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Double Martingale Structure and Existence of ϕ-Moments for Weighted Branching Processes

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Given the infinite Ulam-Harris tree $\mathbb{V} = \cup_{n\geq 0}\mathbb{N}^n$, let $T(v) = (T_i(v))_{i\geq 1}$, $v \in \mathbb{V}$, be a familiy of i.i.d. nonnegative random vectors with generic copy $(T_i)_{i\geq 1}$. Interpret $T_i(v)$ as a weight attached to the edge connecting the nodes v and vi in the tree. Define $L(v)$ as the branch weight for the unique path from the root to v obtained by multiplication of the edge weights. The associated weighted branching process (WBP) is then given by $Z_n \stackrel{\text{def}}{=} \sum_{|v|=n} L(v)$, $n \geq 0$, and forms a nonnegative martingale with a.s. limit W under the normalization assumption $\sum_{i\geq 1} \mathbb{E}T_i = 1$. For regularly varying functions $\phi(x) = x^\alpha \ell(x)$ of order $\alpha \geq 1$ satisfying $\lim_{x\to\infty} x^{-1}\phi(x) = \infty$, the paper provides necessary and sufficient conditions on $(T_i)_{i\geq 1}$ for $\mathbb{E}\phi(W)$ being positive and finite. The double martingale structure of $(Z_n)_{n\geq 0}$ first observed and utilized in [5] for similar results for Galton-Watson processes forms a major tool in our analysis. It further requires results following from the connection between a WBP and an associated random walk and drawing on results from renewal theory. In particular, a pathwise renewal theorem is proved which may also be of interest in its own right.

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1. Introduction and Main Results

The *weighted branching process (WBP)*, first introduced by Rösler [50] in the form defined below, may be viewed as a generalization of the classical Galton-Watson process (GWP). In the case of nonnegative weights (which will be assumed throughout this paper) it is also known under the name *"multiplicative cascade"* (see [43] and the references therein and also [10] for an interesting vector extension), and it is the multiplicative version of the *branching random walk* [15-17] obtained after an exponential transform. This equivalence seems to be sometimes obscured by different viewpoints and formalisms which in turn have stimulated interest in different results and generalizations. Interest in WBP's does not only stem from their natural relevance in the general theory of branching processes but also because they occur in many stochastic models ranging from recursive algorithms and data structures [49], [52], random Cantor sets [47] and infinite particle systems [31] to interval splitting schemes [25], [11] and fragmentation processes [12-14]. These models share the existence of a homogeneous branching mechanism which entails that asymptotic distributions of relevant quantities are often described by a certain type of stochastic fixed-point equation (see (1.9) below) which has therefore been analyzed in a series of papers, see e.g. [26], [29], [41], [50], [51], [22]. WBP's form an important ingredient in many of these works because the a.s. limit of a normalized WBP is a particular solution to such an equation.

The present paper addresses the problem of finding conditions that ensure the existence of certain moments of this limit. Our main results, Theorems 1.2–4, provide necessary and sufficient conditions for the existence of ϕ-moments (beyond $\mathfrak{L}^1$) when ϕ is from a very general class of regularly varying function including, of course, the $\mathfrak{L}^\alpha$-case $\phi_\alpha(x) \stackrel{\text{def}}{=} x^\alpha$ for $\alpha > 1$. For a GWP, these results were obtained more than 30 years ago by Bingham and Doney [23] using analytic methods and again quite recently by the first author and Rösler [5] via a different approach based on a certain *double martingale structure* which is also inherent in WBP's and playing a key role here. As a fruitful and crucial ingredient, this latter approach allows the double use of certain powerful convex function inequalities for martingales. However, to extend the arguments from [5] so as to encompass WBP's as well requires a considerable amount of additional work due to a more complicated double martingale structure. Further explanations will be given below. Additional relevant references containing related but weaker results are the second author's dissertation [38], a paper by Iksanov [32] covering the $\mathfrak{L}^\alpha$-case, and another one by Iksanov und Rösler [33]. The techniques in the last two references are quite different from ours and based on size-biasing and a connection to perpetuities.

Model description. For the definition of a WBP consider the infinite Ulam-Harris tree $\mathbb{V}$ with vertex set $\cup_{n\geq 0}\mathbb{N}^n$ where $\mathbb{N} = \{1, 2, ...\}$ denotes the set of positive integers and $\mathbb{N}^0 \stackrel{\text{def}}{=} \{\emptyset\}$ by convention. Each vertex $v = (v_1, ..., v_n)$ of length $|v| = n$, shortly written as $v_1v_2...v_n$ hereafter, is uniquely connected to the root $\emptyset$ by the path $v|0 \stackrel{\text{def}}{=} \emptyset \to v|1 \to v|2 \to ... \to v|n = v$, where $v|k \stackrel{\text{def}}{=} v_1...v_k$ for $1 \leq k \leq n$. If $w = w_1...w_m$ denotes another vertex we write vw for the concatenation of v and w, i.e. for $v_1...v_nw_1...w_m$. In the context of branching processes v is interpreted as a (potential) individual of the n-th generation. It is the mother of the successors $vi \stackrel{\text{def}}{=} v_1...v_ni$, $i \in \mathbb{N}$, called children, and an ancestor of any vw, $w \in \mathbb{V}$. In places where it occurs $v_1...v_n \stackrel{\text{def}}{=} \emptyset$ is stipulated whenever $n = 0$. Now let $T(v) = (T_i(v))_{i\geq 1}$, $v \in \mathbb{V}$, be a family of i.i.d. infinite random vectors consisting of nonnegative components. A generic copy of these vectors is denoted by $T = (T_i)_{i\geq 1}$ and called generic weight vector. Define $L(\emptyset) \stackrel{\text{def}}{=} 1$ and recursively

$$L(vi) \ \stackrel{\text{def}}{=} \ L(v)T_i(v)$$

for $v = v_1...v_n \in \mathbb{V}$ and $i \in \mathbb{N}$, thus

$$L(v) \ = \ \prod_{j=1}^{n} T_{v_j}(v_1...v_{j-1}).$$

We interpret $T_i(v)$ as a weight attached to the edge connecting v and vi. Then $L(v)$ forms the total weight of the branch from the root to v accumulated under multiplication of the edge weights. Given such a weighted branching model, the associated WBP is defined as

$$Z_n \ \stackrel{\text{def}}{=} \ \sum_{|v|=n} L(v), \quad n \in \mathbb{N}_0, \tag{1.1}$$

and forms a sequence of nonnegative random variables. The simple GWP yields as a special case, namely when $\mathbb{P}(T \in \{0,1\}^{\mathbb{N}}$ and $N < \infty) = 1$, where $N \stackrel{\text{def}}{=} \sum_{i\geq 1} \mathbf{1}_{\{T_i>0\}}$. If Z_1 is integrable with $\mu \stackrel{\text{def}}{=} \mathbb{E}Z_1 = \sum_{i\geq 1} \mathbb{E}T_i$, then all Z_n are so as well and $\mathbb{E}Z_n = \mu^n$ for all $n \geq 0$. Moreover, the normalization

$$W_n \ \stackrel{\text{def}}{=} \ \mu^{-n}Z_n, \quad n \geq 0, \tag{1.2}$$

constitutes a nonnegative martingale with respect to the filtration

$$\mathcal{F}_n \ \stackrel{\text{def}}{=} \ \sigma(L(\emptyset), T(v), |v| < n), \quad n \geq 0, \tag{1.3}$$

and is hence a.s. convergent with limit W having expectation $\mathbb{E}W \leq 1$ (by Fatou's lemma). It is this martingale on which we will focus here by addressing the problem of finding necessary and sufficient conditions on T such that, for a suitable class of regularly varying functions ϕ with $\lim_{x\to\infty} \frac{\phi(x)}{x} = \infty$,

$$0 \ < \ \mathbb{E}\phi(W) \ < \ \infty$$

holds true. For a supercritical GWP, this has been done in [5] the result being that a certain moment condition on Z_1 is equivalent. If $\phi(x) = x$, the latter equals the well-known $(L \log L)$-condition "$\mathbb{E}Z_1 \log Z_1 < \infty$", and the result reduces to the famous Kesten-Stigum theorem [7, Thm. II.2.1]. Theorem 1.1 below provides an extension of this result to WBP.

Let us finally note that, for any $\alpha \geq 0$, the sequence

$$Z_n^{(\alpha)} \ \stackrel{\text{def}}{=} \ \sum_{|v|=n} L(v)^{\alpha}, \quad n \geq 0,$$

is the WBP pertinent to the weighted branching model based on the weight family $(T(v)^\alpha)_{v\in\mathbb{V}}$, where $T(v)^\alpha \stackrel{\text{def}}{=} (T_i(v)^\alpha)_{i\geq 1}$. It forms a super-, respectively submartingale if

$$g(\alpha) \stackrel{\text{def}}{=} \mathbb{E}Z_1^{(\alpha)} = \sum_{i\geq 1}\mathbb{E}T_i^\alpha \quad (\mu = g(1))$$

is ≤ 1, respectively $\in [1,\infty)$. In the special case $\alpha = 0$ (with convention $0^0 \stackrel{\text{def}}{=} 0$) we have that

$$Z_n^{(0)} = \sum_{|v|=n}\mathbf{1}_{\{L(v)>0\}}, \quad n\geq 0 \quad (\Rightarrow N \stackrel{d}{=} Z_1^{(0)})$$

forms an ordinary Galton-Watson process, and we denote by q its extinction probability, also given as the smallest root in [0,1] of the offspring generating function $s \mapsto \mathbb{E}s^{Z_1^{(0)}}$.

Standing assumptions. Notice that $(W_n)_{n\geq 0}$ is again a WBP, the generic weight vector being $(\mu^{-1}T_i)_{i\geq 1}$. It is therefore no loss of generality to make the standing assumption

$$\mu = g(1) = \mathbb{E}Z_1 = \sum_{i\geq 1}\mathbb{E}T_i = 1 \tag{C1}$$

hereafter, thus $W_n = Z_n$ for all $n \geq 0$. The study of ϕ-moments of W clearly makes sense only if $\mathbb{E}W > 0$ which is therefore to be guaranteed at the outset by imposing suitable conditions on T. In the Galton-Watson case the Kesten-Stigum theorem provides us with the dichotomy $\mathbb{E}W = 0$ or $= 1$, with the latter being true iff $\mathbb{E}Z_1\log^+ Z_1 < \infty$. The situation is more complicated in the present situation, but the subsequent theorem, cited from [37], justifies that

$$-\infty < \gamma \stackrel{\text{def}}{=} \sum_{i\geq 1}\mathbb{E}T_i\log T_i < 0 \quad \text{and} \quad \mathbb{E}Z_1\log^+ Z_1 < \infty. \tag{C2}$$

constitutes an appropriate standing assumption for our further analysis.

Theorem 1.1. [37, Theorem 2.7] *Let $(Z_n)_{n\geq 0}$ be a WBP with $\mathbb{E}Z_1 = 1$ and put*

$$\gamma^\pm \stackrel{\text{def}}{=} \sum_{i\geq 1}\mathbb{E}T_i\log^\pm T_i \quad (\Rightarrow \gamma = \gamma^+ - \gamma^-).$$

(a) If $\kappa \stackrel{\text{def}}{=} \mathbb{P}(T\in\{0,1\}^{\mathbb{N}}) < 1$ and $\gamma^- < \infty$, then the following assertions are equivalent:
  - *(i) $\mathbb{P}(W > 0) > 0$;*
  - *(ii) $\mathbb{E}W = 1$;*
  - *(iii) $\mathbb{E}Z_1\log^+ Z_1 < \infty$ and $\gamma < 0$.*

(b) If $\kappa = 1$ and $\mathbb{P}(Z_1 = 1) = 1$, then $W = 1$ a.s.

(c) If $\kappa = 1$ and $\mathbb{P}(Z_1 = 1) < 1$, then $W = 0$ a.s.

(d) If $-\infty \leq \gamma < 0$ and $\mathbb{E}Z_1\log^+ Z_1 < \infty$, then $\mathbb{E}W = 1$.

This result, which may also be derived from similar results in [15] and [46], leaves open what happens in the case where γ^- and $\mathbb{E}Z_1 \log^+ Z_1$ are both infinite (for instance if $\gamma^- = \gamma^+ = \infty$). A more general result including this situation is stated in [33, Prop. 1.1].

