and to observe that $P_{\lambda_n} \leq [F^{m+1}(\mathbb{B}) \wedge (1-\beta)]^{-1} P_{F^{m+1}}$ for all $n \geq 1$.

CASE 1. $\upsilon_n \leq k \leq \upsilon_n + \chi_{n+1}$ for some $n \geq -1$. Then

$$P_\lambda(X_k \in \cdot|\mathcal{G}) \ = \ P_{\lambda_n}(X_k \in \cdot).$$

CASE 2. $k = \upsilon_n + \chi_{n+1} + jm_0 + r < \upsilon_{n+1}$ for some $j \geq 0, n \geq -1$ and $1 \leq r \leq m+1$. Then

$$P_\lambda(X_k \in \cdot|\mathcal{G}) \;=\; \begin{cases} P_{\lambda_n}(X_{r+\chi_{n+1}} \in \cdot|M_{m+1+\chi_{n+1}} \in \mathbb{A}), & \text{if } j=0, \kappa_{n+1}=0 \\ P_{\lambda_n}(X_{r+\chi_{n+1}} \in \cdot|M_{m+1+\chi_{n+1}} \notin \mathbb{A}), & \text{if } j=0, \kappa_{n+1}>0 \\ P_{F^{m+1}(\cdot|\mathbb{A}^c)}(X_r \in \cdot|M_{m+1} \in \mathbb{A}), & \text{if } j \geq 1, \kappa_{n+1}=j \\ P_{F^{m+1}(\cdot|\mathbb{A}^c)}(X_r \in \cdot|M_{m+1} \notin \mathbb{A}), & \text{if } j \geq 1, \kappa_{n+1}>j \end{cases}.$$

CASE 3. $k = \upsilon_n + \chi_{n+1} + (j+1)m_0 - n_0 + r \leq \upsilon_{n+1}$ for some $j \geq 0, n \geq -1$ and $1 \leq r \leq n_0$. Then

$$P_\lambda(X_k \in \cdot|\mathcal{G}) \;=\; \begin{cases} P(Y_r \in \cdot), & \text{if } \kappa_{n+1}=j, \eta_{\upsilon_{n+1}}=1 \\ P(Z_r \in \cdot), & \text{if } \kappa_{n+1}=j, \eta_{\upsilon_{n+1}}=0 \\ P_{F^{m+1}(\cdot|\mathbb{A}^c)}(X_r \in \cdot), & \text{if } \kappa_{n+1}>j \end{cases}. \qquad \Diamond$$

PROOF OF PROPOSITION 4.2. It suffices again to assume all X_n's to be nonnegative.

(a) As before, we consider only $\alpha \geq 1$. By Lemma 4.6 and the conditional Minkowski inequality

$$E_\lambda(S_n^\alpha|\mathcal{G})^{1/\alpha} \;\leq\; \sum_{k=0}^n E_\lambda(X_k^\alpha|\mathcal{G})^{1/\alpha} \;\leq\; C_0^{1/\alpha}\sum_{k=0}^n (E_\lambda X_k^\alpha)^{1/\alpha} \;\leq\; (C_0C_\lambda(\alpha))^{1/\alpha}(n+1)$$

a.s. for all $n \geq 0$. Since T_0 is $\mathcal{G}$-measurable, this further implies

$$E_\lambda(S_n^\alpha|T_0>n) \;=\; E_\lambda(E_\lambda(S_n^\alpha|\mathcal{G})|T_0>n) \;\leq\; C_0C_\lambda(\alpha)(n+1)^\alpha$$

a.s. for all $n \geq 0$. Combining this with (4.5) we finally obtain

$$\begin{aligned} E_\lambda\left(\sum_{n=0}^{T_0-1} S_n^\alpha\right) &= \sum_{n\geq 0} E_\lambda S_n^\alpha \mathbf{1}_{\{T_0>n\}} \\ &= \sum_{n\geq 0} E_\lambda(S_n^\alpha|T_0>n)P(T_0>n) \\ &\leq C_0C_1C_\lambda(\alpha)\sum_{n\geq 0}(n+1)^\alpha\gamma_1^{-n} \;<\; \infty \end{aligned}$$

which is the assertion.

(b) Here we obtain for sufficiently small $\theta > 0$ in a similar manner as in (4.7)

$$\begin{aligned} E_\lambda\left(\sum_{n=0}^{T_0-1} e^{\theta S_n}\right) &= \sum_{n\geq 0} E_\lambda e^{\theta S_n}\mathbf{1}_{\{T_0>n\}} \\ &\leq \sum_{n\geq 0}(E_\lambda e^{2\theta S_n})^{1/2}P(T_0>n)^{1/2} \\ &\leq C_1C_2\sum_{n\geq 0}(\gamma_2/\gamma_1)^{n/2} \;<\; \infty \end{aligned}$$

and thus again the desired result. $\Diamond$

REMARK. All previous moment results remain true when replacing T_0 by the associated first level 1 ladder epoch

$$\hat{T}_0 \overset{\text{def}}{=} T_\phi, \quad \phi \overset{\text{def}}{=} \inf\{n : S_{T_n} > 1\}.$$

This can be easily shown when combining the previous results with

$$E_\nu(S_{T_0}^+)^\alpha < \infty \text{ for } \nu \in \{\lambda, F^{m+1}\} \quad \Rightarrow \quad E_\nu(S_{\hat{T}_0}^+)^\alpha < \infty \text{ for } \nu \in \{\lambda, F^{m+1}\};$$
$$E_\nu(S_{T_0}^-)^\alpha < \infty \text{ for } \nu \in \{\lambda, F^{m+1}\} \quad \Rightarrow \quad E_\nu \phi^\alpha < \infty \text{ for } \nu \in \{\lambda, F^{m+1}\}$$

and similar conclusions for exponential moments, which are well-known facts from standard renewal theory (see [10]).

However, it should be observed for later purposes, notably Proposition 6.3, that $\hat{T}_0$ needs no longer have moments of arbitrary order under P_λ as being true for T_0 (by Lemma 3.5). Indeed, assuming $\mu > 0$ and defining $\phi(x) = \inf\{n \geq 0 : S_{T_n} > x\}$, a straightforward argument in combination with Theorem I.5.2 in [10] gives for $\alpha \geq 1$

$$\begin{aligned} E_\lambda \hat{T}_0^\alpha \;&=\; E_\lambda T_\phi^\alpha \leq \; E_\lambda T_0^\alpha + \int_{(-\infty,1]} E_{F^{m+1}(\cdot|\mathbb{B})} T_{\phi(x)}^\alpha \; P_\lambda(S_{T_0} \in dx) \\ &\leq \; E_\lambda T_0^\alpha + \text{const } E_{F^{m+1}(\cdot|\mathbb{B})} T_0^\alpha \int_{(-\infty,1]} E_{F^{m+1}(\cdot|\mathbb{B})} \phi(x)^\alpha \; P_\lambda(S_{T_0} \in dx). \end{aligned}$$

But the latter expression is finite if $E_\nu(S_{T_0}^-)^\alpha < \infty$ for $\nu \in \{\lambda, F^{m+1}\}$, whence we conclude with Proposition 4.2 (in case $\mu > 0$)

$$C_\lambda^-(\alpha) < \infty \quad \Rightarrow \quad E_\nu \hat{T}_0^\alpha < \infty \text{ for } \nu \in \{\lambda, F^{m+1}\} \tag{4.8}$$

By a similar argument, one can show for $\alpha > 0$ that

$$M_\lambda^-(\alpha) < \infty \quad \Rightarrow \quad E_\nu e^{\theta \hat{T}_0} < \infty \text{ for some } \theta \in (0,\alpha] \text{ and } \nu \in \{\lambda, F^{m+1}\}. \tag{4.9}$$

For the remainder of this section suppose $\mu \in (0,\infty)$. Our next lemma deals with the moments of the X_n's under P_{ν^s}, ν^s the stationary Markov delay distribution defined in (1.4). The notation from there should be recalled, in particular $\vartheta = E_{\xi^*}\sigma_1$.

LEMMA 4.7. *There is a finite constant K such that $C_{\nu^s}^+(\alpha) \leq K\, E(X_1^+)^{\alpha+1}$ and $C_{\nu^s}^-(\alpha) \leq K\, E(X_1^-)^\alpha$ for all $\alpha > 0$.*

PROOF. $F^{m+1} = \vartheta^{-1} E_{\xi^*}(\sum_{n=0}^{\sigma_1 - 1} \mathbf{1}_{\{M_n \in \cdot\}})$ implies $\xi^* = P_{\xi^*}(M_0 \in \cdot) \leq \vartheta F^{m+1}$ and thus for each $n \geq 1$ and $\alpha > 0$

$$\begin{aligned} E_{\nu^s}(X_n^\pm)^\alpha \;&=\; \int E_x(X_n^\pm)^\alpha \; P_{\nu^s}(M_0 \in dx) \;=\; \int E_x(X_n^\pm)^\alpha \; \xi^*(dx) \\ &\leq\; \vartheta \int E_x(X_n^\pm)^\alpha \; F^{m+1}(dx) \;=\; \vartheta E(X_1^\pm)^\alpha \;<\; \infty \end{aligned}$$

since $\vartheta = \mu^>/\mu < \infty$. Notice that this is the same for $E_{\xi^*}(X_n^\pm)^\alpha$. For $n = 0$, $X_0^- = 0$ and a simple computation gives

$$E_{\nu^s}(X_0^+)^\alpha \;=\; \frac{1}{\mu^>}\int_0^\infty t^\alpha P_{\xi^*}(S_{\sigma_1} > t)\, dt \;=\; \frac{E_{\xi^*}S_{\sigma_1}^{\alpha+1}}{(\alpha+1)\mu^>}$$

and this is again bounded by a constant times $E(X_1^+)^{\alpha+1}$ because, by using (4.2) in [4] and the previous estimates,

$$\begin{aligned} E_{\xi^*}S_{\sigma_1}^{\alpha+1} \;&\le\; E_{\xi^*}(X_{\sigma_1}^+)^{\alpha+1} \;\le\; E_{\xi^*}\left(\sum_{n=1}^{\sigma_1+m}(X_n^+)^{\alpha+1}\right) \\ &=\; E_{\xi^*}\left(\sum_{n=1}^{m}(X_n^+)^{\alpha+1}\right) \;+\; E_{\xi^*}\sigma_1 E(X_1^+)^{\alpha+1} \\ &\le\; (m+1)\vartheta E(X_1^+)^{\alpha+1}. \end{aligned}$$

This completes the proof of the lemma. ◊

The moments of the first passage times $\tau(t) = \inf\{n \ge 1 : S_n > t\}$, $t \ge 0$, and the associated stopped sums $S_{\tau(t)}$ are considered in the following proposition which may be viewed as the natural extension of a well-known result for i.i.d. increments due to Gut, see [10], Section III.3. Keep in mind that $\sigma_1 = \tau(0)$.

PROPOSITION 4.8. *Let $\alpha \ge 1$ in parts (a),(b) and $\alpha > 0$ in parts (c),(d) below.*

(a) If $C_\lambda^+(\alpha) < \infty$ and $C_\lambda^-(1) < \infty$, then $E_\lambda S_{\tau(t)}^\alpha \le const\,(t+1)^\alpha$ for all $t \ge 0$.

(b) If $C_\lambda^-(\alpha) < \infty$, then $E_\lambda \tau(t)^\alpha \le const\,(t+1)^\alpha$ for all $t \ge 0$.

(c) If $M_\lambda^+(\alpha) < \infty$ and $C_\lambda^-(1) < \infty$, then $E_\lambda e^{\theta S_{\tau(t)}} \le const\,(t+1)e^{\theta t}$ for all $t \ge 0$ and $\theta \le \alpha$.

(d) If $M_\lambda^-(\alpha) < \infty$, then $E_\lambda e^{\theta\tau(t)} \le g(\theta)e^{r\theta t}$ for all $t \ge 0$ and $\theta \le \theta_0 \le \alpha$ where $r \ge 1$ does not depend on θ and $g(\theta) \to 1$ as $\theta \to 0$.

PROOF. Parts (a) and (b) follow from Theorem 2.3 of [11] in the stationary case $\lambda = F^{m+1}$. For the extension to general λ observe that

$$\begin{aligned} E_\lambda(S_{\tau(t)} - t)^\alpha \;&\le\; E_\lambda(X_{\tau(t)}^+)^\alpha \;\le\; E_\lambda\left(\sum_{k=0}^{\tau(t)+m}(X_k^+)^\alpha\right) \\ &\le\; (m+1)C_\lambda^+(\alpha) + E(X_1^+)^\alpha E_\lambda\tau(t) \;\le\; \text{const}\,(t+1), \end{aligned} \tag{4.10}$$

where (4.2), (4.4) from [4] have been utilized, as well as

$$E_\lambda\tau(t)^\alpha \;\le\; E_\lambda T_{\hat\tau(t)}^\alpha \;\le\; \text{const}\,(t+1)^\alpha \tag{4.11}$$

where $\hat\tau(t) \overset{\text{def}}{=} \inf\{n \ge 0 : S_{T_n} > t\}$. The final inequality is a standard renewal result applied to the ordinary delayed random walk $(S_{T_n} - S_{T_{n-1}})_{n\ge0}$ which, by Proposition 4.1(a), satisfies $E_\lambda(S_{T_0}^-)^\alpha < \infty$ and $E_\lambda(S_{T_n} - S_{T_{n-1}})^\alpha = E_{F^{m+1}(\cdot|\mathbb{B})}S_{T_0}^\alpha < \infty$ for $n \ge 1$ if $C_\lambda^-(\alpha) < \infty$.

Similar arguments lead to (c) and (d): Instead of (4.10), we get

$$\begin{aligned} Ee^{\alpha(S_{\tau(t)}-t)} &\le E_\lambda\left(\sum_{k=0}^{\tau(t)+m} e^{\alpha X_k^+}\right) \\ &\le (m+1)M_\lambda^+(\alpha) + Ee^{\alpha X_1^+}E\tau(t) \ \le\ \text{const}\,(t+1) \end{aligned}$$

for all $t \ge 0$ providing $M_\lambda^+(\alpha) < \infty$. This clearly implies (c).

By Proposition 4.1(b), $M_\lambda^-(\alpha)$ ensures $E_\lambda e^{\theta S_{T_0}^-} < \infty$ and $E_{F^{m+1}(\cdot|\mathbb{B})}e^{\theta S_{T_0}^-} < \infty$ for all $\theta \le \theta_1 \le \alpha$, whence $E_\lambda e^{\theta T_{\hat\tau(t)}} \le g(\theta)e^{r\theta t}$ for all $t \ge 0$, $\theta \le \theta_0 \le \alpha$ and some g as stated above may again be deduced by standard renewal arguments in combination with Theorem III.3.2 in [10]. Since $\tau(t) \le T_{\hat\tau(t)}$ for all $t \ge 0$, (d) follows. ♢

The moments of the occupation measure $U_{\xi^*}^{\sigma_1} = E_{\xi^*}(\sum_{n=0}^{\sigma_1-1}\mathbf{1}_{\{(M_n,S_n)\in\cdot\}})$ will also be of interest, see the proof of Theorems 2.6 and 2.7 at the end of Section 6.

LEMMA 4.9. *Let* $\alpha > 0$.

(a) If $C_\lambda^-(\alpha+1) < \infty$ *then* $\int_{\mathcal{S}^{m+1}\times I\!R}|t|^\alpha\,U_\lambda^{\sigma_1}(dx,dt) = E_\lambda(\sum_{n=0}^{\sigma_1-1}(S_n^-)^\alpha) < \infty$.

(b) If $M_\lambda^-(\alpha) < \infty$ *then* $\int_{\mathcal{S}^{m+1}\times I\!R}e^{\theta|t|}\,U_\lambda^{\sigma_1}(dx,dt) = E_\lambda(\sum_{n=0}^{\sigma_1-1}e^{\theta S_n^-}) < \infty$ *for some* $\theta \in (0,\alpha]$.

PROOF. (a) First note that $C_\lambda^-(\alpha+1) < \infty$ implies $E_\lambda\sigma_1^{\alpha+1} < \infty$ by Proposition 4.8(b). By combining this with Theorem 1.3 in [11], which may easily be adapted to the nonstationary case $\lambda \ne F^{m+1}$, we further obtain $E_\lambda(S_{\sigma_1}^{(-)})^{\alpha+1} \le \text{const}\,E_\lambda\sigma_1^{\alpha+1}C_\lambda^-(\alpha+1)$ where $S_n^{(-)} = \sum_{k=0}^n X_k^-$ should be recalled. Using the inequality

$$\sum_{n=0}^{\sigma_1-1}\mathbf{1}_{\{S_n^->t\}} \ \le\ \sigma_1\mathbf{1}_{\{\sigma_1>t\}} \ +\ t\,\mathbf{1}_{\{S_{\sigma_1}^{(-)}>t\}}$$

we now conclude

$$\begin{aligned} \int_{\mathcal{S}^{m+1}\times I\!R}|t|^\alpha\,U_\lambda^{\sigma_1}(dx,dt) &= E_\lambda\left(\int_0^\infty \alpha t^{\alpha-1}\sum_{n=0}^{\sigma_1-1}\mathbf{1}_{\{S_n^->t\}}\,dt\right) \\ &\le E_\lambda\left(\sigma_1\int_0^\infty \alpha t^{\alpha-1}\mathbf{1}_{(0,\sigma_1)}(t)\,dt\right) + E_\lambda\left(\int_0^\infty \alpha t^{\alpha}\mathbf{1}_{(0,S_{\sigma_1}^{(-)})}(t)\,dt\right) \\ &= E_\lambda\sigma_1^{\alpha+1} + \frac{\alpha}{\alpha+1}E_\lambda(S_{\sigma_1}^{(-)})^{\alpha+1} \ <\ \infty. \end{aligned}$$

(b) The procedure here is similar so that we restrict ourselves to the only critical point, namely an argument why $E_\lambda e^{\theta S_{\sigma_1}^{(-)}} < \infty$ for some $\theta > 0$ follows from $M_\lambda^-(\alpha) < \infty$. Indeed, using Hölder's inequality and (4.4) of Lemma 4.5, we obtain for sufficiently small $\theta > 0$

$$\begin{aligned} E_\lambda e^{\theta S_{\sigma_1}^{(-)}} &= \sum_{n\ge0}E_\lambda e^{\theta S_n^{(-)}}\mathbf{1}_{\{\sigma_1=n\}} \ \le\ \sum_{n\ge0}\left(E_\lambda e^{2\theta S_{n+m}^{(-)}}\right)^{1/2}P_\lambda(\sigma_1=n)^{1/2} \\ &\le (m+1)^{1/2}M_\lambda^-(4(m+1)\theta)^{1/2}\sum_{n\ge1}\left(Ee^{4\theta(m+1)X_1^-}\right)^{(n+m+1)/(4m+4)}P_\lambda(\sigma_1=n)^{1/2} \end{aligned}$$

which is finite because $E_\lambda e^{\theta\sigma_1} < \infty$ by Proposition 4.8(d) for sufficiently small $\theta > 0$. $\diamondsuit$

5. Proof of Theorems 2.1 - 2.3

Proof of Theorem 2.1. (a) It clearly suffices to prove the assertion for $I = [0,1]$. Given $C_\nu^+(\alpha) < \infty$ for $\nu \in \{\lambda, \lambda'\}$, we have $E_\nu(S_{T_0}^+)^\alpha < \infty$ for $\nu \in \{\lambda, \lambda', F^{m+1}\}$ by Proposition 4.1(a). A coupling argument in classical renewal theory (see [14] and the Appendix) gives

$$\|\mathbb{U}_{\lambda|t+I} - \mathbb{U}_{\lambda'|t+I}\| \ \le\ H_{\lambda,\lambda'}(t)$$

for all $t \ge 0$ and a decreasing function $H_{\lambda,\lambda'}$ on $[0,\infty)$ satisfying

$$\int_0^\infty t^{\alpha-1} H_{\lambda,\lambda'}(t)\ dt\ <\ \infty,$$

thus in particular $\lim_{t\to\infty} t^\alpha H_{\lambda,\lambda'}(t) = 0$. Moreover,

$$\sup_{s\in\mathbb{R}} \|\mathbb{U}_{\lambda|s+I} - \mathbb{U}_{\lambda'|s+I}\| \ \le\ \sup_{s\in\mathbb{R}} (\mathbb{U}_\lambda(s+I) \vee \mathbb{U}_{\lambda'}(s+I))\ \le\ \mathbb{U}^*[-1,1]\ <\ \infty. \tag{5.1}$$

For each $\nu \in \{\lambda, \lambda', F^{m+1}(\cdot|\mathbb{B})\}$, $\|U_\nu^{T_0}\| = ET_0 < \infty$ holds by Lemma 3.5 and

$$\int_{[0,\infty)} t^\alpha\ U_\nu^{T_0}(\mathcal{S}^{m+1} \times dt)\ =\ \int_0^\infty \alpha t^{\alpha-1} U_\nu^{T_0}(\mathcal{S}^{m+1} \times (t,\infty))\ dt\ <\ \infty$$

by Proposition 4.2(a) and the subsequent Remark. The latter equation further implies

$$\lim_{t\to\infty} t^\alpha U_\nu^{T_0}(\mathcal{S}^{m+1} \times (t,\infty))\ =\ 0.$$

Using these facts and (3.5) of Lemma 3.3, the assertion follows from

$$\begin{aligned} \|U_{\lambda|t+I} - U_{\lambda'|t+I}\| \ &\le\ \|U^{T_0}_{\lambda|t+I} - U^{T_0}_{\lambda'|t+I}\| \\ &\quad +\ \int_{\mathbb{R}} \|\mathbb{U}_{\lambda|t-y+I} - \mathbb{U}_{\lambda'|t-y+I}\|\ U^{T_0}_{F^{m+1}(\cdot|\mathbb{B})}(\mathcal{S}^{m+1} \times dy) \\ &\le\ \|U^{T_0}_{\lambda|t+I} - U^{T_0}_{\lambda'|t+I}\|\ +\ \sup_{s\ge t/2} \|\mathbb{U}_{\lambda|s+I} - \mathbb{U}_{\lambda'|s+I}\|\, ET_0 \\ &\quad +\ \sup_{s\in\mathbb{R}} \|\mathbb{U}_{\lambda|s+I} - \mathbb{U}_{\lambda'|s+I}\|\, U^{T_0}_{F^{m+1}(\cdot|\mathbb{B})}(\mathcal{S}^{m+1} \times (t/2,\infty)). \end{aligned} \tag{5.2}$$

(b) follows directly from (a) with $\lambda' = \nu^s$ when using Lemma 4.7.