The cases where $(Z_n)_{n\geq 0}$ is a (critical) GWP or a multiplicative random walk with no branching are of no interest here because either $W = 0$ a.s., or $Z_1 = Z_2 = ... = 1$ a.s. Therefore we further assume throughout

$$\mathbb{P}(T \in \{0,1\}^{\mathbb{N}}) \ < \ 1, \tag{C3}$$

$$\mathbb{P}(N \geq 2) \ > \ 0. \tag{C4}$$

It is worth mentioning that, in contrast to many earlier related contributions, we do *not* exclude the possibility

$$\mathbb{P}(N = \infty) \ > \ 0.$$

The same allowance is made in [15-17],[32],[33] and [46] and seems to have first appeared in a paper by Kingman [36] on age-dependent branching processes (Biggins, personal communication). The case where N is a.s. finite has been studied in some detail by Liu [43], [44] (using the name "multiplicative cascade"), and also by Biggins and Kyprianou in a series of papers [15], [16], [17], [19], [20], [39] in the analysis of the branching random walk.

Assuming (C1-4) hereafter and recalling that q denotes the extinction probability of the GWP $(Z_n^{(0)})_{n\geq 0}$, the following implications besides $\mathbb{E}W = 1$ are also valid:

$$\mathbb{E}\sup_{n\geq 0} W_n < \infty, \tag{1.4}$$

$$\mathbb{P}(W = 0) = q < 1, \tag{1.5}$$

$$\mathbb{E}N > 1. \tag{1.6}$$

The most difficult assertion (1.4) follows from (1.18) below proved in the Appendix.

The double martingale structure. Next put $L_v(w) \stackrel{\text{def}}{=} \prod_{i=1}^m T_{w_i}(w_1...w_{i-1})$ for $w = w_1...w_m \in \mathbb{V}$ and $v \in \mathbb{V}$, thus $L_v(w) = \frac{L(vw)}{L(v)}$ if $L(v) > 0$. Then our model assumptions imply that, for each $v \in \mathbb{V}$ with $|v| = m \geq 1$, the sequence

$$Z_n(v) \ \stackrel{\text{def}}{=} \ \sum_{|w|=n} L_v(w), \quad n \geq 0, \tag{1.7}$$

is independent of $\mathcal{F}_m$ defined in (1.3) and forms a copy of $(Z_n)_{n\geq 0} = (Z_n(\emptyset))_{n\geq 0}$, in particular $Z_1(v) \stackrel{d}{=} Z_1 \stackrel{d}{=} \sum_{i\geq 1} T_i$, where $\stackrel{d}{=}$ means equality in distribution. Defining the martingale differences $D_n \stackrel{\text{def}}{=} W_n - W_{n-1}$ for $n \geq 1$, we have

$$D_n \ = \ \sum_{|v|=n-1} L(v)(Z_1(v) - 1) \tag{1.8}$$

which, when conditioned upon $\mathcal{F}_{n-1}$, may be viewed as a weighted sum of i.i.d. mean zero random variables (as $\mathbb{E}(Z_1(v)-1)=0$) and thus as a martingale limit. So $(W_n)_{n\ge 0}$, besides being itself a martingale, has increments also bearing a martingale structure, an observation dating back to at least [6], see also [7]. It will be of crucial importance in our proofs and in fact exploited in a different way than in earlier work (apart from [5] and [38]). The additional complication incurred here is caused by the fact that an estimation of a ϕ-moment of Z_n eventually leads to an estimation of the ϕ-moments of the products $L(v)(Z_1(v)-1)$ which is not as straightforward as one might expect unless ϕ is multiplicative or at least submultiplicative. In the case of a GWP $(Z_n)_{n\ge 0}$, this problem does not occur because the $L(v)$ are 0-1-valued and (1.8) thus simpifies to

$$D_n \;=\; \sum_{j=1}^{Z_{n-1}} (X_{n-1,j}-1)$$

where the $X_{n-1,j}$ are i.i.d. random variables giving the numbers of offspring of the members of the $(n-1)$th generation.

THE STOCHASTIC FIXED-POINT EQUATION SOLVED BY W. It is not difficult to see that the martingale limit W solves a stochastic fixed-point equation. Indeed, by defining $W^{(i)}$ as the a.s. limit of $Z_n(i)$, as $n\to\infty$, and applying Fatou's lemma to the equation $Z_n = \sum_{i\ge 1} T_i(\emptyset) Z_{n-1}(i)$ (called backward equation), we obtain

$$W \;=\; \sum_{i\ge 1} T_i(\emptyset) W^{(i)} \quad \mathbb{P}\text{-a.s.} \tag{1.9}$$

which upon iteration leads to $W=\sum_{|v|=m} L(v)W^{(v)}$ $\mathbb{P}$-a.s. for all $m\ge 1$, where $W^{(v)}$ is given as the a.s. limit of $Z_n(v)$, as $n\to\infty$. The $W^{(v)}$, $|v|=m$, are i.i.d. copies of W and independent of the $L(v)$, $|v|\le m$. Any distribution ν on $[0,\infty)$ such that (1.9) holds true in distribution with $W\stackrel{d}{=}\nu$ constitutes a fixed point of the so called *smoothing transform* $K:\mathcal{D}\to\overline{\mathcal{D}}$, $K(\nu)\stackrel{\text{def}}{=}\mathbb{P}(\sum_{i\ge 1} T_i X_i \in \cdot)$, where $\mathcal{D}$, $\overline{\mathcal{D}}$ denote the sets of probability distributions on $[0,\infty)$, respectively $[0,\infty]$, and where $X_1, X_2, \ldots$ are i.i.d. random variables with distribution ν and independent of T. The smoothing transform and its pertinent fixed-point equation have received considerable interest in the literature due to its connections to various interesting models in applied probability mentioned earlier, see e.g. [29], [47], [50], [26], [32], [41], [22] and [1].

THE CLASS OF FUNCTIONS ϕ. Nondecreasing functions $\phi:[0,\infty)\to[0,\infty)$ that are regularly varying at infinity with $\lim_{x\to\infty}\frac{\phi(x)}{x}=\infty$ form a natural class beyond the standard one $\{\phi_\alpha : \alpha>1\}$ when aiming at moment results. On the other hand, regular variation does not appear to be the appropriate property for the application of powerful martingale inequalities which rather require convexity. But since $\mathbb{E}\phi(W)<\infty$ is equivalent to $\mathbb{E}\psi(W)<\infty$ for any nondecreasing $\psi:[0,\infty)\to[0,\infty)$ of the same asymptotic order ($\phi\asymp\psi$), which means that

$$0 \;<\; \liminf_{x\to\infty}\frac{\phi(x)}{\psi(x)} \;\le\; \limsup_{x\to\infty}\frac{\phi(x)}{\psi(x)} \;<\; \infty,$$

this obstacle may be overcome by finding, to any given regularly varying ϕ, a function ψ of the same asymptotic order and with the needed convexity properties. This has been elaborated in greater detail in [5] and we will take advantage of the results from there. Besides convexity, *submultiplicativity* forms another property that will be useful in our analysis due to the multiplicative structure of the branch weights $L(v)$. A function ϕ is submultiplicative if $\phi(xy) \leq \phi(x)\phi(y)$ for all $x, y \geq 0$. This property is shared by all ϕ_α but does not generally hold for regularly varying functions. Section 2 contains all necessary facts about regular variation, convexity and submultiplicativity including a definition of the relevant classes of convex functions. At this point we confine ourselves to a collection of some notation and those facts that are needed for the statement of our results.

For $\alpha \geq 0$, let $\mathfrak{R}_\alpha$ be the class of locally bounded functions $\phi : [0,\infty) \to [0,\infty)$ which are regularly varying at infinity with exponent α (slowly varying in case $\alpha = 0$), so $\phi(x) = x^\alpha \ell(x)$ with slowly varying part ℓ, thus having the form (see [24, Theorem 1.3.1])

$$\ell(x) \;=\; c(x)\exp\left(\int_{[1,1\vee x]} \frac{\varepsilon(u)}{u}\,\lambda\!\!\lambda(du)\right), \quad x \geq 0, \tag{1.10}$$

where $c(x)$ is measurable, nonnegative with $\lim_{x\to\infty} c(x) = c \in (0,\infty)$, $\varepsilon(u)$ is measurable, locally bounded with $\lim_{u\to\infty}\varepsilon(u) = 0$, and $\lambda\!\!\lambda$ denotes Lebesgue measure. We call ℓ *normalized* if its representation (1.10) (which is clearly not unique) may be chosen with $c(x) \equiv 1$, thus $\ell(x) = 1$ for all $x \in [0,1]$. In any case,

$$\ell(x) \;\asymp\; \exp\left(\int_{[1,1\vee x]} \frac{\varepsilon(u)}{u}\,\lambda\!\!\lambda(du)\right), \quad x \to \infty, \tag{1.11}$$

and the right-hand normalization will be shown to be submultiplicative if ε is further nonincreasing on $[1,\infty)$ (and thus nonnegative), see Lemma 2.4. We denote by $\mathfrak{R}_0^*$ the class of all such $\ell \in \mathfrak{R}_0$ with nonincreasing ε. Notice for this case that slow variation in combination with submultiplicativity yields

$$\ell(x) \;=\; \lim_{y\to\infty}\frac{\ell(x)\ell(y)}{\ell(y)} \;\geq\; \lim_{y\to\infty}\frac{\ell(xy)}{\ell(y)} \;=\; 1 \tag{1.12}$$

for all $x > 0$. For arbitrary normalized $\ell \in \mathfrak{R}_0$, call $\ell^*(x) = \exp\left(\int_{[1,1\vee x]}\frac{\varepsilon^*(s)}{s}\,\lambda\!\!\lambda(ds)\right) \in \mathfrak{R}_0^*$ a *submultiplicative cap* of ℓ, if

$$\lim_{x\to\infty}\int_{[1,x]}\frac{(\varepsilon(s)-\varepsilon^*(s))^+}{s}\,\lambda\!\!\lambda(ds) \;<\; \infty. \tag{1.13}$$

Denote by $\mathfrak{R}_0^*[\ell]$ the class of all such functions. (1.13) particularly ensures $\limsup_{x\to\infty}\frac{\ell(x)}{\ell^*(x)} < \infty$, see Lemma 2.5, and the choice $\varepsilon^*(s) \stackrel{\text{def}}{=} \operatorname{ess\,sup}_{t\geq s}\varepsilon(t)$ shows that $\mathfrak{R}_0^*[\ell] \neq \emptyset$.

Given any $\phi \in \mathfrak{R}_\alpha$, the smooth variation theorem [24, Thm. 1.8.2] ensures the existence of a function $\psi \in \mathfrak{R}_\alpha$ which is smooth (infinitely often differentiable) on $(0,\infty)$ and satisfies $\phi \asymp \psi$. If $\alpha > 0$ and $\alpha \notin \mathbb{N}$ then ψ can also be chosen such that all its derivatives are monotone

[24, Theorem 1.8.3] which implies the most useful fact (in view of the above discussion) that ϕ and all its derivatives are either convex or concave. However, if α is an integer, a similar conclusion fails without further ado due to the slowly varying part and motivates the subsequent definitions. For any measurable $\phi : [0,\infty) \to [0,\infty)$, we put

$$\mathbb{U}\phi(x) \stackrel{\text{def}}{=} \int_{[1,1\vee x]} \frac{\phi(y)}{y}\, \lambda\!\!\!\lambda(dy) \quad \text{and} \quad \mathbb{H}\phi(x) \stackrel{\text{def}}{=} \int_{(x,\infty)} \frac{\phi(y)}{y}\, \lambda\!\!\!\lambda(dy) \tag{1.14}$$

and note that $\mathbb{H}\phi$ is finite only if $\frac{\phi(x)}{x}$ is integrable on $[0,\infty)$ which we will stipulate hereafter whereever such a function appears. On the other hand, $\mathbb{U}\phi$ is always well defined, and we point out that, if $\ell(x) = \exp\left(\int_{[1,1\vee x]} \frac{\varepsilon(u)}{u}\, \lambda\!\!\!\lambda(du)\right) \in \mathfrak{R}_0^*$, then $\ell = \mathbb{U}\ell_0$ for some $\ell_0 \in \mathfrak{R}_0$ iff $\varepsilon \in \mathfrak{R}_0$, see Lemma 2.4.