(c) Using part (a) (with $\alpha+1$ instead of α), we infer the inequality

$$t^\alpha \|U_{\lambda|[t,\infty)} - U_{\lambda'|[t,\infty)}\|\ \le\ t^\alpha \sum_{n\ge\lfloor t\rfloor} \|U_{\lambda|n+I} - U_{\lambda'|n+I}\|\ \le\ t^\alpha K_{\lambda,\lambda'}(t) \sum_{n\ge\lfloor t\rfloor} n^{-\alpha-1}$$

for a suitable function $K_{\lambda,\lambda'}(t)$ convergent to 0 as $t \to \infty$. Moreover,

$$\int_0^\infty t^{\alpha-1} \|U_{\lambda|[t,\infty)} - U_{\lambda'|[t,\infty)}\|\ dt\ \le\ \sum_{n\ge 0} (n+1)^{\alpha-1} \|U_{\lambda|[n,\infty)} - U_{\lambda'|[n,\infty)}\|$$

$$\le \sum_{n\ge 0}(n+1)^{\alpha-1}\sum_{k\ge n}\|U_{\lambda|k+I}-U_{\lambda'|k+I}\|$$

$$= \sum_{k\ge 0}\|U_{\lambda|k+I}-U_{\lambda'|k+I}\|\sum_{n=0}^{k}(n+1)^{\alpha-1}$$

$$\le \sum_{k\ge 0}(k+1)^{\alpha}\|U_{\lambda|k+I}-U_{\lambda'|k+I}\|$$

$$\le \sum_{k\ge 0}\int_k^{k+1}(t+1)^{\alpha}\|U_{\lambda|t-1+2I}-U_{\lambda'|t-1+2I}\|\;dt$$

$$= \int_0^{\infty}(t+1)^{\alpha}\|U_{\lambda|t-1+2I}-U_{\lambda'|t-1+2I}\|\;dt\;<\;\infty.$$

This proves the assertion.

(d) is again just a specialization of (c).

(e) Here the moment assumptions guarantee $E_\nu S_{T_0}^+ < \infty$ for $\nu \in \{\lambda,\nu^s,F^{m+1}(\cdot|\mathbb{B})\}$ whence classical renewal theory (see [14] and the Appendix) yields $\|\mathbb{U}_\lambda - \mathbb{U}_{\nu^s}\| < \infty$. Moreover, $U_\nu^{T_0}$ is a finite measure with total mass $ET_0 < \infty$ for every distribution ν on $\mathcal{S}^{m+1}\times I\!R$. Assertion (2.1) now easily follows from (5.2) with $t=0$, $I=[0,\infty)$ and $\lambda'=\nu^s$. ◊

Proof of Theorem 2.2. (a) The arguments are very similar to those for Theorem 2.1(a), but t is negative here. Given $C_\lambda^-(\alpha) < \infty$ and $E(X_1^-)^{\alpha+1} < \infty$, Proposition 4.1(a) implies $E_\lambda(S_{T_0}^-)^\alpha < \infty$ and $E(S_{T_0}^-)^{\alpha+1} < \infty$. This can further be used (see the Appendix) to obtain

$$\mathbb{U}_\lambda(t+I)\;\le\;H_\lambda(t)$$

for all $t\le 0$ and an increasing function H_λ on $(-\infty,0]$ satisfying

$$\int_{-\infty}^{0}|t|^{\alpha-1}H_\lambda(t)\;dt\;<\;\infty,$$

thus in particular $\lim_{t\to-\infty}|t|^\alpha H_\lambda(t)=0$. By Proposition 4.2(a) and the subsequent Remark,

$$\int_{(-\infty,0]}|t|^\alpha\;U^{T_0}_{F^{m+1}(\cdot|\mathbb{B})}(\mathcal{S}^{m+1}\times dt)\;=\;\int_{-\infty}^{0}\alpha|t|^{\alpha-1}U^{T_0}_{F^{m+1}(\cdot|\mathbb{B})}(\mathcal{S}^{m+1}\times(-\infty,t])\;dt\;<\;\infty.$$

Now one can easily conclude the asserted result from the inequality

$$\begin{aligned}U_\lambda(\mathcal{S}^{m+1}\times t+I)\;&\le\;U_\lambda^{T_0}(\mathcal{S}^{m+1}\times t+I)\;+\;ET_0\sup_{s\le t/2}\mathbb{U}_\lambda(s+I)\\&\quad+\;U^{T_0}_{F^{m+1}(\cdot|\mathbb{B})}(\mathcal{S}^{m+1}\times(-\infty,t/2))\sup_{s\in I\!R}\mathbb{U}_\lambda(s+I).\end{aligned}\tag{5.3}$$

which in turn follows from (3.5) of Lemma 3.3.

(b) This is shown by the same argument as Theorem 2.1(c).

(c) The moment assumptions give here $E_\lambda S_{T_0}^- < \infty$ and $E(S_{T_0}^-)^2 < \infty$. It is a well-known fact from ordinary renewal theory that under these conditions $\|\mathbb{U}_\lambda^-\| = \mathbb{U}_\lambda(-\infty,0] < \infty$ and

$\mathbb{U}_\lambda(-\infty,x] \le K(x+1)$ for all $x \ge 0$ and some constant K. By another appeal to (3.5) of Lemma 3.3, we thus infer

$$\begin{aligned}\|U_\lambda^-\| &\le \|(U_\lambda^{T_0})^-\| + \int_{I\!R} \mathbb{U}_\lambda(-\infty,-x]\, U^{T_0}_{F^{m+1}(\cdot|\mathbb{B})}(\mathcal{S}^{m+1}\times dx)\\ &\le \|(U_\lambda^{T_0})^-\| + \|\mathbb{U}_\lambda^-\|\,\|(U^{T_0}_{F^{m+1}(\cdot|\mathbb{B})})^+\| + \int_{(-\infty,0]} K(|x|+1)\, U^{T_0}_{F^{m+1}(\cdot|\mathbb{B})}(\mathcal{S}^{m+1}\times dx)\\ &\le (K+1+\|\mathbb{U}_\lambda^-\|)ET_0 + K\int_{(-\infty,0]} |x|\, U^{T_0}_{F^{m+1}(\cdot|\mathbb{B})}(\mathcal{S}^{m+1}\times dx) < \infty,\end{aligned}$$

where the latter integral is finite by Proposition 4.2(a). ♢

Proof of Theorem 2.3. Here it suffices to note that the assertions are proved similarly to those of Theorem 2.1(c),(d) and Theorem 2.2(b), of course, with the help of Proposition 4.1(b) and 4.2(b). ♢

6. Coupling at Ladder Epochs and Proof of Theorems 2.4 - 2.7

In order to prove convergence rate results for the ladder variable sequence $(M_n^>, S_n^>)_{n\ge0}$, the regeneration scheme of Section 3 cannot be used directly because the T_n need not be ladder epochs and therefore do not provide a regeneration scheme for the above sequences as well. However, it can still be employed for the following coupling construction, unfortunately at the price of stronger moment conditions than possibly necessary. Although the technical details of the construction are rather involved, its basic outline is simple and may be described as follows:

First we construct two coupled versions $(M_n', S_n')_{n\ge0}$ and $(M_n'', S_n'')_{n\ge0}$ of $(M_n,S_n)_{n\ge0}$ with different initial distributions. This is accomplished by using regeneration lemma 3.2. Hence there are a.s. finite random times τ' and τ'', in fact regeneration times for the respective sequences, such that $(M'_{\tau'+n}, S'_{\tau'+n})_{n\ge0} = (M''_{\tau''+n}, S''_{\tau''+n})_{n\ge0}$. The coupling process

$$(\hat M_n, \hat S_n) \stackrel{\text{def}}{=} \begin{cases} (M_n', S_n'), & \text{if } 0\le n\le \tau' \\ (M''_{n-\tau'+\tau''}, S''_{n-\tau'+\tau''}), & \text{if } n\ge\tau' \end{cases}, \quad n\ge 0,$$

then provides us with a copy of $(M_n', S_n')_{n\ge0}$ which concides with $(M_n'', S_n'')_{n\ge0}$ after time τ'. In order to see that the ladder epochs of $(\hat M_n, \hat S_n)_{n\ge0}$ and $(M_n'', S_n'')_{n\ge0}$ eventually coincide, notice that $\tau'+\psi$, where

$$\psi \stackrel{\text{def}}{=} \inf\{n\ge\tau' : \hat S_n > \max\{S_1', ..., S_{\tau'}', S_1'', ..., S_{\tau''}''\},$$

is a joint ladder epoch. It is this extra amount of time ψ it takes to synchronize the ladder epochs of $(\hat M_n, \hat S_n)_{n\ge0}$ and $(M_n'', S_n'')_{n\ge0}$ which has led to the stronger moment assumptions in our theorems.

Turning to the details, let $(M_n, X_n)_{n\ge0}$, with regeneration epoch sequence $(T_n)_{n\ge0}$, be as constructed in the previous section. Put

$$G_\lambda \stackrel{\text{def}}{=} P_\lambda(S_{T_0}\in\cdot) \quad\text{and}\quad G \stackrel{\text{def}}{=} P_{F^{m+1}(\cdot|\mathbb{B})}(S_{T_0}\in\cdot).$$

We summarize the main properties of the regeneration scheme described in Section 3:

(R.1) $(S_{T_n})_{n\ge 0}$ is an ordinary delayed 1-arithmetic or absolutely continuous random walk with delay distribution G_λ and increment distribution G under P_λ.

(R.2) $(M_{T_n})_{n\ge -1}$ forms a sequence of independent random variables which are identically distributed as $F^{m+1}(\cdot|\mathbb{B})$ for $n\ge 0$.

(R.3) For each $n\ge 0$, M_{T_n} is independent of $(M_k,X_k)_{0\le k\le T_n-n_0}, S_{T_n}-S_{T_n-n_0}$ and thus in particular independent of $(T_k,S_{T_k})_{0\le k\le n}$.

(R.4) $\max_{1\le k\le n_0}|X_{T_n-n_0+k}|\le t_0$ for each $n\ge 0$.