Main results. Our standing assumptions (C1-4) will always be in force throughout unless stated otherwise. We are now ready to state the main results on the existence of ϕ-moments for W to be proved in this article. Given $\phi(x) = x^\alpha \ell(x) \in \mathfrak{R}_\alpha$, $\alpha \geq 1$, the first theorem deals with the case where the slowly varying part ℓ is submultiplicative and thus particularly nondecreasing. They are followed by two further theorems which cover the situation where ℓ is not submultiplicative, or $\alpha > 1$ and $\ell(x) \to 0$, as $x \to \infty$.

Theorem 1.2. *Let $\alpha \geq 1$ and $\ell(x) = \exp\left(\int_{[1,1\vee x]} \frac{\varepsilon(u)}{u}\, \lambda\!\!\!\lambda(du)\right) \in \mathfrak{R}_0^*$, that is ε is nonincreasing and vanishing at ∞. Suppose $\varepsilon \in \mathfrak{R}_0$ if $\alpha \in \{2^n : n \geq 0\}$ and further ℓ be unbounded if $\alpha = 1$. Then the following assertions are equivalent:*

(a) for $\alpha > 1$: $\mathbb{E}Z_1^\alpha \ell(Z_1) < \infty$ and $g(\alpha) < 1$.
for $\alpha = 1$: $\mathbb{E}Z_1 \mathbb{U}\ell(Z_1) < \infty$, where $\mathbb{U}\ell(x) = \int_{[1,1\vee x]} y^{-1}\ell(y)\, \lambda\!\!\!\lambda(dy)$.

(b) $0 < \mathbb{E}W^\alpha \ell(W) < \infty$.

The reader may wonder about the extra condition imposed on ε in case where α is a dyadic power. As mentioned earlier, $\phi(x) = x^\alpha \ell(x) \in \mathfrak{R}_\alpha$ for any positive $\alpha \notin \mathbb{N}$ may be chosen in such a way (up to asymptotic equivalence) that it be infinitely often differentiable with all derivatives being either convex or concave. However, for $\alpha \in \mathbb{N}$, the latter requires an extra condition on the slowly varying part ℓ. Since convexity (or at least monotonicity) plays an important role in the study of ϕ-moments it seems difficult to get away without such ado. Even in the simpler situation of normalized GWP's, Bingham and Doney [23] needed an extra condition on ℓ for their analytic treatment of ϕ-moments and tail probabilities if $\alpha \in \mathbb{N}$. Due to a different approach here, which is based upon the repeated use of convex function inequalities for martingales, we will need convexity of ϕ and its iterates $\phi(x^{1/2^m}) \in \mathfrak{R}_{\alpha/2^m}$ for all $m \in \mathbb{N}$ such that $\alpha/2^m \geq 1$. As a consequence, an additional assumption on ℓ of the form as stated above is only required when α is a dyadic power. Although it must be admitted that any such conditions form a nuisance in the statement of the results and might be removable by an even more elaborate analysis, the gain of generality is relatively small in view of the fact that most relevant special cases like $\phi(x) = x^\alpha \log^\beta(1+x)$ do satisfy the imposed extra conditions.

Our next theorem extends the previous one to general $\phi \in \mathfrak{R}_\alpha$, $\alpha \geq 1$, provided that the slowly varying part, w.l.o.g. assumed to be normalized, satisfies an extra condition involving a submultiplicative cap.

THEOREM 1.3. *Let $\alpha \geq 1$, $\ell(x) = \exp\left(\int_{[1,1\vee x]} \frac{\varepsilon(u)}{u}\, \lambda\!\!\lambda(du)\right) \in \mathfrak{R}_0$ and suppose $\varepsilon \in \mathfrak{R}_0$ if $\alpha \in \{2^n : n \geq 1\}$ and further ℓ be unbounded if $\alpha = 1$. Provided that, for some $\ell^* \in \mathfrak{R}_0^*[\ell]$, $\sum_{i\geq 1} \mathbb{E}T_i^\alpha \ell^*(T_i) < \infty$ if $\alpha > 1$, respectively $\sum_{i\geq 1} \mathbb{E}T_i \mathbb{U}\ell^*(T_i) < \infty$ if $\alpha = 1$, assertions (a) and (b) of Theorem 1.2 remain to be equivalent.*

The reader may check by a careful look at the proof that the above proviso is actually only needed for the implication "(a)$\Rightarrow$(b)", but not for the converse.

Clearly, $\ell^* \equiv 1 \in \mathfrak{R}_0^*[\ell]$ whenever $\ell \in \mathfrak{R}_0$ vanishes at infinity. In this case, the following simplification of Theorem 1.3 can be stated.

THEOREM 1.4. *Let $\alpha > 1$, $\ell(x) = \exp\left(\int_{[1,1\vee x]} \frac{\varepsilon(u)}{u}\, \lambda\!\!\lambda(du)\right) \in \mathfrak{R}_0$ with $\lim_{x\to\infty} \ell(x) = 0$ and suppose $\varepsilon \in \mathfrak{R}_0$ if $\alpha \in \{2^n : n \geq 1\}$. Then the assertions (a) and (b) of Theorem 1.2 remain to be equivalent.*

Note that Theorem 1.2 comprises the particularly important $\mathfrak{L}^\alpha$-case (choose $\ell \equiv 1$), which has also been settled in a recent paper by Iksanov [32] using different methods. Here this special case is stated as Theorem 3.1 in Section 3. Its proof, also given there along with further references, hinges exclusively on an exploitation of the double martingale structure of $(W_n)_{n\geq 0}$ and is not, in contrast to the proof of the general result given in Section 7, complicated by the use of stopping lines and renewal theory for weighted branching models as developed to the necessary extent in Sections 4-6. Not covered by our results are α-moments of W for negative α (conditioned upon $W > 0$) for which different methods are needed, see [9], [44] and the references given there.

For supercritical GWP's, where all T_i are 0 or 1, the condition $\sum_{i\geq 1} \mathbb{E}T_i^\alpha \ell^*(T_i) < \infty$ ($\alpha > 1$), respectively $\sum_{i\geq 1} \mathbb{E}T_i \mathbb{U}\ell^*(T_i) < \infty$ ($\alpha = 1$) reduces to $g(\alpha) < \infty$ and is automatically satisfied under assertion (a) as well as (b) of Theorem 1.2. Theorem 1.3 in this special case has essentially been obtained by Bingham and Doney [23] using analytic methods and been reproved as Corollary 2.3 in [5] by similar methods as in the present paper.

Our final result is a supplement to the previous ones and only stated for completeness.

COROLLARY 1.5. *Suppose $\phi \in \mathfrak{R}_\alpha$, $\alpha \geq 1$, is also convex. Then the following conditions are equivalent:*

$$0 < E\phi(W) < \infty; \tag{1.15}$$

$$\sup_{n\geq 0} E\phi(W_n) < \infty; \tag{1.16}$$

$$E\phi(W^*) < \infty, \tag{1.17}$$

where $W^* \stackrel{\text{def}}{=} \sup_{n\geq 0} W_n$.

The equivalence of (1.16), (1.17) holds true for any ϕ-integrable submartingale $(W_n)_{n\geq 0}$, but the equivalence with (1.15) hinges on the tail inequality

$$P(W^* > ax) \ \leq \ C\,P(W > x), \quad x \geq 0 \tag{1.18}$$

for suitable $a, C > 0$ which is well-known for supercritical normalized GWP's with positive limit (see [5, Lemma II.2.6]) and has been extended by Biggins [16] to branching random walks. A further extension involving stopping lines is stated as Lemma A.1 in the Appendix and can be proved by an adaptation of Biggins' argument.

The further organization of the paper is as follows. As already mentioned, Section 2 provides the necessary details about how regular variation links to convexity and submultiplicativity for the functions ϕ appearing in our results. Section 3 treats the $\mathfrak{L}^\alpha$-case but further contains a series of lemmata that are also relevant for the more general results. The definition of *homogeneous stopping lines* for weighted branching models and the connection between WBP and renewal theory are the subject of Section 4, followed by the introduction of an imbedded model based on ladder epochs in Section 5 which builds on this connection. Section 6 provides a pathwise renewal theorem for weighted branching models which is crucial for the proof of Theorem 1.2, "(b)$\Rightarrow$(a)" in the case $\alpha = 1$. These results are also of interest in their own right. The proofs of Theorems 1.2-4 are then presented in Section 7, followed by an Appendix.

2. Regular variation, convexity and submultiplicativity

A regular varying function is generally neither convex nor smooth. Since, on the other hand, our approach relies on the application of certain convex function inequalities, we first collect a number of facts which link regular variation, convexity, submultiplicativity and other useful properties. Apart from those concerning submultiplicativity, these facts are essentially taken from the Sections 2 and 3 in [5], and in part from [38].

Let us stipulate hereafter that any function ϕ defined on $[0,\infty)$ is extended to the real line by putting $\phi(x) \stackrel{\text{def}}{=} \phi(-x)$ for $x < 0$. The usual primed notation for derivatives of a convex or concave function on $(0,\infty)$ is always to be understood in the right-hand sense if the latter differs from the left-hand one. Now let $\mathfrak{C}_0$ be the class of convex differentiable functions ϕ which are (strictly) increasing on $[0,\infty)$ with $\phi(0) = 0$ and concave derivative ϕ' on $(0,\infty)$ satisfying $\lim_{x\downarrow 0}\phi'(x) = 0$. Obviously, each $\phi_\alpha(x) = x^\alpha$, $1 < \alpha \leq 2$, belongs $\mathfrak{C}_0$, but the identity function ϕ_1 does not. We further note for each $\phi \in \mathfrak{C}_0$ that ϕ' is nondecreasing and positive on $(0,\infty)$ and that $\liminf_{x\to\infty}\frac{\phi(x)}{x} > 0$. For $n \geq 1$, we define recursively

$$\mathfrak{C}_n \ \stackrel{\text{def}}{=} \ \{\mathbb{S}\phi \in \mathcal{G} : \phi \in \mathfrak{C}_{n-1}\} \ = \ \mathbb{S}\mathfrak{C}_{n-1},$$

where the operator $\mathbb{S}$ is given by $\mathbb{S}\phi(x) \stackrel{\text{def}}{=} \phi(x^2)$, thus $\mathbb{S}^n\phi(x) = \phi(x^{2^n})$ for $n \geq 1$. The functions ϕ to be considered throughout shall be elements from one of these classes, i.e. from

$\mathfrak{C} \stackrel{\text{def}}{=} \cup_{n\ge 0}\mathfrak{C}_n$, and they are clearly always differentiable and convex, so $\mathbb{S}:\mathfrak{C}\to\mathfrak{C}$. As two further useful properties of functions in $\mathfrak{C}$ we mention (see [5, Lemmata 3.3 and 3.4])

$$\phi(2x) \;\le\; C\,\phi(x), \quad x\ge 0, \tag{2.1}$$

for some $C = C_\phi \in (0,\infty)$, and

$$\phi\in\mathfrak{C}_n \quad\Rightarrow\quad \liminf_{x\to\infty}\frac{\phi(x)}{x^{2^n}} > 0 \text{ and } \limsup_{x\to\infty}\frac{\phi(x)}{x^{2^{n+1}}} < \infty \tag{2.2}$$

for each $n\ge 0$. Note that (2.1) and the monotonicity of ϕ yield

$$\phi(ax) \;\le\; C\phi(x), \quad x\ge 0, \tag{2.3}$$

for any $a>0$ and some $C = C_{\phi,a}\in(0,\infty)$.

Next define (slightly differing from [5])

$$\mathfrak{C}_0^* \;\stackrel{\text{def}}{=}\; \{\phi\in\mathfrak{C}_0 : \phi''(0)\in(0,\infty)\}$$

and $\mathfrak{C}^* \stackrel{\text{def}}{=} \cup_{n\ge 0}\mathfrak{C}_n^*$, where $\mathfrak{C}_n^* \stackrel{\text{def}}{=} \mathbb{S}^n\mathfrak{C}_0^*$ for $n\ge 1$. Notice that $\phi''(0)=0$ for any $\phi\in\mathfrak{C}^*\backslash\mathfrak{C}_0^*$. Lemma 3.3 in [5] asserts that to each $\phi\in\mathfrak{C}_0$ there exists a function $\hat\phi\in\mathfrak{C}_0^*$ such that $\phi\sim\hat\phi$, the latter having the usual meaning $\lim_{x\to\infty}\frac{\phi(x)}{\hat\phi(x)}=1$.

Given any slowly varying function ℓ, recall from (1.14) the definition of the functions $\mathbb{U}\ell$ and $\mathbb{H}\ell$ and that the everywhere finiteness of them is stipulated wherever they appear. The function $\mathbb{U}\ell$ is nondecreasing, while $\mathbb{H}\ell$ is nonincreasing. Furthermore, if $\ell_0,\ell_1\in\mathfrak{R}_0$ satisfy $\ell_0\asymp\ell_1$ and both, $\mathbb{U}\ell_0$ and $\mathbb{U}\ell_1$, are everywhere finite, then $\mathbb{U}\ell_0\asymp\mathbb{U}\ell_1$. Finally, if $\mathbb{U}\ell$ is everywhere finite then, by Karamata's theorem [24, Proposition 1.5.9a], $\mathbb{U}\ell$ is also slowly varying and grows faster than ℓ, i.e. $\lim_{x\to\infty}\frac{\mathbb{U}\ell(x)}{\ell(x)}=\infty$.

The following lemma links regularly varying functions and the function classes just introduced and may be proved by combining Lemmata 2.1 and 3.3 of [5].

Lemma 2.1. *Given $\phi(x)=x^\alpha\ell(x)\in\mathfrak{R}_\alpha$ for some $\alpha\ge 1$, the following assertions hold true:*

(a) If $2^n<\alpha<2^{n+1}$ for some $n\ge 0$, then $\phi\asymp\psi$ for some $\psi\in\mathfrak{C}_n^*\cap\mathfrak{R}_\alpha$.

(b) If $\alpha=2^n$ for some $n\ge 0$ and $\ell\asymp\mathbb{U}\ell_0$ for some $\ell_0\in\mathfrak{R}_0$, then $\phi\asymp\psi$ for some $\psi\in\mathfrak{C}_n^*\cap\mathfrak{R}_\alpha$.

(c) If $\alpha=2^n$ for some $n\ge 1$ and $\ell\asymp\mathbb{H}\ell_0$ for some $\ell_0\in\mathfrak{R}_0$, then $\phi\asymp\psi$ for some $\psi\in\mathfrak{C}_{n-1}^*\cap\mathfrak{R}_\alpha$.

The second lemma collects further relevant properties shared by all elements of $\mathfrak{C}^*$ and summarizes Lemmata 3.3 and 3.4 of [5].

Lemma 2.2. *Let $\psi\in\mathfrak{C}_n^*$ for some $n\ge 0$. Then the following assertions hold true:*

(a) $\frac{\psi(x)}{x^{2^n}}$ *is nondecreasing and* $\frac{\psi(x)}{x^{2^{n+1}}}$ *is nonincreasing in* $x \geq 0$.

(b) $\lim_{x\downarrow 0} \frac{\psi(x)}{x^{2^n}} = (\mathbb{S}^{-n}\psi)'(0) = 0$ *and* $\lim_{x\downarrow 0} \frac{\psi(x)}{x^{2^{n+1}}} = \frac{1}{2}(\mathbb{S}^{-n}\psi)''(0) \in (0,\infty)$.

The lemma shows that any $\psi \in \mathfrak{C}_n^*$ $(n \geq 0)$ satisfies

$$\psi(s) \;=\; O(s^{2^{n+1}}), \quad s \downarrow 0, \tag{2.4}$$

so a fortiori

$$\psi(s) \;=\; o(s^{\alpha}), \quad s \downarrow 0, \tag{2.5}$$

whenever $0 < \alpha < 2^{n+1}$.

Given any nondecreasing convex function $\phi : [0,\infty) \to [0,\infty)$, we next define the operator $\mathbb{L}$ through

$$\mathbb{L}\phi(x) \;\stackrel{\text{def}}{=}\; \int_0^x \int_0^s \frac{\phi'(r)}{r}\, dr\, ds, \quad x \geq 0. \tag{2.6}$$

If $\phi \in \mathfrak{C}^*$, then $\mathbb{L}\phi$ is everywhere finite, i.e. $\mathfrak{C}^* \subset \{\phi \in \mathfrak{C} : \mathbb{L}\phi(z) < \infty \text{ for all } z \geq 0\}$. $\mathbb{L}\phi$ will be of importance in our analysis in the case $\phi \in \mathfrak{C}_0^*$. Therefore the subsequent lemma collects a number of properties of the function $\mathbb{L}\phi$ associated with $\phi \in \mathfrak{C}^*$. For the proof we refer once again to [5, Lemmata 2.2 and 3.5].

LEMMA 2.3. *Let* $\phi \in \mathfrak{C}_n^*$ *for some* $n \geq 0$. *Then* $\mathbb{L}\phi$ *is everywhere finite and convex and satisfies*

$$\liminf_{x\to\infty} \frac{\mathbb{L}\phi(x)}{\phi(x)} \;>\; 0$$

as well as

$$\liminf_{x\to\infty} \frac{\mathbb{L}\phi(x)}{x\log x} \;>\; 0.$$

If $n \geq 1$, *then* $2\phi(x/2) \leq \mathbb{L}\phi(x) \leq \phi(x)$ *for all* $x \geq 0$, *in particular* $\mathbb{L}\phi \asymp \phi$ *by (2.1), whereas in case* $n = 0$, $\mathbb{L}\phi \geq \phi$. *More specifically, if* $\phi(x) = x^{\alpha}\ell(x) \in \mathfrak{C}^* \cap \mathfrak{R}_{\alpha}$ *for some* $\alpha > 1$, *then*

$$\mathbb{L}\phi(x) \;\sim\; \frac{\phi(x)}{\alpha - 1},$$

while in case $\phi(x) = x\ell(x) \in \mathfrak{C}_0^* \cap \mathfrak{R}_1$

$$\mathbb{L}\phi(x) \;\sim\; x\,\mathbb{U}\ell(x) \;=\; x\int_{(0,x]} \frac{\ell(s)}{s}\, \lambda\!\!\lambda(ds) \tag{2.7}$$

and

$$\lim_{x\to\infty} \frac{\mathbb{L}\phi(x)}{\phi(x)} \;=\; \infty.$$

REMARK. Any increasing convex function $\phi : [0,\infty) \to [0,\infty)$ satisfies the relation $\phi(x) \leq x\phi'(x) \leq \phi(2x)$ (see [5]). If $\phi \in \mathfrak{C}$ then ϕ also satisfies (2.1) and therefore

$$\phi(x) \;\asymp\; x\phi'(x). \tag{2.8}$$

Moreover, if ϕ and ψ are asymptotically equivalent elements ($\phi \asymp \psi$) of $\{\varphi \in \mathfrak{C}_0 : \mathbb{L}\varphi < \infty\}$, then $\mathbb{L}\phi, \mathbb{L}\psi$ belong to $\mathfrak{C}_0$ as well (by Lemma 3.5 of [5]) and $\mathbb{L}\phi \asymp \mathbb{L}\psi$. For the last assertion observe that, by (2.8), $\mathbb{L}\phi \asymp \mathbb{L}\psi$ holds true iff

$$(\mathbb{L}\phi)'(x) \;=\; \int_0^x \frac{\phi'(s)}{s}\,ds \;\asymp\; \int_0^x \frac{\psi'(s)}{s}\,ds \;=\; (\mathbb{L}\psi)'(x)$$

which is readily verified when combining (2.8), $\phi \asymp \psi$ with $\frac{\phi'(s)}{s} \asymp \frac{\phi(s)}{s^2}$ and $\frac{\psi'(s)}{s} \asymp \frac{\psi(s)}{s^2}$.

We finally turn to the property of submultiplicativity, more precisely to the question which slowly varying functions ℓ are also submultiplicative. Recalling (1.11) it suffices for our purposes to consider normalized $\ell \in \mathfrak{R}_0$, thus of the form

$$\ell(x) \;=\; \exp\left(\int_{[1,1\vee x]} \frac{\varepsilon(s)}{s}\;\lambda\!\!\lambda(ds)\right) \tag{2.9}$$

with some locally integrable, asymptotically vanishing $\varepsilon : [1,\infty) \to \mathbb{R}$. Put $\hat{\ell}(x) \stackrel{\text{def}}{=} \log \ell(e^x)$ for $x \in \mathbb{R}$, that is

$$\hat{\ell}(x) \;=\; \int_{[1,1\vee e^x]} \frac{\varepsilon(s)}{s}\;\lambda\!\!\lambda(ds) \;=\; \int_{[0,0\vee x]} \varepsilon(e^u)\;\lambda\!\!\lambda(du), \quad x \in \mathbb{R},$$

and observe that ℓ is submultiplicative iff $\hat{\ell}$ is subadditive. Recall from the Introduction that $\mathfrak{R}_0^*$ consists of those normalized $\ell \in \mathfrak{R}_0$ with nonincreasing ε.

LEMMA 2.4. *For each* $\ell(x) = \exp\left(\int_{[1,x\vee 1]} \frac{\varepsilon(s)}{s}\;\lambda\!\!\lambda(ds)\right) \in \mathfrak{R}_0^*$, *the following assertions hold true*

(a) ℓ *is submultiplicative.*
(b) $\ell^a \in \mathfrak{R}_0^*$ *for each* $a > 0$.
(c) $\ell(x) = \int_0^x \ell'(y)\,dy$ *with* $\ell'(x) = \frac{\varepsilon(x)}{x}\ell(x)\mathbf{1}_{(1,\infty)}(x) = o(\ell(x))$, *as* $x \to \infty$.
(d) $\ell = \mathbb{U}\ell_0$ *with* $\ell_0 \in \mathfrak{R}_0$ *iff* $\varepsilon \in \mathfrak{R}_0$.

PROOF. We show the subadditivity of any $\hat{\ell}$ with $\ell \in \mathfrak{R}_0^*$. If $x \vee y \le 0$, then $\hat{\ell}(x+y) = \hat{\ell}(x) + \hat{\ell}(y) = 0$. Otherwise, suppose w.l.o.g. $x = x \vee y > 0$. Then, by the monotonicity of ε,

$$\begin{aligned}
\hat{\ell}(x+y) \;&=\; \int_{[0,0\vee(x+y)]} \varepsilon(e^u)\;\lambda\!\!\lambda(du) \\
&\le\; \int_{[0,x]} \varepsilon(e^u)\;\lambda\!\!\lambda(du) \;+\; \int_{[x,x\vee(x+y)]} \varepsilon(e^u)\;\lambda\!\!\lambda(du) \\
&=\; \int_{[0,x]} \varepsilon(e^u)\;\lambda\!\!\lambda(du) \;+\; \int_{[0,0\vee y]} \varepsilon(e^{u+x})\;\lambda\!\!\lambda(du) \\
&\le\; \hat{\ell}(x) + \hat{\ell}(y)
\end{aligned}$$

which is the desired conclusion for (a). The other assertions of the lemma are easily verified and details therefore omitted. $\diamondsuit$

LEMMA 2.5. *For any normalized $\ell \in \mathfrak{R}_0$ and $\ell^* \in \mathfrak{R}_0^*[\ell]$, it holds true that*

$$\ell(xy) \;\le\; C\ell(x)\ell^*(y) \tag{2.10}$$

for all $x, y \ge 1$ and some $C > 0$, in particular $\ell(y) \le C\ell^*(y)$ for all $y \ge 1$.

PROOF. As usual, write $\ell^*(x) = \exp\left(\int_{[1,x\vee 1]} \frac{\varepsilon^*(s)}{s}\,\lambda\!\!\lambda(ds)\right)$ and put

$$C \stackrel{\text{def}}{=} \sup_{x\ge 1} \exp\left(\int_{[1,x]} \frac{(\varepsilon(s)-\varepsilon^*(s))^+}{s}\,\lambda\!\!\lambda(ds)\right),$$

which is finite by (1.13). We then infer for all $x, y \ge 1$

$$\begin{aligned}\frac{\ell(xy)}{\ell(x)} \;&=\; \exp\left(\int_{[x,xy]} \frac{\varepsilon(s)}{s}\,\lambda\!\!\lambda(ds)\right)\\ &\le\; C\exp\left(\int_{[x,xy]} \frac{\varepsilon^*(s)}{s}\,\lambda\!\!\lambda(ds)\right) \;=\; C\,\frac{\ell^*(xy)}{\ell^*(x)} \;\le\; C\ell^*(y),\end{aligned}$$

and thus (2.10), from which the second assertion follows by putting $x = 1$ ($\ell(1) = 1$). $\diamondsuit$

3. AUXILIARY LEMMATA AND THE $\mathfrak{L}^\alpha$-CASE

The purpose of this section is to present a proof of Theorem 1.2 specialized to ordinary moments of order $\alpha > 1$, i.e. to the $\mathfrak{L}^\alpha$-case. This is a situation where our method of exploiting the double martingale structure works in a particular transparent and instructive way, due to the fact that the function $\phi_\alpha(x) = x^\alpha$, beyond being in $\mathfrak{C}^* \cap \mathfrak{R}_\alpha$, is also multiplicative, viz $\phi_\alpha(xy) = \phi_\alpha(x)\phi_\alpha(y)$. However, besides settling the $\mathfrak{L}^\alpha$-case, the result being stated as Theorem 3.1 below, we will prove a number of auxiliary lemmata that will also be needed later when proving Theorem 1.2 in full generality in Section 7.