Since these facts remain unaffected when switching to the level 1 ladder epochs of $(S_{T_n})_{n\ge 0}$ by considering $(\hat T_n, M_{\hat T_n}, S_{\hat T_n})_{n\ge -1}$, where

$$\hat T_n \stackrel{\text{def}}{=} \inf\{T_k>\hat T_{n-1} : S_{T_k}-S_{\hat T_{n-1}}>1\},$$

it is no loss of generality to assume hereafter G_λ, G be concentrated on $(1,\infty)$. The reason for taking level 1 instead of level 0 as usual is only a simplification in the proof of Lemma 6.2 below. We refer to our remark preceding 4.7 for the fact that the moment results of Section 4 are still applicable. In the following, *we confine ourselves to the little more complicated case of absolutely continuous G.*

Given arbitrary initial distributions λ,λ' on $\mathcal{S}^{m+1}\times I\!R$, we proceed by several steps:

STEP 1. Following Lindvall's approach for absolutely continuous renewal processes, we first give a construction of an exact coupling $(\hat S_{1,n},\hat S_{2,n})_{n\ge 0}$ for the distributions of $(S_{T_n})_{n\ge 0}$ under P_λ and $P_{\lambda'}$, i.e.

$$\begin{aligned} I\!P_{\lambda,\lambda'}((\hat S_{1,n})_{n\ge 0}\in\cdot) &= P_\lambda((S_{T_n})_{n\ge 0}\in\cdot),\\ I\!P_{\lambda,\lambda'}((\hat S_{2,n})_{n\ge 0}\in\cdot) &= P_{\lambda'}((S_{T_n})_{n\ge 0}\in\cdot). \end{aligned}$$

and

$$(\hat S_{1,n})_{n\ge\hat\tau_{1,\zeta}} = (\hat S_{2,n})_{n\ge\hat\tau_{2,\zeta}}. \tag{6.1}$$

for a suitable coupling pair $(\hat\tau_{1,\zeta},\hat\tau_{2,\zeta})$. We have to do so in some detail because of the moment considerations further below. Let $(\tilde S_{0,n})_{n\ge 0}$ and $(\tilde S_{2,n})_{n\ge 0}$ be two independent renewal processes with delay distributions G_λ and $G_{\lambda'}$, respectively, and common increment distribution G under $I\!P_{\lambda,\lambda'}$, say. Put $\tilde X_{i,n}\stackrel{\text{def}}{=}\tilde S_{i,n}-\tilde S_{i,n-1}$ for $n\ge 1$ and $i=0,2$. The backward and forward recurrence time processes of $(\tilde S_{i,n}-\tilde S_{i,0})_{n\ge 1}$ are denoted by $(\tilde B_{i,t})_{t\ge 0}$ and $(\tilde F_{i,t})_{t\ge 0}$. Of course, $(\tilde B_{0,t},\tilde F_{0,t})_{t\ge 0}$ and $(\tilde B_{2,t},\tilde F_{2,t})_{t\ge 0}$ have the same distribution which does not depend on λ,λ'. Absolute continuity of G yields the existence of $c_1,c_2,t^*>0$ such that

$$I\!P_{\lambda,\lambda'}(\tilde B_t\in\cdot) \ \ge\ c_1\mathbf{1}_{[0,c_2]}\lambda\!\!\!\lambda_0 \tag{6.2}$$

for all $t\ge t^*$, [14], Lemma III.5.1. Let Q_t, h_t denote the distribution and absolutely continuous component density of $\tilde B_t$, thus $h_t\ge c_1\mathbf{1}_{[0,c_2]}$ for $t\ge t^*$. For $a\ge 0$, define G_a by

$$G_a(x,x+b] \stackrel{\text{def}}{=} \begin{cases} \dfrac{G(a+x,a+x+b]}{G(a,\infty)}, & \text{if } G(a,\infty)>0\\ \delta_0(x,x+b], & \text{otherwise}\end{cases}.$$

Hence G_a is the conditional distribution of $\tilde{F}_t$ under $\tilde{B}_t = a$ for every t. Now put $(V_{0,0}, V_{2,0}) \stackrel{\text{def}}{=} (\tilde{S}_{0,0}, \tilde{S}_{2,0})$, $W_0 \stackrel{\text{def}}{=} t^* + (V_{0,0} \vee V_{2,0})$ and

$$\hat{\tau}_{i,0} \stackrel{\text{def}}{=} \inf\{n \geq 1 : \tilde{S}_{i,n} - \tilde{S}_{i,0} > W_0 - V_{i,0}\}.$$

Given $\tilde{B}_{i,W_0-V_{i,0}} = b_i$, the forward recurrence time $\tilde{F}_{i,W_0-V_{i,0}}$ has distribution G_{b_i} because $W_0 - V_{i,0}$ is independent of $(\tilde{S}_{i,n} - \tilde{S}_{i,0})_{n\geq 0}$. Let $(V_{0,1}, V_{2,1})$ be a maximal coupling (see [14], p. 19) with these conditional marginals, independent of $(\tilde{S}_{i,\hat{\tau}_{i,0}+n} - \tilde{S}_{i,\hat{\tau}_{i,0}})_{n\geq 0}^{i=0,2}$, and define

$$\hat{S}_{i,0} \stackrel{\text{def}}{=} \tilde{S}_{i,0},\ \hat{X}_{i,n} \stackrel{\text{def}}{=} \tilde{X}_{i,n} \text{ for } 0 \leq n < \hat{\tau}_{i,0} \quad \text{and} \quad \hat{X}_{i,\hat{\tau}_{i,0}} \stackrel{\text{def}}{=} \tilde{B}_{i,W_0-V_{i,0}} + V_{i,1}.$$

The same procedure is next applied to the post-$\hat{\tau}_{i,0}$-processes $(\tilde{S}_{i,\hat{\tau}_{i,0}+n} - \tilde{S}_{i,\hat{\tau}_{i,0}})_{n\geq 0}$ with backward recurrence times $(\tilde{B}_t^1)_{t\geq 0}$: Put $W_1 \stackrel{\text{def}}{=} t^* + V_{0,1} \vee V_{2,1}$,

$$\hat{\tau}_{i,1} \stackrel{\text{def}}{=} \inf\{n \geq \hat{\tau}_{i,0} + 1 : \tilde{S}_{i,n} - \tilde{S}_{i,\hat{\tau}_{i,0}} > W_1 - V_{i,1}\},$$

let $(V_{0,2}, V_{2,2})$ be a maximal coupling with conditional marginals G_{b_1}, G_{b_2}, given $(\tilde{B}^1_{i,W_1-V_{i,1}}, \tilde{B}^1_{i,W_1-V_{i,1}}) = (b_1, b_2)$, which is independent of $(\tilde{S}_{i,\hat{\tau}_{i,1}+n} - \tilde{S}_{i,\hat{\tau}_{i,1}})_{n\geq 0}^{i=0,2}$. Define

$$\hat{X}_{i,n} \stackrel{\text{def}}{=} \tilde{X}_{i,n} \text{ for } \hat{\tau}_{i,0} < n < \hat{\tau}_{i,1} \quad \text{and} \quad \hat{X}_{i,\hat{\tau}_{i,1}} \stackrel{\text{def}}{=} \tilde{B}^1_{i,W_1-V_{i,1}} + V_{i,2}.$$

It is clear how the construction continues leading to strictly increasing sequences $(\hat{\tau}_{i,k})_{k\geq 0}$ of random times such that

$$\hat{X}_{i,n} \stackrel{\text{def}}{=} \tilde{X}_{i,n} \text{ for } \hat{\tau}_{i,k} < n < \hat{\tau}_{i,k+1} \quad \text{and} \quad \hat{X}_{i,\hat{\tau}_{i,k}} \stackrel{\text{def}}{=} \tilde{B}^k_{i,W_k-V_{i,k}} + V_{i,k+1}$$

where the meaning of $W_k, V_{i,k}$ and $\tilde{B}_t^k$ should now be clear. For each $i = 0, 2$, the resulting renewal process $(\hat{S}_{i,n})_{n\geq 0}$ is a copy of $(\tilde{S}_{i,n})_{n\geq 0}$ and a coupling of both occurs at $(\hat{\tau}_{0,\zeta}, \hat{\tau}_{2,\zeta})$, i.e.

$$\hat{S}_{0,\hat{\tau}_{0,\zeta}} = \hat{S}_{2,\hat{\tau}_{2,\zeta}},$$

where

$$\zeta \stackrel{\text{def}}{=} \inf\{k \geq 1 : V_{0,k} = V_{2,k}\}.$$

As shown in [14], the absolute continuity of G (notably (6.2)) implies $\mathbb{P}_{\lambda,\lambda'}(\zeta > n) \leq \kappa^n$ for some $\kappa \in (0, 1)$ and all $n \geq 0$. The coupling process $(\hat{S}_{1,n})_{n\geq 0}$ takes the form

$$\hat{S}_{1,n} \stackrel{\text{def}}{=} \begin{cases} \hat{S}_{0,n}, & \text{if } n \leq \hat{\tau}_{0,\zeta} \\ \hat{S}_{2,\hat{\tau}_{2,\zeta}+n-\hat{\tau}_{0,\zeta}}, & \text{if } n \geq \hat{\tau}_{0,\zeta} \end{cases}.$$

Put also

$$\hat{\tau}_{1,n} \stackrel{\text{def}}{=} \begin{cases} \hat{\tau}_{0,n}, & \text{if } n \leq \zeta \\ \hat{\tau}_{0,\zeta} + \hat{\tau}_{2,n} - \hat{\tau}_{2,\zeta}, & \text{if } n \geq \zeta \end{cases}.$$

Step 1 is herewith complete.

STEP 2. Our next task is to define regeneration epochs $T_{i,n}$ for $i = 1, 2$ and $n \geq 0$. To that end notice that by (R.1)

$$\begin{aligned} P_\lambda(T_0 \in \cdot|(S_{T_k})_{k\geq -1}) &= \boldsymbol{K}_1^\lambda(S_{T_0}, \cdot), \\ P_\lambda(T_n - T_{n-1} \in \cdot|(S_{T_k})_{k\geq -1}) &= \boldsymbol{K}_1(S_{T_n} - S_{T_{n-1}}, \cdot), \quad n \geq 1, \end{aligned}$$

for suitable kernels $\boldsymbol{K}_1, \boldsymbol{K}_1^\lambda$. Put $T_{1,-1} = T_{2,-1} = \hat{S}_{i,-1} \stackrel{\text{def}}{=} 0$ (as stipulated), $\mathbb{K}_{1,0} \stackrel{\text{def}}{=} \boldsymbol{K}_1^\lambda$, $\mathbb{K}_{2,0} \stackrel{\text{def}}{=} \boldsymbol{K}_1^{\lambda'}$ and $\mathbb{K}_{1,n} = \mathbb{K}_{2,n} \stackrel{\text{def}}{=} \boldsymbol{K}_1$ for $n \geq 1$. Generate $T_{i,n} - T_{i,n-1}$, given $(\hat{S}_{1,k}, \hat{S}_{2,k}, V_{0,k}, V_{2,k})_{k\geq 0}$, according to $\mathbb{K}_{i,n}(\hat{S}_{i,n} - \hat{S}_{i,n-1}, \cdot)$ for $n \geq 0$ and $i = 1, 2$. By (6.1) this can obviously be done in such a way that

$$T_{1,\hat{\tau}_{1,\zeta}+n} - T_{1,\hat{\tau}_{1,\zeta}+n-1} = T_{2,\hat{\tau}_{2,\zeta}+n} - T_{2,\hat{\tau}_{2,\zeta}+n-1} \tag{6.3}$$

for all $n \geq 1$. Put $\tau_{i,n} \stackrel{\text{def}}{=} T_{i,\hat{\tau}_{i,n}}$.