THEOREM 3.1. *For $\alpha > 1$, the following assertions are equivalent:*

(a) $\mu(\alpha) \stackrel{\text{def}}{=} \mathbb{E}Z_1^\alpha < \infty$ *and* $g(\alpha) < 1$.

(b) $0 < \mathbb{E}W^\alpha < \infty$.

The same result has recently been obtained by Iksanov [32, Proposition 4] via different methods based on spinal trees. Earlier versions under varying restrictions on α or $N = \sum_{i\ge 1} \mathbf{1}_{\{T_i>0\}}$ also appeared in [16],[17],[40] and [53]. If $\alpha \in \mathbb{N}$ and $\sup_{i\ge 1} T_i \le 1$ a.s., our result coincides with Theorem 2.1 of Mauldin and Williams [47] who used their result for the calculation of the Hausdorff dimension of the limit set in a random recursive construction.

The proof of the theorem requires a number of preparations. Recall from the Introduction that $Z_n^{(\alpha)} = \sum_{|v|=n} L(v)^\alpha$ and put

$$W_n^{(\alpha)} \stackrel{\text{def}}{=} g(\alpha)^{-n} Z_n^{(\alpha)} \quad [\Rightarrow\ W_n = W_n^{(1)} = Z_n^{(1)} = Z_n],$$

$$D_n^{(\alpha)} \stackrel{\text{def}}{=} W_n^{(\alpha)} - W_{n-1}^{(\alpha)} = g(\alpha)^{-n} \sum_{|v|=n-1} L(v)^\alpha \big(Z_1^{(\alpha)}(v) - g(\alpha)\big) \quad [\Rightarrow\ D_n = D_n^{(1)}],$$

and

$$\overline{D}_n^{(\alpha)} \stackrel{\text{def}}{=} g(\alpha)^n D_n^{(\alpha)}$$

for $n \geq 1$ and $\alpha > 0$ with $g(\alpha) < \infty$. All variables with index 0 are defined as 1 unless stated otherwise. Both, $(Z_n^{(\alpha)})_{n\geq 0}$ and $(W_n^{(\alpha)})_{n\geq 0}$, are WBP's with generic weight sequences $(T_i^\alpha)_{i\geq 1}$ and $(g(\alpha)^{-1}T_i)_{i\geq 1}$, respectively. Moreover, each $(W_n^{(\alpha)})_{n\geq 0}$ forms a nonnegative martingale exhibiting the double martingale structure explained in the Introduction for $(W_n)_{n\geq 0}$. This will form the key to our analysis.

The following simple lemma on the function $g(\alpha)$ can be stated without proof. We only note for part (b), that $g'(1) = \sum_{i\geq 1} \mathbb{E}T_i \log T_i = \gamma < 0$ by (C2).

LEMMA 3.2. *(a) If $g(\alpha) < \infty$ for some $\alpha > 1$, then g is strictly convex on $[1, \alpha]$.*
(b) If $g(\alpha) < 1$ for some $\alpha > 1$, then $g(\beta) < 1$ for all $\beta \in (1, \alpha)$.

The second lemma asserts that W has always unbounded support under our standing assumptions, thereby ruling out the possibility of $\mathbb{E}W^\alpha < \infty$ being a trivial fact.

LEMMA 3.3. *Suppose that $\mathbb{E}W = 1$ and $\mathbb{P}(Z_1 = 1) < 1$. Then W is unbounded in the sense that $\mathbb{P}(W > t) > 0$ for all $t > 0$.*

PROOF. Since $\mathbb{E}Z_1 = 1$ and $\mathbb{P}(Z_1 = 1) < 1$, we find some $\varepsilon > 0$ and $m \in \mathbb{N}$ such that $\delta \stackrel{\text{def}}{=} \mathbb{P}(Z_1' > 1 + \varepsilon) > 0$, where

$$Z_n' \stackrel{\text{def}}{=} \sum_{v \in \{1,\dots,m\}^n} L(v), \quad n \geq 0.$$

Since $Z_n' = \sum_{|v|=n-1} L(v) Z_1'(v) \geq \sum_{v\in\{1,\dots,m\}^{n-1}} L(v) Z_1'(v)$ with all $Z_1'(v)$ being independent copies of Z_1 and also independent of Z_{n-1}', we infer

$$\begin{aligned}
\mathbb{P}(Z_n' > (1+\varepsilon)^n) &\geq \mathbb{P}(Z_{n-1}' > (1+\varepsilon)^{n-1}, Z_1'(v) > 1+\varepsilon), v \in \{1, ..., m\}^{n-1}) \\
&\geq \delta^m \mathbb{P}(Z_{n-1}' > (1+\varepsilon)^{n-1})
\end{aligned}$$

which inductively yields $\mathbb{P}(Z_n' > (1+\varepsilon)^n) \geq \delta^{nm} > 0$ and thus, using $W_n \geq Z_n'$,

$$\mathbb{P}(W^* > (1+\varepsilon)^n) \ \geq\ \mathbb{P}(W_n > (1+\varepsilon)^n) \ \geq\ \mathbb{P}(Z_n' > (1+\varepsilon)^n) \ >\ 0$$

for all $n \geq 1$. Finally, the assertion follows by an appeal to the tail inequality (1.18). $\diamondsuit$

REMARK. In view of the previous lemma it is worthwile to point out that Biggins and Grey [18] have shown (also assuming $\mathbb{E}W = 1$ and $\mathbb{P}(Z_1 = 1) < 1$) that the distribution of W restricted to $(0, \infty)$ is continuous, and that it even has a continuous Lebesgue density provided that $\mathbb{E}N < \infty$. The latter may be omitted in the particular case of *homogeneous branching* *random walks* (see [42, Section 0] or [45, Section 8] for a model description), as shown by Liu [42], see also [44].

The key lemma for the proof of Theorem 3.1 stated next will also be needed later for the proof of Theorem 1.2. It is therefore stated for a larger set of functions than only ϕ_α. An extension of this lemma will be presented in Section 7. Denote by $\mathfrak{Z}$ the class of all even nonnegative functions ψ which are continuous, nondecreasing on $[0,\infty)$ and satisfy the growth condition (2.3). Notice that for any $\psi \in \mathfrak{Z}$ and $m \in \mathbb{Z}$, the function $\mathbb{S}^m\psi(x) = \psi(|x|^{2^m})$ lies in $\mathfrak{Z}$, too. It is further to be mentioned that $\mathfrak{C} \subset \mathfrak{Z}$. Given two expressions A, B, we write $A \ll B$ if $B < \infty$ implies $A < \infty$. In what follows, C always denotes a suitable finite constant which may differ from line to line.

LEMMA 3.4. *Let $m \in \mathbb{N}$ and $\psi \in \mathfrak{Z}$. Suppose that $\mathbb{E}\psi(Z_1) < \infty$, $\mu(2^m) < \infty$ and $g(2^m) < 1$. Then*

$$\sup_{n\ge 0} \mathbb{E}\psi(W_n) \ \ll \ Q(m,\psi) \ \stackrel{\text{def}}{=} \ Q_1(m,\psi) + Q_2(m,\psi),$$

where

$$Q_1(m,\psi) \ \stackrel{\text{def}}{=} \ \mathbb{E}\mathbb{S}^{-m}\psi\left(\sum_{n\ge 0} \overline{D}_n^{(2^m)}\right) \quad \textit{and} \quad Q_2(m,\psi) \ \stackrel{\text{def}}{=} \ \sum_{l=0}^{m-1}\sum_{n\ge 0} \mathbb{E}\mathbb{S}^{-l}\psi\left(\overline{D}_n^{(2^l)}\right).$$

Furthermore,

$$0 \ \le \ \sum_{k\ge 0} \overline{D}_k^{(2^m)} \ < \infty \quad \mathbb{P}\text{-}a.s.$$

PROOF. The proof runs by induction over m and hinges on a repeated application of the Burkholder-Davis-Gundy inequality (abbreviated as BDG-inequality hereafter), see [28, Theorem 11.3.2].

If $m = 1$, a first application of that inequality yields

$$\mathbb{E}\psi(W) \ \le \ \sup_{n\ge 0}\mathbb{E}\psi(W_n) \ \ll \ C\left[\mathbb{E}\mathbb{S}^{-1}\psi\left(\sum_{n\ge 0}\mathbb{E}(D_n^2|\mathcal{F}_{n-1})\right) + \sum_{n\ge 0}\mathbb{E}\psi(D_n)\right].$$

As $g(1) = 1$ and $D_n = \overline{D}_n^{(1)}$,

$$\sum_{n\ge 0}\mathbb{E}\psi(D_n) \ = \ Q_2(1,\psi).$$

Moreover, if $n \ge 1$, the independence of $T(v)$ and $T(w)$ for $v \ne w$ ensures

$$\mathbb{E}(D_n^2|\mathcal{F}_{n-1}) \ = \ \sum_{|v|=n-1}\sum_{|w|=n-1} L(v)L(w)\,\mathbb{E}\left(\sum_{i\ge 1}T_i(v) - 1\right)\left(\sum_{i\ge 1}T_i(w) - 1\right)$$

$$
\begin{aligned}
&= \mathbb{E}\left(\sum_{i\geq 1} T_i - 1\right)^2 \sum_{|v|=n-1} L(v)^2 \\
&\leq \mu(2)g(2)^{n-1} W_{n-1}^{(2)} \\
&= \mu(2)g(2)^{n-1} \sum_{k=0}^{n-1} D_k^{(2)} \quad \mathbb{P}\text{-a.s.},
\end{aligned}
$$

whence

$$
\begin{aligned}
\sum_{n\geq 0} \mathbb{E}(D_n^2|\mathcal{F}_{n-1}) &\leq \mu(2) \sum_{n\geq 0} g(2)^{n-1} \sum_{k=0}^{n} D_k^{(2)} \\
&= \frac{\mu(2)}{1-g(2)} \sum_{k\geq 0} g(2)^k D_k^{(2)} \\
&= \frac{\mu(2)}{1-g(2)} \sum_{k\geq 0} \overline{D}_k^{(2)} \quad \mathbb{P}\text{-a.s.}
\end{aligned}
$$

Now use $\sup_{k\geq 0} \mathbb{E}|D_k^{(2)}| \leq 2$ to infer

$$
\mathbb{E}\left(\sum_{k\geq 0} \left|\overline{D}_k^{(2)}\right|\right) = \sum_{k\geq 0} g(2)^k\, \mathbb{E}|D_k^{(2)}| \leq 2\sum_{k\geq 0} g(2)^k < \infty,
$$

in particular $\sum_{k\geq 0} \overline{D}_k^{(2)} \in [0,\infty)$ a.s. Finally, in view of $D_0^2 = 1$ and (2.3),

$$
\begin{aligned}
\sup_{n\geq 0} \mathbb{E}\psi(W_n) &\leq C\left[\mathbb{E}\mathbb{S}^{-1}\psi\left(1 + \frac{\mu(2)}{1-g(2)} \sum_{k\geq 0} \overline{D}_k^{(2)}\right) + Q_2(1,\psi)\right] \\
&\ll Q_1(1,\psi) + Q_2(1,\psi).
\end{aligned}
$$