STEP 3. The final step is to define two coupled sequences $(M_{1,n}, S_{1,n})_{n\geq 0}$, $(M_{2,n}, S_{2,n})_{n\geq 0}$ which are copies of $(M_n, S_n)_{n\geq 0}$ under P_λ and $P_{\lambda'}$, respectively. Put $X_{i,n} = S_{i,n} - S_{i,n-1}$ for $n \geq 1$, as usual.

From (R.3) in the previous section, we infer the existence of a kernel $\boldsymbol{K}_2$ satisfying

$$P_\lambda((M_{T_n+k}, X_{T_n+k})_{k\geq 0} \in \cdot|(M_0, S_0), (T_j, S_{T_j})_{j\geq 0}) \;=\; \boldsymbol{K}_2((T_{j+1} - T_j, S_{T_{j+1}} - S_{T_j})_{j\geq n}), \cdot)$$

for all $n \geq 0$ and λ. Generate $(M_{1,\tau_{1,\zeta}+k}, X_{1,\tau_{1,\zeta}+k})_{k\geq 0} = (M_{2,\tau_{2,\zeta}+k}, X_{2,\tau_{2,\zeta}+k})_{k\geq 0}$, given ζ, $(T_{i,k}, \hat{S}_{i,k}, \hat{\tau}_{i,k})_{k\geq 0}^{i=1,2}$, $(V_{0,k}, V_{2,k})_{k\geq 0}$, according to $\boldsymbol{K}_2((T_{1,\tau_{1,\zeta}+k} - T_{1,\tau_{1,\zeta}}, \hat{S}_{1,\tau_{1,\zeta}+k} - \hat{S}_{1,\tau_{1,\zeta}})_{k\geq 0}, \cdot)$ (a reasonable definition in view of (6.1) and (6.3)).

The regeneration scheme in the previous section further yields the existence of a kernel $\boldsymbol{K}_3$ such that

$$\begin{aligned} P_\lambda((M_k, X_k)_{0\leq k\leq T_n} \in \cdot|(M_0, S_0),&(T_j, S_{T_j})_{j\geq 1}, (M_j, S_j)_{j\geq T_n}) \\ &= \boldsymbol{K}_3((M_0, M_{T_n}), (T_j, S_{T_j})_{1\leq j\leq n}, \cdot) \end{aligned}$$

for all $n \geq 0$ and λ. Let $(M_{1,0}, S_{1,0})$ and $(M_{2,0}, S_{2,0})$ be independent random vectors with distribution λ and λ' under $\mathbb{P}_{\lambda,\lambda'}$. Given these and all other variables generated so far, we generate $(M_{i,k}, X_{i,k})_{0\leq k\leq \tau_{i,\zeta}}$ according to $\boldsymbol{K}_3((M_{i,0}, M_{i,\tau_{i,\zeta}}), (T_{i,j}, \hat{S}_{i,T_{i,j}})_{1\leq j\leq \hat{\tau}_{i,\zeta}}, \cdot)$ for $i = 1, 2$. This completes the definition of $(M_{i,n}, S_{i,n})_{n\geq 0}$ for $i = 1, 2$. The main properties are summarized below:

$$\begin{aligned} \mathbb{P}_{\lambda,\lambda'}((M_{1,n}, S_{1,n})_{n\geq 0} \in \cdot) &= P_\lambda((M_n, S_n)_{n\geq 0} \in \cdot); \\ \mathbb{P}_{\lambda,\lambda'}((M_{2,n}, S_{2,n})_{n\geq 0} \in \cdot) &= P_{\lambda'}((M_n, S_n)_{n\geq 0} \in \cdot); \\ \mathbb{P}_{\lambda,\lambda'}((M_{1,0}, S_{1,0}, M_{2,0}, S_{2,0}) \in \cdot) &= \lambda \otimes \lambda'; \\ (S_{i,T_{i,n}})_{n\geq 0} &= (\hat{S}_{i,n})_{n\geq 0} \text{ for } i = 1, 2; \\ (M_{1,n}, S_{1,n})_{n\geq \tau_{1,\zeta}} &= (M_{2,n}, S_{2,n})_{n\geq \tau_{2,\zeta}}. \end{aligned}$$

Defining the filtrations

$$\mathfrak{F}_{i,n} \stackrel{\text{def}}{=} \sigma((S_{i,k})_{0\leq k\leq n}, (T_{i,k}\mathbf{1}_{\{T_{i,k}\leq n\}})_{k\geq 0}, (\mathbf{1}_{\{\tau_{i,k}\leq n\}}(V_{0,k}, V_{2,k}))_{k\geq 0}), \quad n \geq 0$$

for $i=1,2$, it can be easily checked that

(F.1) the $T_{i,k},\tau_{i,k}$ as well as $\tau_{i,\zeta}$ are stopping times with respect $(\mathfrak{F}_{i,n})_{n\ge 0}$;
(F.2) $(T_{i,k}-T_{i,k-1},S_{T_{i,k}}-S_{T_{i,k-1}})_{k>n}$ and $\mathfrak{F}_{i,T_{i,n}}$ are independent for all $n\ge 0$.
(F.3) $(T_{i,k}-T_{i,k-1},S_{T_{i,k}}-S_{T_{i,k-1}})_{k>\zeta}$ and $\mathfrak{F}_{i,T_{i,\zeta}}$ are independent.
for each $i=1,2$.

Let $(M_{i,n}^>,S_{i,n}^>)_{n\ge 0}$ be the Markov renewal process of strictly ascending ladder heights associated with $(M_{i,n},S_{i,n})_{n\ge 0}$. The process of forward recurrence times is denoted by $(\hat{M}_{i,t}, R_{i,t})_{t\ge 0}$, i.e.

$$(\hat{M}_{i,t},R_{i,t}) \stackrel{\text{def}}{=} (M_{i,\tau(t)},S_{i,\tau_i(t)}-t),\quad \tau_i(t) \stackrel{\text{def}}{=} \inf\{n\ge 0: S_{i,n}>t\}.$$

Let $(\hat{M}_t,R_t)_{t\ge 0}$ be that process for $(M_n,S_n)_{n\ge 0}$ and put $\lambda_t \stackrel{\text{def}}{=} P_\lambda((\hat{M}_t,R_t)\in\cdot)$ for each $t\ge 0$.

So far we have not yet shown that our construction also provides an exact coupling for the afore-mentioned ladder variable sequences. Indeed, for $\tau_{i,\zeta}$ needs not be a ladder epoch for $(M_{i,n},S_{i,n})_{n\ge 0}$, we have to look for a pair $(\tau_1^*,\tau_2^*)=(\tau_{1,\zeta}+\psi,\tau_{2,\zeta}+\psi)$, ψ a random time, such that $\tau_{i,\zeta}+\psi$ is one for $i=1,2$. Since $S_{1,\tau_{1,\zeta}}=S_{2,\tau_{2,\zeta}}>0$ and the maximal upward excursion of $(S_{i,n})_{0\le n\le\tau_{i,\zeta}}$ is bounded by

$$\sum_{n=0}^{\tau_{i,\zeta}} X_{i,n}^+ \;\le\; \sum_{n=0}^{\tau_{i,\zeta}-n_0} X_{i,n}^+ + n_0t_0 \stackrel{\text{def}}{=} Z_i$$

(for the inequality recall (R.4)), an obvious admissible choice for ψ is

$$\psi \stackrel{\text{def}}{=} \tau^*(Z_1\vee Z_2-S_{i,\tau_{i,\zeta}}),$$

where

$$\tau^*(t) \stackrel{\text{def}}{=} \inf\{n\ge 0: S_{i,\tau_{i,\zeta}+n}-S_{i,\tau_{i,\zeta}}>t\}$$

(does not depend on $i=1,2$). We then have

$$R_{1,t}=R_{2,t}\quad \text{for all } t\ge S^* \stackrel{\text{def}}{=} S_{1,\tau_1^*}=S_{2,\tau_2^*},$$

and therefore

$$\begin{aligned}\|\lambda_t-\lambda_t'\| &= \|I\!\!P_{\lambda,\lambda'}((\hat{M}_{1,t},R_{1,t})\in\cdot)-I\!\!P_{\lambda,\lambda'}((\hat{M}_{2,t},R_{2,t})\in\cdot)\|\\ &\le \|P_{\lambda,\lambda'}((\hat{M}_{1,t},R_{1,t})\in\cdot,S^*>t)-I\!\!P_{\lambda,\lambda'}((\hat{M}_{2,t},R_{2,t})\in\cdot,S^*>t)\| \\ &\le I\!\!P_{\lambda,\lambda'}(S^*>t)\end{aligned} \tag{6.4}$$

for all $t>0$. Dealing with moments of S^* below we first show two auxiliary lemmata:

Lemma 6.1. *$M_{\tau_{i,\zeta}}$ and $(Z_1,Z_2,S_{i,\tau_{i,\zeta}})$ are independent under $I\!\!P_{\lambda,\lambda'}$ for $i=1,2$.*

Proof. The assertion follows directly from our coupling construction, regeneration property (R.3) and the definition of the Z_i, $i=1,2$. ♢

Lemma 6.2. *Let $(\hat{\tau}_{i,n})_{n\ge 0}$, $i=1,2$, and ζ be as defined further above and let $\alpha>0$.*

(a) If $C^+_\lambda(\alpha)<\infty$ *and* $C^+_{\lambda'}(\alpha)<\infty$, *then* $\mathbb{E}_{\lambda,\lambda'}\hat{\tau}^\alpha_{i,\zeta}<\infty$ *for* $i=1,2$.
(b) If $M^+_\lambda(\alpha)<\infty$ *and* $M^+_{\lambda'}(\alpha)<\infty$, *then* $\mathbb{E}_{\lambda,\lambda'}e^{\theta\hat{\tau}_{i,\zeta}}<\infty$ *for some* $\theta\in(0,\alpha]$ *and* $i=1,2$.

Proof. (a) Since ζ has geometrically decreasing tail under $I\!P_{\lambda,\lambda'}$ it is enough to prove $\mathbb{E}_{\lambda,\lambda'}\hat{\tau}^\alpha_{i,n}\le \text{const}\,(n+1)^{\alpha+2}$ for all $n\ge 0$ and $i=0,2$. Notice that $\hat{\tau}_{i,n}-\hat{\tau}_{i,n-1}=\inf\{k\ge 1:\tilde{S}_{i,\hat{\tau}_{i,n-1}+k}-\tilde{S}_{i,\hat{\tau}_{i,n-1}}>W_n-V_{i,n}\}$ is the first passage time $\Phi_{i,n}(W_n-V_{i,n})$ beyond level $W_n-V_{i,n}$ for the ordinary renewal process $(\tilde{S}_{i,\hat{\tau}_{i,n-1}+k}-\tilde{S}_{i,\hat{\tau}_{i,n-1}})_{k\ge 0}$ (independent of $W_n-V_{i,n}$ and with increment distribution $G=P_{F^{m+1}(\cdot|\mathbb{B})}(S_{T_0}\in\cdot)$) and hence a well-studied object. Setting $\Phi(t)\stackrel{\text{def}}{=}\inf\{n\ge 1:\tilde{S}_{0,n}-\tilde{S}_{0,0}>t\}$, we thus have

$$I\!P_{\lambda,\lambda'}(\hat{\tau}_{i,n}-\hat{\tau}_{i,n-1}\in\cdot|\mathfrak{G}_{n-1},W_n=w,V_{i,n}=v)\;=\;I\!P_{\lambda,\lambda'}(\Phi_{i,n}(w-v)\in\cdot)\;=\;I\!P_{\lambda,\lambda'}(\Phi(w-v)\in\cdot)$$

where $\mathfrak{G}_{-1}\stackrel{\text{def}}{=}\sigma(V_{0,0},V_{2,0})$ and

$$\mathfrak{G}_n\;\stackrel{\text{def}}{=}\;\sigma((V_{0,k},V_{2,k})_{0\le k\le n+1},(\hat{\tau}_{i,k})^{i=1,2}_{0\le k\le n-1},(\tilde{S}_{i,k})^{i=0,2}_{0\le k\le\hat{\tau}_{i,n-1}})$$

for $n\ge 0$. Furthermore, $G(1,\infty)=1$ clearly implies $\Phi(w)\le w+1$. Use Proposition 4.1(a) and the subsequent remark to infer $\mathbb{E}_{\lambda,\lambda'}\tilde{S}^\alpha_{i,n}<\infty$ as well as $\mathbb{E}_{\lambda,\lambda'}W^\alpha_n<\infty$ for all $n\ge 0$ and $i=0,2$ from $C^+_\lambda(\alpha)<\infty$ and $C^+_{\lambda'}(\alpha)<\infty$. As shown in [14], III.6, even $\mathbb{E}_{\lambda,\lambda'}W^\alpha_n\le\text{const}\,(n+1)$ holds under these assumptions. Combining these facts, we conclude

$$\begin{aligned}\mathbb{E}_{\lambda,\lambda'}\hat{\tau}^\alpha_{i,n}\;&\le\;(n+1)^\alpha\sum_{k=0}^n\mathbb{E}_{\lambda,\lambda'}(\hat{\tau}_{i,k}-\hat{\tau}_{i,k-1})^\alpha\\&=\;(n+1)^\alpha\sum_{k=0}^n\mathbb{E}_{\lambda,\lambda'}E(\Phi_{i,k}(W_k-V_{i,k})^\alpha|\mathfrak{G}_{k-1})\\&\le\;(n+1)^\alpha\int\mathbb{E}_{\lambda,\lambda'}\Phi(w)^\alpha\;\sum_{k=0}^nI\!P_{\lambda,\lambda'}(W_k\in dw)\\&\le\;(n+1)^\alpha\sum_{k=0}^n\mathbb{E}_{\lambda,\lambda'}(W_k+1)^\alpha\\&\le\;\text{const}(n+1)^\alpha\sum_{k=0}^n(k+1)\;\le\;\text{const}\,(n+1)^{\alpha+2}\end{aligned}$$

as claimed.

(b) Here it suffices to verify $\mathbb{E}_{\lambda,\lambda'}e^{\theta\hat{\tau}_{i,n}}\le g(\theta)^n$ for all sufficiently small $\theta>0$ and a suitable function g satisfying $g(\theta)\to 1$ as $\theta\to 0$. By Proposition 4.1(b), $M^+_\lambda(\alpha)<\infty$ and $M^+_{\lambda'}(\alpha)<\infty$ implies $\mathbb{E}_{\lambda,\lambda'}e^{\theta\tilde{S}_{i,n}}<\infty$ as well as $\mathbb{E}_{\lambda,\lambda'}e^{\theta W_n}<\infty$ for some $\theta\in(0,\alpha]$, all $n\ge 0$ and $i=0,2$. It can further be shown that $E(e^{\theta W_n}|\mathfrak{G}_{n-2})\le g(\theta)$ $I\!P_{\lambda,\lambda'}$-a.s. for all $n\ge 0$, g a function as claimed above and $\mathfrak{G}_{-2}$ the trivial σ-field. Indeed, W_n conditioned upon $(V_{0,n-1},V_{2,n-1})=(v_1,v_2)$ is distributed as the maximum of the two forward recurrence times $\tilde{F}_{t^*+v_1\vee v_2-v_1},\tilde{F}_{t^*+v_1\vee v_2-v_2}$ and the family $e^{\theta(\tilde{F}_{1,t}+\tilde{F}_{2,t})},t\ge 0$ is uniformly integrable, in particular L_1-bounded for all $\theta\in(0,\theta_0]$, $\theta_0>0$. By combining these facts with $\Phi(w)\le w+1$ we obtain

$$\mathbb{E}_{\lambda,\lambda'}e^{\theta\hat{\tau}_{i,n}}\;\le\;\mathbb{E}_{\lambda,\lambda'}e^{\theta(\phi_{i,0}(W_0)+\ldots+\Phi_{i,n}(W_n))}$$

$$
\begin{aligned}
&\le \mathbb{E}_{\lambda,\lambda'} e^{\theta(W_0+\ldots+W_n+n+1)} \\
&\le e^{\theta(n+1)} \mathbb{E}_{\lambda,\lambda'} e^{\theta(W_0+\ldots+W_{n-1})} E(e^{\theta W_n}|\mathfrak{G}_{n-2}) \\
&\le g(\theta) e^{\theta(n+1)} \mathbb{E}_{\lambda,\lambda'} e^{\theta(W_0+\ldots+W_{n-1})} \le \ldots \le \Big(g(\theta)e^{\theta}\Big)^{n+1}
\end{aligned}
$$

for all sufficiently small θ which is the desired conclusion. ◇

Now we are ready to prove

PROPOSITION 6.3. *Let S^* be as defined above.*
(a) For each $\alpha \ge 1$, $C_\lambda(\alpha) < \infty$ and $C_{\lambda'}(\alpha) < \infty$ imply $\mathbb{E}_{\lambda,\lambda'}(S^*)^\alpha < \infty$.
(b) For each $\alpha > 0$, $M_\lambda(\alpha) < \infty$ and $M_{\lambda'}(\alpha) < \infty$ imply $\mathbb{E}_{\lambda,\lambda'} e^{\theta S^*} < \infty$ for some $\theta \in (0,\alpha]$.

PROOF. This time we only prove (a). Setting $S_{i,n}^{(+)} \stackrel{\text{def}}{=} \sum_{k=0}^n X_{i,k}^+$, we have

$$
\begin{aligned}
S^* &\le Z_1 + Z_2 + S_{1,\tau_{1,\zeta}+\tau^*(Z_1\vee Z_2)} - S_{1,\tau_{1,\zeta}} \\
&\le S_{1,\tau_{1,\zeta}}^{(+)} + S_{2,\tau_{2,\zeta}}^{(+)} + (S_{1,\tau_{1,\zeta}+\tau^*(Z_1\vee Z_2)} - S_{1,\tau_{1,\zeta}}) + 2n_0 t_0
\end{aligned}
\tag{6.5}
$$

Observe that $S_{1,\tau_{1,\zeta}}^{(+)} \le \sum_{k=0}^{\hat\tau_{1,\zeta}} Y_{1,k}$, where $Y_{1,k} \stackrel{\text{def}}{=} \sum_{j=T_{i,k-1}}^{T_{i,k}-n_0} X_{1,j}^+ + n_0 t_0$. Under $I\!P_{\lambda,\lambda'}$, the latter variables are independent for $k \ge 0$ and identically distributed for $k \ge 1$ as $S_{T_0-n_0}^{(+)} + n_0 t_0$ under $P_{F^{m+1}(\cdot|\mathbb{B})}$. Moreover, $Y_{1,0},\ldots,Y_{1,n}$ are $\mathfrak{F}_{1,T_{1,n}}$-measurable, $(Y_{1,k})_{k>n}$ is independent of $\mathfrak{F}_{1,T_{1,n}}$ and $\hat\tau_{1,\zeta}$ a stopping time with respect to $(\mathfrak{F}_{1,T_{1,n}})_{n\ge 0}$. Consequently, we infer from Theorem I.5.2 in [10] that

$$
\begin{aligned}
\mathbb{E}_{\lambda,\lambda'}(S_{1,\tau_{1,\zeta}}^{(+)})^\alpha &\le \text{const}\left(\mathbb{E}_{\lambda,\lambda'} Y_{1,0}^\alpha + \mathbb{E}_{\lambda,\lambda'}\left(\sum_{k=1}^{\hat\tau_{1,\zeta}} Y_{1,k}\right)^\alpha\right) \\
&\le \text{const}\Big(E_\lambda(S_{T_0}^{(+)} + n_0 t_0)^\alpha + E_{F^{m+1}(\cdot|\mathbb{B})}(S_{T_0}^{(+)} + n_0 t_0)^\alpha\, \mathbb{E}_{\lambda,\lambda'} \hat\tau_{1,\zeta}^\alpha\Big).
\end{aligned}
$$

We get $\mathbb{E}_{\lambda,\lambda'}\hat\tau_{1,\zeta}^\alpha < \infty$ by Lemma 6.2. $C_\lambda^-(\alpha) < \infty$ yields $E_\nu T_0^\alpha < \infty$ for $\nu \in \{\lambda, F^{m+1}(\cdot|\mathbb{B})\}$ (see (4.8)) and then together with $C_\lambda^+(\alpha) < \infty$ also $E_{F^{m+1}(\cdot|\mathbb{B})}(S_{T_0}^{(+)})^\alpha < \infty$, by Theorem 1.3(ii) in [11]. It is this conclusion which needs the stronger $C_\lambda(\alpha) < \infty$ instead of $C_\lambda^+(\alpha) < \infty$. Clearly, the same arguments show $\mathbb{E}_{\lambda,\lambda'}(S_{2,\tau_{2,\zeta}}^{(+)})^\alpha < \infty$ under $C_{\lambda'}(\alpha) < \infty$. Hence, in view of (6.5), it remains to prove

$$
\mathbb{E}_{\lambda,\lambda'}(S_{1,\tau_{1,\zeta}+\tau^*(Z_1\vee Z_2)} - S_{1,\tau_{1,\zeta}})^\alpha < \infty
$$

under $C_\lambda(\alpha) + C_{\lambda'}(\alpha) < \infty$. Lemma 6.1 and the strong Markov property lead to

$$
\begin{aligned}
&\mathbb{E}_{\lambda,\lambda'}(S_{1,\tau_{1,\zeta}+\tau^*(Z_1\vee Z_2)} - S_{1,\tau_{1,\zeta}})^\alpha \\
&\quad = \int_{I\!R}\int_{\mathcal{S}^{m+1}} E_x S_{\tau(z)}^\alpha\; I\!P_{\lambda,\lambda'}(M_{\tau_{i,\zeta}} \in dx)\; I\!P_{\lambda,\lambda'}(Z_1 \vee Z_2 \in dz) \\
&\quad = \int_{I\!R} E_{F^{m+1}(\cdot|\mathbb{B})} S_{\tau(z)}^\alpha\; I\!P_{\lambda,\lambda'}(Z_1 \vee Z_2 \in dz)
\end{aligned}
$$

where $\tau(t) = \inf\{n \ge 0 : S_n > t\}$. Now use Proposition 4.8 for

$$E_{F^{m+1}(\cdot|\mathbb{B})} S^\alpha_{\tau(z)} \ \le\ \text{const}\,(z+1)^\alpha$$

whence

$$\begin{aligned}\mathbb{E}_{\lambda,\lambda'}(S_{1,\tau_{1,\varsigma}+\tau^*(Z_1\vee Z_2)} - S_{1,\tau_{1,\varsigma}})^\alpha \ &\le\ \text{const}\,\mathbb{E}_{\lambda,\lambda'}(Z_1 \vee Z_2 + 1)^\alpha \\ &\le\ \text{const}\,\mathbb{E}_{\lambda,\lambda'}(S^{(+)}_{1,\tau_{1,\varsigma}} \vee S^{(+)}_{2,\tau_{2,\varsigma}} + 1)^\alpha \ <\ \infty.\end{aligned}$$