Now suppose the claim be proved for some $m \in \mathbb{N}$, put $r \stackrel{\text{def}}{=} 2^m$ and assume that $g(2^{m+1}) < 1$, $\mu(2^{m+1}) < \infty$ and $\sum_{k\geq 0} \overline{D}_k^{(r)} \in [0,\infty)$ a.s. Note that $g(r) < 1$ (Lemma 3.2(b)) and $\mu(r) < \infty$. Hence, by the inductive hypothesis, $\sup_{n\geq 0} \mathbb{E}\psi(W_n) \ll Q_1(m,\psi) + Q_2(m,\psi)$. Another application of the BDG-inequality in combination with Fatou's lemma gives the estimate

$$
\begin{aligned}
Q_1(m,\psi) &= \mathbb{E}\mathbb{S}^{-m}\psi\left(\sum_{k\geq 0} \overline{D}_k^{(r)}\right) \\
&\leq \liminf_{n\to\infty} \mathbb{E}\mathbb{S}^{-m}\psi\left(\sum_{k=0}^{n} \overline{D}_k^{(r)}\right) \\
&\leq C\left[\mathbb{E}\mathbb{S}^{-m-1}\left(\sum_{k\geq 0} \mathbb{E}\left(\overline{D}_k^{(r)^2}\Big|\mathcal{F}_{k-1}\right)\right) + \sum_{k\geq 0} \mathbb{E}\mathbb{S}^{-m}\left(\overline{D}_k^{(r)}\right)\right],
\end{aligned}
$$

because the sequence $\left(\sum_{k=0}^{m}\overline{D}_k^{(r)}\right)_{m\ge 0}$ forms a martingale. Similarly to the case $m=1$, it follows for $k\ge 1$

$$\begin{aligned}
\mathbb{E}\left(\overline{D}_k^{(r)^2}\Big|\mathcal{F}_{k-1}\right) &\le \sum_{|v|=k-1} L(v)^{2r}\mathbb{E}\left(\sum_{i\ge 1}T_i^r - g(r)\right)^2\\
&\le \sum_{|v|=k-1} L(v)^{2r}\mathbb{E}\left(\sum_{i\ge 1}T_i^r\right)^2\\
&\le \mathbb{E}Z_1^{2r}\sum_{|v|=k-1} L(v)^{2r}\\
&= \mu(2r)g(2r)^{k-1}W_{k-1}^{(2r)} \quad \mathbb{P}\text{-a.s.}
\end{aligned}$$

where $\sum_{i\ge 1}T_i^r\le Z_1^r$ has been utilized. Consequently, writing $W_k^{(2r)}=\sum_{l=0}^{k}D_l^{(2r)}$,

$$\begin{aligned}
\sum_{k\ge 1}\mathbb{E}\left(\overline{D}_k^{(r)^2}\Big|\mathcal{F}_{k-1}\right) &\le \mu(2r)\sum_{k\ge 0} g(2r)^k\sum_{l=0}^{k}D_l^{(2r)}\\
&= \frac{\mu(2r)}{1-g(2r)}\sum_{k\ge 0} g(2r)^k D_k^{(2r)}\\
&= \frac{\mu(2r)}{1-g(2r)}\sum_{k\ge 0}\overline{D}_k^{(2r)} \quad \mathbb{P}\text{-a.s.}
\end{aligned}$$

By using $\sup_{k\ge 0}\mathbb{E}|D_k^{(2r)}|\le 2$, we find $\sum_{k\ge 0}g(2r)^k\mathbb{E}|D_k^{(2r)}|<\infty$ and thus $\sum_{k\ge 0}\overline{D}_k^{(2r)}\in[0,\infty)$ a.s. To finish the proof, it suffices to verify that

$$Q(m,\psi) \ll Q(m+1,\psi).$$

Suppose that $Q(m+1,\psi)<\infty$. Then the finiteness of $Q_2(m,\psi)$ is easily obtained from

$$\infty > Q(m+1,\psi) \ge Q_2(m+1,\psi) \ge Q_2(m,\psi).$$

Left with $Q_1(m,\psi)$, note that, by the previous findings,

$$Q_1(m,\psi) \le C(I_1(m,\psi)+I_2(m,\psi))$$

with

$$\begin{aligned}
I_1(m,\psi) &\stackrel{\text{def}}{=} \mathbb{E}\mathbb{S}^{-m-1}\psi\left(1+\frac{\mu(2r)}{1-g(2r)}\sum_{k\ge 0}\overline{D}_k^{(2r)}\right)\\
&\ll \mu(2^{m+1})\vee\mathbb{E}\mathbb{S}^{-m-1}\psi\left(\sum_{k\ge 0}\overline{D}_k^{(2r)}\right)\\
&\ll Q_1(m+1,\psi) < \infty
\end{aligned}$$

and

$$I_2(m,\psi) \stackrel{\text{def}}{=} \sum_{k\ge 0} \mathbb{E}\mathbb{S}^{-m}\psi\left(\overline{D}_k^{(r)}\right) \le Q_2(m+1,\psi) < \infty,$$

as required. $\diamondsuit$

REMARK. Lemma 3.4. reveals some additional technical difficulty not encountered in the situation of supercritical GWP's (see [5]): When estimating moments of type $\mathbb{E}\psi(W)$ by means of the BDG-inequality, processes depending on the random variables $\overline{D}_n^{(2^m)}$, $n \ge 0$, come into play, where $m \ge 1$. In the Galton-Watson case, the weights $L(v)$, $v \in \mathbb{V}$, take only the values 0 or 1 with the effect that the underlying process remains unchanged. This entails the pleasant technical simplification that the result of such estimations can be expressed in terms of the original process rather than of the random variables we had to introduce (cf. Lemmata 4.1 and 4.2 in [5]). On the other hand, our calculations comprise the case of supercritical GWP's.

Our final preparative lemma is a technical prerequisite.

LEMMA 3.5. *Suppose that $(\overline{Z}_n)_{n\ge 0}$ is a WBP with generic weight vector $(\overline{T}_i)_{i\ge 1}$ such that, for some $q > 1$,*

$$\overline{g}(1) \vee \overline{g}(q) < 1,$$

where $\overline{g}(\alpha) \stackrel{\text{def}}{=} \sum_{i\ge 1} \mathbb{E}\overline{T}_i^{\alpha}$. Then there exists another WBP $(\widehat{Z}_n)_{n\ge 0}$ with generic weight vector $(\widehat{T}_i)_{i\ge 1}$ such that

$$\overline{Z}_1 = c + \widehat{Z}_1 \quad \text{and} \quad \overline{Z}_n \le \widehat{Z}_n \quad \mathbb{P}\text{-a.s.}$$

for some $c > 0$ and all $n \ge 0$, and hence $\mathbb{E}f(\widehat{Z}_1) \ll \mathbb{E}f(\overline{Z}_1)$ for any $f \in \mathfrak{Z}$. Furthermore,

$$\widehat{g}(1) < 1 \quad \text{and} \quad \widehat{g}(q) < \widehat{g}(1)^q,$$

where $\widehat{g}(\alpha) \stackrel{\text{def}}{=} \sum_{i\ge 1} \mathbb{E}\widehat{T}_i^{\alpha}$.

PROOF. To begin with, fix $c > 0$ such that $1 > (c+\overline{g}(1))^q > \frac{1}{2}(\overline{g}(q)+1) \in (0,1)$. For $m \in \mathbb{N}$, let $(\widehat{Z}_{m,n})_{n\ge 0}$ be the WBP with weights

$$\widehat{T}_{m,i}(v) \stackrel{\text{def}}{=} \begin{cases} \dfrac{c}{m}, & \text{if } 1 \le i \le m, \\ \overline{T}_{i-m}(v), & \text{if } i \ge m+1. \end{cases}, \quad v \in \mathbb{V}.$$

Then $\widehat{Z}_{m,n} \ge \overline{Z}_n$ a.s. for all $n \ge 0$ and $\widehat{Z}_{m,1} = c + \overline{Z}_1$. Put $\widehat{g}_m(\alpha) \stackrel{\text{def}}{=} \sum_{i\ge 1} \mathbb{E}\widehat{T}_{m,i}^{\alpha}$, which satisfies

$$\widehat{g}_m(\alpha) = m\left(\frac{c}{m}\right)^{\alpha} + \overline{g}(\alpha), \quad \alpha \ge 1,$$

in particular $\widehat{g}_m(1) = c + \overline{g}(1) < 1$. Now choose m so large that

$$\widehat{g}_m(q) = \overline{g}(q) + \frac{c^q}{m^{q-1}} < \frac{\overline{g}(q)+1}{2} < (c+\overline{g}(1))^q = \widehat{g}_m(1)^q.$$

The assertions of the lemma follow upon choosing $(\widehat{Z}_n)_{n\ge 0} = (\widehat{Z}_{m,n})_{n\ge 0}$ for such m. $\diamondsuit$

PROOF OF THEOREM 3.1. We first show "(a)$\Rightarrow$(b)", thus assuming $\mathbb{E}Z_1^\alpha < \infty$ and $g(\alpha) < 1$ for some $\alpha > 1$. In analogy to Theorem 1.1 in [5], this is done by distinguishing the cases $\alpha \in (2^m, 2^{m+1}]$.

STEP 1. (see also [53, Theorem 6]) If $\alpha \in (1,2]$, the function $\phi_\alpha(x) = x^\alpha$ is convex with concave derivative on $(0,\infty)$. Hence an application of the classical von Bahr-Esseen inequality [8, Thm. 2] (abbreviated as BE-inequality hereafter) to the nonnegative martingale $(W_n)_{n\ge 0}$ combined with $W_0 = 1$ gives

$$\mathbb{E}W_n^\alpha \ \le\ 1 \ +\ 2\sum_{k=1}^n \mathbb{E}|D_k|^\alpha$$

for all $n \ge 1$, and then

$$\mathbb{E}W^\alpha \ \le\ \sup_{n\ge 0}\mathbb{E}W_n^\alpha \ \le\ 1 \ +\ 2\sum_{k\ge 1}\mathbb{E}|D_k|^\alpha.$$

Next we use that each D_k may itself be viewed as a martingale limit. To be more precise, fix k and an enumeration $(v^j)_{j\ge 1}$ of $\mathbb{N}^{k-1}$, define $\Theta_0 \stackrel{\text{def}}{=} 0$ and (see (1.8))

$$\Theta_n \ \stackrel{\text{def}}{=}\ \sum_{j=1}^n L(v^j)(Z_1(v^j) - 1), \quad n \ge 1,$$

and observe that $D_k = \lim_{n\to\infty}\Theta_n$. The limit is independent of the chosen enumeration because all summands are nonnegative. With $\mathcal{H}_0 \stackrel{\text{def}}{=} \mathcal{F}_{k-1}$ and $\mathcal{H}_n \stackrel{\text{def}}{=} \sigma(\mathcal{F}_{k-1}, (T(v^j)_{1\le j\le n})$ for $n \ge 1$, it is now obvious that $(\Theta_n, \mathcal{H}_n)_{n\ge 0}$ forms a martingale. Hence another application of the BE-inequality together with Fatou's lemma and (4.3) leads to

$$\begin{aligned}
\mathbb{E}|D_k|^\alpha \ &\le\ \liminf_{n\to\infty}\ \mathbb{E}|\Theta_n|^\alpha \\
&\le\ 2\sum_{j\ge 1}\mathbb{E}\left|L(v^j)(Z_1(v^j) - 1)\right|^\alpha \\
&=\ 2\,\mathbb{E}|Z_1 - 1|^\alpha \sum_{|v|=k-1} L(v)^\alpha \\
&=\ 2\,\mathbb{E}|Z_1 - 1|^\alpha g(\alpha)^{k-1}
\end{aligned}$$

if $k \ge 2$, and this estimate also holds in case $k = 1$, for $D_1 = Z_1 - 1$. Consequently, as $\mathbb{E}|Z_1 - 1|^\alpha \ll \mathbb{E}Z_1^\alpha$ and $g(\alpha) < 1$,

$$\mathbb{E}W^\alpha \ \le\ \sup_{n\ge 0}\mathbb{E}W_n^\alpha \ \le\ 1 \ +\ 4\,\mathbb{E}|Z_1 - 1|^\alpha \sum_{n\ge 1} g(\alpha)^{n-1} \ <\ \infty.$$

STEP 2. Now let $m \ge 0$ and suppose that the claim is proved for all $\beta \in (1, 2^{m+1}]$. Pick $\alpha \in (2^{m+1}, 2^{m+2}]$ and assume $g(\alpha) < 1$ and $\mathbb{E}Z_1^\alpha < \infty$. Since $\mathbb{E}Z_1^{2^{m+1}} \ll \mathbb{E}Z_1^\alpha < \infty$ and