◊

Proof of Theorems 2.4 and 2.5. Recall from Section 2 that $U_{\lambda|J} = U_\lambda(\cdot\cap(\mathcal{S}^{m+1}\times J))$ for intervals $J \subset I\!R$. Let $I = (0,1]$. By Corollary 3.4, for all distributions λ on $\mathcal{S}^{m+1} \times I\!R$

$$\sup_{t\in I\!R} U^>_\lambda(\mathcal{S}^{m+1}\times t+I) \ \le\ \sup_{t\in I\!R} U_\lambda(\mathcal{S}^{m+1}\times t+I) \ \stackrel{\text{def}}{=}\ H(1) \ <\ \infty. \tag{6.6}$$

Put $\tau^>(t) = \inf\{n \ge 0 : S^>_n > t\}$ and notice $R_t = S^>_{\tau^>(t)} - t$ as well as

$$\sum_{n\ge0} \mathbf{1}_{\{M^>_n\in\cdot,S^>_n\in t+J\}} \ =\ \sum_{n\ge0} \mathbf{1}_{\{M^>_{\tau^>(t)+n}\in\cdot,R_t+(S^>_{\tau^>(t)+n}-S^>_{\tau^>(t)})\in J\}}$$

for every $J \subset (0,\infty)$. Using this, the strong Markov property, (6.4) and (6.6), we infer

$$\begin{aligned}\|U^>_{\lambda|t+I} - U^>_{\lambda'|t+I}\| \ &=\ \|U^>_{\lambda_t|I} - U^>_{\lambda'_t|I}\| \\ &=\ \left\| \int_{\mathcal{S}^{m+1}\times(0,\infty)} U^>_{s,x|I}\ (\lambda_t - \lambda'_t)(ds,dx)\right\| \\ &\le\ H(1)\,\|\lambda_t - \lambda'_t\| \\ &\le\ H(1)\,I\!P_{\lambda,\lambda'}(S^* > t)\end{aligned} \tag{6.7}$$

for all $t > 0$ and then further

$$\begin{aligned}\|U^>_{\lambda|(t,\infty)} - U^>_{\lambda'|(t,\infty)}\| \ &\le\ \sum_{n\ge0} \|U^>_{\lambda|t+n+I} - U^>_{\lambda'|t+n+I}\| \\ &\le\ H(1)\sum_{n\ge0} P_{\lambda,\lambda'}(S^* > t+n) \\ &\le\ H(1)\sum_{n\ge0}\int_{t+n-1}^{t+n} P_{\lambda,\lambda'}(S^* > s)\ ds \\ &=\ H(1)\int_{(t-1,\infty)} P_{\lambda,\lambda'}(S^* > s)\ ds\end{aligned} \tag{6.8}$$

for all initial distributions λ, λ' on $\mathcal{S}^{m+1} \times I\!R$. All assertions are now easily verified when combining (6.7) and (6.8) with the moment results of Proposition 6.3. We thus omit further details. ◊

Proof of Theorems 2.6 and 2.7. (a) Given λ, λ' with $C^-_\lambda(\alpha) < \infty$ and $C^-_{\lambda'}(\alpha) < \infty$, consider the following coupling model: Let $Y'_{-m}, ..., Y'_0, Y_{-m}, ..., Y_0, ...$ be $(\mathcal{S}, \mathfrak{S})$-valued random variables on a probability space $(\Omega, \mathfrak{A}, I\!P_{\lambda,\lambda'})$ such that

$$I\!P_{\lambda,\lambda'}^{((M_0,S_0),(M_0',S_0'),(Y_n)_{n\ge 1})} \;=\; \lambda\otimes\lambda'\otimes F^{\infty}.$$

where $M_0 \stackrel{\rm def}{=} (Y_{-m},...,Y_0)$ and $M_0' \stackrel{\rm def}{=} (Y_{-m}',...,Y_0')$. Put further

$$\begin{aligned} M_1 &= (Y_{-m+1},...,Y_1), \quad M_1' = (Y_{-m+1}',...,Y_0',Y_1) \\ &\vdots \\ M_m &= (Y_0,...,Y_m), \quad M_m' = (Y_0',Y_1,...,Y_m) \\ M_n &= M_n' = (Y_{n-m},...,Y_n) \quad \text{for } n\ge m+1 \end{aligned}$$

and then

$$X_n \stackrel{\rm def}{=} \varphi(M_n), \quad X_n' = \varphi(M_n') \quad \text{for } n\ge 1.$$

Obviously, $(M_n,S_n)_{n\ge 0}$ and $(M_n',S_n)_{n\ge 0}$ are (φ,F)-md MRWs with initial distributions λ,λ', respectively, under $I\!P_{\lambda,\lambda'}$ and

$$(M_n,X_n)_{n\ge m+1} \;=\; (M_n',X_n')_{n\ge m+1}.$$

The ladder epoch MRWs of $(M_n,S_n)_{n\ge 0}$ and $(M_n',S_n')_{n\ge 0}$ are denoted by $(M_n^>,\sigma_n)_{n\ge 0}$ and $(M_n^{>\prime},\sigma_n')_{n\ge 0}$, respectively, where $\sigma_0=\sigma_0'=0$ should be recalled. Let further $(R_n)_{n\ge 0}$ and $(R_n')_{n\ge 0}$ be the associated sequences of forward recurrence times and put $\lambda_n \stackrel{\rm def}{=} I\!P_{\lambda,\lambda'}(R_n\in\cdot)$, $\lambda_n' \stackrel{\rm def}{=} I\!P_{\lambda,\lambda'}(R_n'\in\cdot)$.

Since the first regeneration time T_0 as constructed in Section 3 does not depend on the first $m{+}1$ values of $(M_n,X_n)_{n\ge 0}$ it can here be defined in such a way that it is a regeneration time for both chains $(M_n,X_n)_{n\ge 0}$ and $(M_n',X_n')_{n\ge 0}$. As a consequence, $(M_n,M_n',X_n,X_n')_{0\le n\le T_0-n_0}$ and $(M_{T_0},M_{T_0'})\sim F^{2m+2}(\cdot|\mathbb{B})$ are independent. The important observation is now that the downward excursions $\max_{0\le n\le T_0}S_n-S_{T_0}, \max_{0\le n\le T_0}S_n'-S_{T_0}'$ of the two MRWs at T_0 are both bounded by

$$S_{T_0}^{(-)}\vee S_{T_0}^{(-)'} \;=\; \sum_{n=0}^{T_0}X_n^-\vee\sum_{n=0}^{T_0}X_n'^{\,-}$$

which, by (R.4), is further bounded by

$$Z \;\stackrel{\rm def}{=}\; S_{T_0-n_0}^{(-)}\vee S_{T_0-n_0}^{(-)'}+2n_0t_0.$$

Consequently, a joint ladder epoch occurs at $T_0+\Phi(Z)$ where

$$\Phi(t) \;\stackrel{\rm def}{=}\; \inf\{n\ge 0: S_{T_0+n}-S_{T_0}>t\},$$

and leads to the conclusion that

$$(M_n,R_n)_{n\ge T_0+\Phi(Z)} \;=\; (M_n',R_n')_{n\ge T_0+\Phi(Z)}$$

and thereby to (compare (6.7) and (6.8))

$$\sup_{n\in\mathbb{N}_0} |V_\lambda^>\{n\} - V_{\lambda'}^>\{n\}| \;\le\; \mathbb{P}_{\lambda,\lambda'}(T_0+\Phi(Z)>n), \tag{6.9}$$

$$\|V_{\lambda|[n,\infty)}^> - V_{\lambda'|[n,\infty)}^>\| \;\le\; \sum_{k\ge n} \mathbb{P}_{\lambda,\lambda'}(T_0+\Phi(Z)>k) \tag{6.10}$$

for all $n\ge 0$. Instead of (6.6) we have used here the trivial inequality

$$\sup_{n\in\mathbb{N}_0} V_\lambda^>\{n\} \;\le\; 1.$$

The proof is now obviously completed by providing suitable moment results for $T_0+\Phi(Z)$. Since the distribution of T_0 is always geometrically bounded (Lemma 3.5), only $\Phi(Z)$ remains to be considered. But the independence of Z and $(M_{T_0+n}, S_{T_0+n}-S_{T_0})_{n\ge 0} = (M'_{T_0+n}, S'_{T_0+n} - S'_{T_0})_{n\ge 0}$ in combination with Proposition 4.8(b) (if $C_\lambda^-(\alpha)+C_{\lambda'}^-(\alpha)<\infty$) gives

$$\mathbb{E}_{\lambda,\lambda'}\Phi(Z)^\alpha \;=\; \int E_{F^{m+1}(\cdot|\mathbb{B})}\tau(z)^\alpha\; \mathbb{P}_{\lambda,\lambda'}(Z\in dz) \;\le\; \text{const}\,\mathbb{E}_{\lambda,\lambda'}Z^\alpha$$

and a similar inequality for $\mathbb{E}_{\lambda,\lambda'}e^{\theta\Phi(Z)}$, $\theta>0$ if $M_\lambda^-(\alpha)+M_{\lambda'}^-(\alpha)<\infty$. The assertions of Theorem 2.6(a),(c) and 2.7(a) are now easily verified because, by Proposition 4.1, $C_\lambda^-(\alpha)+C_{\lambda'}^-(\alpha)<\infty$ further implies $\mathbb{E}_{\lambda,\lambda'}Z^\alpha<\infty$ and $M_\lambda^-(\alpha)+M_{\lambda'}^-(\alpha)<\infty$ further implies $\mathbb{E}_{\lambda,\lambda'}e^{\theta Z}<\infty$ for sufficiently small $\theta>0$.

(b),(d),(e) The use the former coupling construction for the comparison of $V_\lambda^>$ with $V_*^> = \vartheta^{-1}\xi^*\otimes\lambda\!\!\lambda_1^+$ requires a modification of the previous arguments. The first step is to define a distribution λ' on $\mathcal{S}^{m+1}\times\mathbb{R}$ such that $P_{\lambda'}((M_\rho,\rho)\in\cdot) = \phi^s$ for a suitable stopping time ρ satisfying $P_{\lambda'}(\rho\in\{\sigma_n : n\ge 0\})=1$. We define

$$\lambda'(C) \;\stackrel{\text{def}}{=}\; \vartheta^{-1}E_{\xi^*}\left(\sum_{n=0}^{\sigma_1-1}\mathbf{1}_{\{(M_n,S_n)\in C\}}\right)$$

for $C\in\mathfrak{S}^{m+1}\otimes\mathfrak{B}$ and claim that $\rho=\tau(-S_0)$ has the desired properties. Since $P_{\lambda'}(S_0\le 0)=1$, ρ is indeed a ladder epoch for $(M_n,S_n)_{n\ge 0}$. Moreover,

$$\begin{aligned}
P_{\lambda'}(M_\rho\in A,\rho=k) \;&=\; \vartheta^{-1}E_{\xi^*}\left(\sum_{n=0}^{\sigma_1-1}\mathbf{1}_{\{M_{n+k}\in A,\tau(-S_n)=k\}}\right)\\
&=\; \vartheta^{-1}E_{\xi^*}\left(\sum_{n=0}^{\sigma_1-1}\mathbf{1}_{\{M_{n+k}\in A,\sigma_1-n=k\}}\right)\\
&=\; \vartheta^{-1}E_{\xi^*}\left(\sum_{n=0}^{\sigma_1-1}\mathbf{1}_{\{M_{\sigma_1}\in A,\sigma_1=n+k\}}\right)\\
&=\; \vartheta^{-1}\sum_{n\ge 0}P_{\xi^*}(\sigma_1>n, M_{\sigma_1}\in A,\sigma_1=n+k)\\
&=\; \vartheta^{-1}P_{\xi^*}(M_1^>\in A,\sigma_1\ge k) \;=\; \phi^s(A\times\{k\})
\end{aligned} \tag{6.11}$$

for all $A\in\mathfrak{S}^{m+1}$ and $k\in\mathbb{N}$ which proves the other asserted property of ρ.

The second step is to verify that $E(X_1^-)^{\alpha+1}<\infty$ implies $C_{\lambda'}^-(\alpha)<\infty$. Since $\lambda'(\cdot\times I\!R)=F^{m+1}$ and $P_{\lambda'}((X_n)_{n\ge 1}\in\cdot)=P_{\lambda'(\cdot\times I\!R)}((X_n)_{n\ge 1}\in\cdot)$, we infer $C_{\lambda'}^-(\alpha)=E_{\lambda'}(S_0^-)^\alpha\vee E(X_1^-)^\alpha$. It hence remains to show $E_{\lambda'}(S_0^-)^\alpha<\infty$ providing $E(X_1^-)^{\alpha+1}<\infty$. Note that the latter implies $C_{\xi^*}^-(\alpha+1)<\infty$ for $\xi^*\le\vartheta F^{m+1}$. The definition of λ' gives

$$E_{\lambda'}(S_0^-)^\alpha \;=\; \vartheta^{-1}E_{\xi^*}\left(\sum_{n=0}^{\sigma_1-1}(S_n^-)^\alpha\right)$$

and this is indeed finite under the former condition by Lemma 4.9(a). The same type of argument shows that $Ee^{\alpha X_1^-}<\infty$ for some $\alpha>0$ implies $M_{\lambda'}^-(\theta)<\infty$ for some $\theta\in(0,\alpha]$.

Now one can use Theorem 2.6(a),(c) and 2.7(a) to infer the assertions of all other parts of these theorems, however, with $V_*^>=\vartheta^{-1}\xi^*\otimes\lambda\!\!\lambda_1$ replaced by $V_{\lambda'}^>$. What hence remains to be done in order to get the same results without this replacement is to show (as $n\to\infty$)

$$|V_{\lambda'}^>\{n\}-V_*^>\{n\}| \;=\; o(n^{-\alpha}),\quad \text{respectively}\quad o(e^{-\theta n})\text{ for some }\theta>0, \tag{6.12}$$

providing $E(X_1^-)^{\alpha+1}<\infty$, respectively $Ee^{\alpha X_1^-}<\infty$ for some $\alpha>0$. But

$$V_{\lambda'}^>-V_*^> \;\le\; E_{\lambda'}\left(\sum_{k=0}^{\rho-1}\mathbf{1}_{\{(M_k,k)\in\cdot\}}\right)$$

and therefore

$$0\le\; V_{\lambda'}^>\{n\}-V_*^>\{n\} \;=\; E_{\lambda'}\left(\sum_{k=0}^{\rho-1}\mathbf{1}_{\{k=n\}}\right) \;=\; P_{\lambda'}(\rho>n).$$

Use (6.11) with $A=\mathcal{S}^{m+1}$ to see that

$$E_{\lambda'}\rho^\alpha \;\le\; \text{const}\,E_{\xi^*}\sigma_1^{\alpha+1} \;<\; \infty$$

by Proposition 4.8(b) for $E(X_1^-)^{\alpha+1}<\infty$ also gives $C_{\xi^*}^-(\alpha+1)<\infty$. A similar argument shows $E_{\lambda'}e^{\theta\rho}<\infty$ for some $\theta>0$ if $Ee^{\alpha X_1^-}<\infty$ for some $\alpha>0$. (6.12) is now a trivial consequence. $\diamondsuit$

Appendix

We finally want to collect some basic facts from standard renewal theory that have been used somewhere before. Let $(S_n)_{n\ge 0}$ be an ordinary random walk with i.i.d. increments $X_1,X_2,\ldots$ having positive mean μ and a delay S_0 which is independent of $(X_n)_{n\ge 1}$. Let G be the increment distribution and λ that of S_0 under P_λ, also called initial distribution of $(S_n)_{n\ge 0}$. We only write P for P_0. The renewal measure of $(S_n)_{n\ge 0}$ under P_λ is denoted by $\mathbb{U}_\lambda$, i.e. $\mathbb{U}_\lambda=\lambda*\mathbb{U}$ with $\mathbb{U}\overset{\text{def}}{=}\sum_{n\ge 0}G^{*(n)}$. Let $(\sigma_n,S_n^>)_{n\ge 0}$ be the sequence of strictly ascending ladder epochs and ladder heights associated with $(S_n)_{n\ge 0}$ and put $\mu^>\overset{\text{def}}{=}ES_1^>,\mathbb{U}^>\overset{\text{def}}{=}\sum_{n\ge 0}P(S_n^>\in\cdot)$ and $\mathbb{U}_\lambda^>\overset{\text{def}}{=}\lambda*\mathbb{U}^>$, the renewal measure of $(S_n^>)_{n\ge 0}$ under P_λ.

Suppose $(S_n)_{n\geq 0}$, and thus also $(S_n^>)_{n\geq 0}$, is 1-arithmetic or spread-out. As usual, we consider without further notice only initial distributions λ on $\mathbb{Z}$ in the 1-arithmetic case. By using a coupling of forward recurrence times (to some extent described in Section 6) and the inequality

$$\sup_{t\geq 0} \mathbb{U}^>(t+I) \;\leq\; \mathbb{U}^>(I), \tag{A.1}$$

one can show that

$$\|\mathbb{U}^>_{\lambda|t+I} - \mathbb{U}^>_{\lambda'|t+I}\| \;\leq\; \mathbb{U}^>(I)\mathbb{P}_{\lambda,\lambda'}(T>t) \tag{A.2}$$

where $\mathbb{P}_{\lambda,\lambda'}$ is the underlying probability measure in a suitable coupling model, T the coupling time and $I=[0,1]$. Provided $E_\nu(S_0^+)^\alpha<\infty$ for $\nu\in\{\lambda,\lambda'\}$ and $E(X_1^+)^\alpha<\infty$, it can be shown that $\mathbb{E}_{\lambda,\lambda'}T^\alpha<\infty$.

In order to get a similar bound for $\|\mathbb{U}_\lambda-\mathbb{U}_{\lambda'}\|$ we first note that

$$\mathbb{U} \;=\; \mathbb{U}^{\sigma_1} * \mathbb{U}^>, \quad \mathbb{U}^{\sigma_1} \;\stackrel{\text{def}}{=}\; E_0\left(\sum_{n=0}^{\sigma_1-1} \mathbf{1}_{\{S_n\in\cdot\}}\right). \tag{A.3}$$

Moreover, letting $Z\stackrel{\text{def}}{=}\min_{n\geq 0}(S_n-S_0)$ and $\vartheta=E\sigma_1$,

$$P_\lambda(Z\in\cdot) \;=\; \vartheta^{-1}\mathbb{U}^{\sigma_1}, \tag{A.4}$$

for every λ, see [13], Lemma 2, so that

$$\mathbb{U} \;=\; \vartheta E\mathbb{U}^>(\cdot - Z) \quad \text{and} \quad \mathbb{U}_\lambda \;=\; \vartheta E\mathbb{U}^>_\lambda(\cdot - Z). \tag{A.5}$$

By using this in (A.2), we obtain

$$\begin{aligned} \|\mathbb{U}_{\lambda|t+I} - \mathbb{U}_{\lambda'|t+I}\| \;&\leq\; \vartheta E\Big(\|\mathbb{U}^>_{\lambda|t-Z+I} - \mathbb{U}^>_{\lambda'|t-Z+I}\|\Big) \\ &\leq\; \vartheta\mathbb{U}^>(I)\mathbb{P}_{\lambda,\lambda'}(T+\hat{Z}>t) \;\leq\; \vartheta\mathbb{U}^>(I)\mathbb{P}_{\lambda,\lambda'}(T>t), \end{aligned} \tag{A.6}$$

where $\hat{Z}$ is a copy of Z independent of the coupling time T. Hence we get the same coupling bound and thereby the same convergence rate results as in (A.2) if t tends to ∞.

Since $E|Z|^\alpha<\infty$ iff $E(X_1^-)^{\alpha+1}<\infty$, see e.g. Theorem IV.4.9 in [10], and by using (A.1), (A.5) further yields an appropriate estimate for the convergence of $\mathbb{U}_\lambda(t+I)$ to 0 as t tends to $-\infty$. Indeed,

$$\begin{aligned} \mathbb{U}_\lambda(t+I) \;&\leq\; \vartheta E\mathbb{U}^>_\lambda(t-Z+I) \;=\; \vartheta E_\lambda\mathbb{U}^>(t-Z-S_0+I) \\ &\leq\; \vartheta\mathbb{U}^>(I)P_\lambda(Z+S_0<t+1), \end{aligned} \tag{A.7}$$

and the final probability is of order $o(|t|^{-\alpha})$ as $t\to-\infty$ if $E(X_1^-)^{\alpha+1}<\infty$ and $E_\lambda(S_0^-)^\alpha<\infty$.

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