Lemma 3.2(b) ensures $g(2^{m+1})<1$, Lemma 3.4 shows that it is enough to prove

$$Q_1(m+1,\phi_\alpha)+Q_2(m+1,\phi_\alpha) \;=\; \mathbb{E}\left(\sum_{k\ge 0}\overline{D}_k^{(2^{m+1})}\right)^{\alpha/2^{m+1}} + \sum_{l=0}^{m}\sum_{k\ge 0}\mathbb{E}\left|\overline{D}_k^{(2^l)}\right|^{\alpha/2^l} \;<\; \infty.$$

To this end, put $s\overset{\text{def}}{=}2^{m+1}$ and $\beta\overset{\text{def}}{=}\alpha/s\in(1,2]$. Then, as the sequence $(\sum_{k=0}^n\overline{D}_k^{(s)},\mathcal{F}_n)_{n\ge 0}$ forms a martingale with $\overline{D}_0^{(s)}=1$, another application of the BE-inequality and Fatou's lemma provide us with

$$\begin{aligned}Q_1(m+1,\phi_\alpha) \;&\le\; \liminf_{n\to\infty}\mathbb{E}\left|\sum_{k=0}^{n}\overline{D}_k^{(s)}\right|^{\beta}\\ &\le\; 1\;+\;2\liminf_{n\to\infty}\sum_{k=1}^{n}\mathbb{E}\left|\overline{D}_k^{(s)}\right|^{\beta} \;=\; 1\;+\;2\sum_{k\ge 1}\mathbb{E}\left|\overline{D}_k^{(s)}\right|^{\beta}.\end{aligned}$$

By viewing each $\overline{D}_k^{(s)}$ as a martingale limit, we can use the BE-inequality once more. Using similar arguments as in Step 1, we conclude for each $k\ge 1$

$$\begin{aligned}\mathbb{E}\left|\overline{D}_k^{(s)}\right|^{\beta} \;&\le\; 2\sum_{|v|=k-1}\mathbb{E}L(v)^{s\beta}\mathbb{E}\left|Z_1^{(s)}-g(s)\right|^{\beta}\\ &\le\; 2\,\mathbb{E}(1+Z_1^{(s)})^{\beta}g(\alpha)^{k-1}\end{aligned}$$

and also

$$\mathbb{E}(1+Z_1^{(s)})^{\beta} \;\ll\; \mathbb{E}\left(\sum_{i\ge 1}T_i^s\right)^{\beta} \;\le\; \mathbb{E}\left(\sum_{i\ge 1}T_i\right)^{s\beta} \;=\; \mathbb{E}Z_1^{\alpha} \;<\; \infty.$$

Putting these estimates together, we get

$$Q_1(m+1,\phi_\alpha) \;\le\; 1\;+\;4\,\mathbb{E}(1+Z_1^{(s)})^{\beta}\sum_{k\ge 1}g(\alpha)^{k-1} \;<\; \infty.$$

As to $Q_2(m+1,\phi_\alpha)$, it suffices to prove that, for each $l\in\{0,...,m\}$,

$$U(l,\alpha) \;\overset{\text{def}}{=}\; \sum_{k\ge 2}\mathbb{E}\left|\overline{D}_k^{(2^l)}\right|^{\alpha/2^l} \tag{3.1}$$

is finite because

$$\mathbb{E}\left|\overline{D}_1^{(2^l)}\right|^{\alpha/2^l} \;=\; \mathbb{E}\left|Z_1^{(2^l)}-g(2^l)\right|^{\alpha/2^l} \;\le\; \mathbb{E}\left(1+Z_1^{(2^l)}\right)^{\alpha/2^l} \;\ll\; \mathbb{E}Z_1^{\alpha} \;<\; \infty.$$

If $k\ge 2$, we make use again of the fact that $\overline{D}_k^{(2^l)}$ may be viewed as a martingale limit, namely of $(\sum_{j=1}^n Y_j,\mathcal{H}_n)_{n\ge 0}$ with

$$Y_j \;\overset{\text{def}}{=}\; L(v^j)^p\left(Z_1(v^j)^p-g(p)\right),\quad j\ge 1,$$

where $p \stackrel{\text{def}}{=} 2^l$. Put $\delta \stackrel{\text{def}}{=} \alpha/p > 2$. Then, by the BDG-inequality, for some constant $C \in (0,\infty)$ not depending on k

$$\mathbb{E}\left|\overline{D}_k^{(p)}\right|^\delta \;=\; \mathbb{E}\left|\sum_{j\ge 1} Y_j\right|^\delta \;\le\; C(J_1(k,p,\delta) + J_2(k,p,\delta)), \tag{3.2}$$

where

$$\begin{aligned} J_1(k,p,\delta) &\stackrel{\text{def}}{=} \mathbb{E}\left(\sum_{j\ge 1}\mathbb{E}(Y_j^2|\mathcal{H}_{j-1})\right)^{\delta/2} \;=\; \mathbb{E}\left(\sum_{j\ge 1} L(v^j)^{2p}\mathbb{E}\left(Z_1^{(p)} - g(p)\right)^2\right)^{\delta/2} \\ &\le \mathbb{E}\left(\sum_{|v|=k-1} L(v)^{2p}\mathbb{E}Z_1^{2p}\right)^{\delta/2} \;=\; \mu(2p)^{\delta/2}\mathbb{E}\left(\sum_{|v|=k-1} L(v)^{2p}\right)^{\delta/2} \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} J_2(k,p,\delta) &\stackrel{\text{def}}{=} \sum_{j\ge 1}\mathbb{E}|Y_j|^\delta \;=\; \sum_{|v|=k-1}\mathbb{E}L(v)^{p\delta}\mathbb{E}\left|Z_1^{(p)} - g(p)\right|^\delta \\ &\le \mathbb{E}\left(1 + Z_1^{(p)}\right)^\delta \sum_{|v|=k-1}\mathbb{E}L(v)^\alpha \;=\; \mathbb{E}\left(1 + Z_1^{(p)}\right)^\delta g(\alpha)^{k-1}. \end{aligned} \tag{3.4}$$

Noting that $\mathbb{E}(1 + Z_1^{(p)})^\delta \ll \mathbb{E}(Z_1^{(p)})^\delta \le \mathbb{E}Z_1^{p\delta} = \mathbb{E}Z_1^\alpha < \infty$, we infer

$$U_2(l,\alpha) \;\stackrel{\text{def}}{=}\; \sum_{k\ge 2} J_2(k,p,\delta) \;\ll\; \sum_{k\ge 2} g(\alpha)^{k-1} \;<\; \infty.$$

In view of (3.1) and (3.2) it remains to verify that

$$U_1(l,\alpha) \;\stackrel{\text{def}}{=}\; \sum_{k\ge 2} J_1(k,p,\delta) \;<\; \infty.$$

We start by observing that

$$J_1(k,p,\alpha) \;\le\; \mu(2p)^{\delta/2}\mathbb{E}\left(\sum_{|v|=k-1} L(v)^{2p}\right)^{\delta/2} \;=\; \mu(2p)^{\delta/2}\mathbb{E}\overline{Z}_{k-1}^{\delta/2}$$

where $(\overline{Z}_n)_{n\ge 0}$ denotes the WBP with generic weight sequence $(\overline{T}_i)_{i\ge 1} \stackrel{\text{def}}{=} (T_i^{2p})_{i\ge 1}$. As in Lemma 3.5, write $\overline{g}(u) = \sum_{i\ge 1}\mathbb{E}\overline{T}_i^u = \sum_{i\ge 1}\mathbb{E}T_i^{2pu}$ for $u \ge 1$ and note that $\delta/2 = \alpha/2p > 1$. By assumption and Lemma 3.2. $\overline{g}(1)\vee\overline{g}(\delta/2) = g(2p)\vee g(\alpha) < 1$. Hence Lemma 3.5 ensures the existence of another WBP $\widehat{Z}_n)_{n\ge 0}$ with generic weight sequence $(\widehat{T}_i)_{i\ge 1}$ such that $\overline{Z}_n \le \widehat{Z}_n$ for all $n \ge 0$, $\mathbb{E}\widehat{Z}_1^{\delta/2} \ll \mathbb{E}\overline{Z}_1^{\delta/2}$ and $\widehat{g}(1) < 1$, $\widehat{g}(\delta/2) < \widehat{g}(1)^{\delta/2}$, where $\widehat{g}$ has the obvious meaning. Put

$$\widehat{W}_n \;\stackrel{\text{def}}{=}\; \widehat{Z}_n/\mathbb{E}\widehat{Z}_n \;=\; \widehat{Z}_n/\widehat{g}(1)^n, \quad n \ge 0,$$

and notice that $\mathbb{E}\widehat{W}_n = 1$ for all $n \geq 0$. This leads to

$$J_1(k,p,\delta) \ \leq\ \mu(2p)^{\delta/2}\mathbb{E}\overline{Z}_{k-1}^{\delta/2} \ \leq\ \mu(2p)^{\delta/2}\mathbb{E}\widehat{Z}_{k-1}^{\delta/2} \ \leq\ \mu(2p)^{\delta/2}\widehat{g}(1)^{(k-1)\delta/2}\sup_{n\geq 0}\mathbb{E}\widehat{W}_n^{\delta/2}.$$

Evidently, $(\widehat{W}_n)_{n\geq 0}$ can also be viewed as a normalized WBP with generic weight sequence $(\widehat{T}_i/\widehat{g}(1))_{i\geq 1}$. Applying the inductive hypothesis to $(\widehat{W}_n)_{n\geq 0}$ instead of $(W_n)_{n\geq 0}$, it follows that

$$\sup_{n\geq 0}\mathbb{E}\widehat{W}_n^{\delta/2} \ <\ \infty,$$

because

(1) $\delta/2 = \alpha/2p \in (1, 2^{m+1}]$,
(2) $\mathbb{E}\widehat{W}_1^{\delta/2} = \widehat{g}(1)^{-\delta/2}\mathbb{E}\widehat{Z}_1^{\delta/2} \ll \mathbb{E}\overline{Z}_1^{\delta/2} = \mathbb{E}(\sum_{i\geq 1} T_i^{2p})^{\delta/2} \leq \mathbb{E}Z_1^{p\delta} = \mathbb{E}Z_1^{\alpha} < \infty$, and
(3) $\sum_{i\geq 1}\mathbb{E}\left(\frac{\widehat{T}_i}{\widehat{g}(1)}\right)^{\delta/2} = \widehat{g}(1)^{-\delta/2}\widehat{g}(\delta/2) < 1$.

To finish the proof, observe that

$$U_1(l,\alpha) \ =\ \sum_{k\geq 2} J_1(k,p,\alpha) \ \leq\ \mu(2p)^{\delta/2}\sup_{n\geq 0}\mathbb{E}\widehat{W}_n^{\delta/2}\sum_{k\geq 1}\widehat{g}(1)^{k\delta/2} \ <\ \infty,$$

because $\mu(2p) \ll \mu(\alpha) < \infty$ and $\widehat{g}(1)^{\delta/2} < \widehat{g}(1) < 1$.

Turning to "(b)⇒(a)" suppose that $\mathbb{E}W^\alpha \in (0,\infty)$ for some $\alpha > 1$. By using the fixed-point equation (1.9) for W and the superadditivity of ϕ_α, we infer

$$W^\alpha \ \geq\ \sum_{i\geq 1} T_i(\emptyset)^\alpha {W^{(i)}}^\alpha \quad \mathbb{P}\text{-a.s.}$$

and even strict inequality with positive probability as a consequence of (C4), the independence of $T(\emptyset)$ and $W^{(1)}, W^{(2)}, \ldots$ and the fact that $\mathbb{E}W > 0$. Therefore

$$\mathbb{E}W^\alpha \ >\ \sum_{i\geq 1}\mathbb{E}\left(T_i(\emptyset)^\alpha {W^{(i)}}^\alpha\right) \ =\ g(\alpha)\,\mathbb{E}W^\alpha,$$

showing $g(\alpha) < 1$ because $\mathbb{E}W^\alpha$ is positive. The latter argument has also been employed in [29] and [40] in the context of stochastic fixed-point equations. ♢

4. The associated multiplicative random walk

In the following, the measure Λ defined by

$$\Lambda(A) \ \stackrel{\text{def}}{=}\ \mathbb{E}\left(\sum_{i\geq 1} T_i\delta_{T_i}(A)\right) \ =\ \sum_{i\geq 1}\mathbb{E}T_i\mathbf{1}_A(T_i), \quad A \in \mathfrak{B}(\mathbb{R}),$$

will play an important role and thus studied in some detail. We first note that it is a probability measure on $(0,\infty)$ due to our standing assumption $g(1) = \sum_{i\geq 1}\mathbb{E}T_i = 1$. Let $X_1, X_2, \ldots$ be i.i.d. random variables with common distribution Λ and denote by by $(M_n)_{n\ge 0}$ the associated multiplicative random walk starting at 1, i.e. $M_0 \stackrel{\text{def}}{=} 1$ and

$$M_n \stackrel{\text{def}}{=} \prod_{i=1}^{n} X_i, \quad n \ge 1.$$

The following result provides the connection of $(M_n)_{n\ge 0}$ to WBP's and is due to Biggins and Kyprianou [20, Lemma 4.1(iii)] if $\mathbb{P}(N < \infty) = 1$. Since the extension to the present situation does not require any extra argument, it is stated without proof. In the case where $\mathbb{P}(N \le k) = 1$ for some $k \in \mathbb{N}$, the very same random walk, more precisely its additive version $\log M_n$, was already used by Durrett and Liggett [29].

LEMMA 4.1. *The following assertions hold true under (C1).*

(a) For all $n \ge 0$ and measurable functions $g : \mathbb{R}^n \to [0,\infty)$,

$$\mathbb{E}g(M_0, ..., M_n) \;=\; \mathbb{E}\left(\sum_{|v|=n} L(v)g(L(v|0), ..., L(v|n-1), L(v))\right), \tag{4.1}$$

in particular

$$\mathbb{E}f(M_n) \;=\; \mathbb{E}\left(\sum_{|v|=n} L(v)f(L(v))\right), \tag{4.2}$$

$$g(\alpha)^n \;=\; \mathbb{E}\left(\sum_{|v|=n} L(v)^{\alpha}\right) \;=\; \mathbb{E}M_n^{\alpha-1} \tag{4.3}$$

for all measurable $f : \mathbb{R} \to [0,\infty)$, $n \in \mathbb{N}_0$ and $\alpha > 1$.

(b) Let $\Psi : \mathbb{R}^2 \to [0,\infty)$ be a measurable function. If, for fixed $n \in \mathbb{N}_0$, $\Pi(v), v \in \mathbb{N}^n$, denote i.i.d. real-valued random variables with generic copy Π such that

- *$(\Pi(v))_{v\in\mathbb{N}^n}$ is independent of $\mathcal{F}_n$, and*
- *Π is independent of $(M_n)_{n\ge 0}$,*

then

$$\mathbb{E}\left(\sum_{|v|=n} L(v)\Psi(L(v), \Pi(v))\right) \;=\; \mathbb{E}\Psi(M_n, \Pi). \tag{4.4}$$

The following common partial order relations $\prec$ and $\preceq$ on $\mathbb{V}$ will be needed hereafter: Write $v \prec w$ if $v \ne w$ and v belongs to the ancestral line of w, while $v \preceq w$ also allows $v = w$. Moreover, $v \prec (\preceq)\, C$ for any $C \subset \mathbb{V}$ shall mean $v \prec (\preceq)\, w$ for some $w \in C$.

We will now extend the previous lemma to a certain class of stopping lines, called *homogeneous stopping lines (HSL)* hereafter. For this purpose let $\underline{\sigma} : [0,\infty)^{\mathbb{N}_0} \to \mathbb{N}_0 \cup \{\infty\}$,

$$\underline{\sigma}(x_0, x_1, ...) \stackrel{\text{def}}{=} \inf\{n \ge 0 : (x_0, ..., x_n) \in B_n\}$$

be a formal stopping rule, where $B_n \in \mathfrak{B}(\mathbb{R}^{n+1})$ for $n \geq 0$ and $\inf \emptyset \stackrel{\text{def}}{=} \infty$. For each $\mathbf{v} = (v_1, v_2, ...) \in \mathbb{N}^{\mathbb{N}}$ (viewed as the boundary of $\mathbb{V}$), we further define

$$\underline{\sigma}_{\mathbf{v}} \stackrel{\text{def}}{=} \underline{\sigma}(\boldsymbol{L}(\mathbf{v})), \quad \boldsymbol{L}(\mathbf{v}) \stackrel{\text{def}}{=} (L(\emptyset), L(v_1), L(v_1v_2), ...),$$

and then

$$\mathcal{S} \stackrel{\text{def}}{=} \{\mathbf{v}|\underline{\sigma}_{\mathbf{v}} : \mathbf{v} \in \mathbb{N}^{\mathbb{N}}\} \cap \mathbb{V} = \{\mathbf{v}|\underline{\sigma}_{\mathbf{v}} : \mathbf{v} \in \mathbb{N}^{\mathbb{N}}, \underline{\sigma}_{\mathbf{v}} < \infty\},$$

where $\mathbf{v}|0 \stackrel{\text{def}}{=} \emptyset$, $\mathbf{v}|n \stackrel{\text{def}}{=} v_1...v_n$ for $n \in \mathbb{N}$, and $\mathbf{v}|\infty \stackrel{\text{def}}{=} \mathbf{v}$. We call $\mathcal{S}$ the *HSL associated with* $\underline{\sigma}$. It consists of all nodes $v \in \mathbb{V}$ that are obtained as stopping places when applying the same rule $\underline{\sigma}$ along all infinite sequences of branch weights $\boldsymbol{L}(\mathbf{v})$. Notice that $\mathcal{S}$ may be empty. Define

$$\mathcal{F}_{\mathcal{S}} \stackrel{\text{def}}{=} \sigma(L(\emptyset), T(v), v \prec \mathcal{S}) \quad \text{and} \quad Z_{\mathcal{S}} \stackrel{\text{def}}{=} \sum_{v \in \mathcal{S}} L(v).$$

Stopping lines, also called optional lines, have been used in various works on branching models, e.g. [27], [39] [20] and [21]. Jagers [34] has the most general definition of an optional line and provides also the basic framework, while [21] contains the definition that is closest to that of an HSL and called very simple line there.

Given two stopping rules $\underline{\sigma}_1, \underline{\sigma}_2$ with associated HSL $\mathcal{S}_1, \mathcal{S}_2$, let $\mathcal{S}_1 \wedge \mathcal{S}_2$ be the HSL associated with $\underline{\sigma}_1 \wedge \underline{\sigma}_2$. In case $\underline{\sigma}_2 = n$ for some $n \in \mathbb{N}_0$, we simply write $\mathcal{S}_1 \wedge n$. Finally, we put $\sigma \stackrel{\text{def}}{=} \underline{\sigma}(M_0, M_1, ...)$. In slight abuse of terminology, but justified by the next lemma, we call $\mathcal{S}$ hereafter also the HSL associated with σ. Recall from the Introduction that $W^{(v)}$ equals the a.s. limit of $Z_n(v) = \sum_{|w|=n} L_v(w)$, as $n \to \infty$. The following lemma may be essentially derived from related results in [21, Section 14]. Its proof is therefore omitted.

LEMMA 4.2. *Given any HSL $\mathcal{S}$ associated with a stopping rule $\underline{\sigma}$, the following assertions hold true under (C1).*

(a) For all $n \geq 0$ and measurable functions $g : \mathbb{R}^n \to [0, \infty)$,

$$\int_{\{\sigma=n\}} g(M_0, ..., M_n)\, d\mathbb{P} = \mathbb{E}\left(\sum_{v \in \mathcal{S}, |v|=n} L(v) g(L(v|0), ..., L(v|n-1), L(v))\right), \tag{4.5}$$

in particular

$$\mathbb{P}(\sigma = n) = \mathbb{E}\left(\sum_{v \in \mathcal{S}, |v|=n} L(v)\right) \quad \text{and} \quad \mathbb{P}(\sigma < \infty) = \mathbb{E}Z_{\mathcal{S}}. \tag{4.6}$$

Moreover,

$$\mathbb{E}f(M_\sigma)\mathbf{1}_{\{\sigma<\infty\}} = \mathbb{E}\left(\sum_{v \in \mathcal{S}} L(v) f(L(v))\right) \tag{4.7}$$

for all measurable $f : \mathbb{R} \to [0, \infty)$ and $n \in \mathbb{N}_0$.

(b) *Suppose that* $\Psi : \mathbb{R}^2 \to [0,\infty)$ *is a measurable function. If* $\Pi(v), v \in \mathbb{V}$, *denotes a family of i.i.d. real-valued random variables with generic copy* Π *such that*

- $(\Pi(v))_{v\in\mathbb{N}^n}$ *is independent of* $\mathcal{F}_n$ *for each* $n \in \mathbb{N}_0$, *and*
- Π *is independent of* $(M_n)_{n\geq 0}$,

then

$$\mathbb{E}\left(\sum_{v\in\mathcal{S}} L(v)\Psi(L(v),\Pi(v))\right) \;=\; \mathbb{E}\Psi(M_\sigma,\Pi)\mathbf{1}_{\{\sigma<\infty\}} \tag{4.8}$$

and

$$\mathbb{E}\left(\sum_{v\prec\mathcal{S}} L(v)\Psi(L(v),\Pi(v))\right) \;=\; \mathbb{E}\left(\sum_{n=0}^{\sigma-1}\Psi(M_n,\Pi)\right). \tag{4.9}$$

(c) *If* $\mathbb{P}(\sigma < \infty) = 1$ *and (C2) holds true, then*

$$W \;=\; \sum_{v\in\mathcal{S}} L(v)W^{(v)}, \tag{4.10}$$

and $(Z_{\mathcal{S}\wedge n})_{n\geq 0}$ *forms a uniformly integrable martingale with respect to* $(\mathcal{F}_{\mathcal{S}\wedge n})_{n\geq 0}$ *satisfying* $Z_{\mathcal{S}\wedge n} = \mathbb{E}(W|\mathcal{F}_{\mathcal{S}\wedge n})$ *a.s. for all* $n \geq 0$ *as well as* $Z_{\mathcal{S}} = \mathbb{E}(W|\mathcal{F}_{\mathcal{S}})$ *a.s.*

5. A weighted branching model derived from ladder epochs

Unlike the previous two lemmata which only required condition (C1) (apart from 4.2(c)) we continue to assume from now on (C1-4) as we did before. Then the additive random walk $(\log M_n)_{n\geq 0}$ has finite negative drift because, by (4.1) and (C2),

$$\mathbb{E}\log M_1 \;=\; \mathbb{E}\left(\sum_{i\geq 1} T_i \log T_i\right) \;=\; \gamma \;\in\; (-\infty, 0).$$

Hence $M_n \to 0$ a.s. Note also that each M_n is a.s. (strictly) positive as following from (4.2) with $f = \mathbf{1}_{(0,\infty)}$.

Now fix any $a \in (0,1]$ with $\mathbb{E}\log(M_1/a) < 0$, put $S_0 \stackrel{\text{def}}{=} 0$ and $S_n \stackrel{\text{def}}{=} \log M_n - n\log a$ for $n \geq 1$, and let $(\sigma_n^<)_{n\geq 0}$ denote the sequence of a.s. finite strictly descending ladder epochs associated with $(S_n)_{n\geq 0}$, so $\sigma_0^< \stackrel{\text{def}}{=} 0$ and

$$\sigma_n^< \;\stackrel{\text{def}}{=}\; \inf\{k > \sigma_{n-1}^< : S_k - S_{\sigma_{n-1}^<} < 0\} \;=\; \inf\left\{k > \sigma_{n-1}^< : \frac{M_k}{a^k} < \frac{M_{\sigma_{n-1}^<}}{a^{\sigma_{n-1}^<}}\right\}$$

for $n \geq 1$. Let $\mathcal{S}_n^<$ denote the HSL associated with $\sigma_n^<$, put $\mathbb{V}^< \stackrel{\text{def}}{=} \cup_{n\geq 0}\mathcal{S}_n^<$ and, for each $n \geq 0$ and $v \in \mathcal{S}_n^<$,

$$T^<(v) \;\stackrel{\text{def}}{=}\; (L_v(w))_{vw\in\mathcal{S}_{n+1}^<},$$

where for sake of definiteness the positions of the $L_v(w)$ in the right-hand sequence are in decreasing order as to their size. The reader should observe that $T^<(v)$ is a.s. an *infinite* sequence because $L_v(i) \to 0$, as $i \to \infty$, and $vi \in \mathbb{S}_{n+1}^<$ for all $i \geq 1$ with $L_v(i) < a$. As a consequence of our model assumptions, the family

$$\{T^<(v),\ v \in \mathbb{V}^<\}$$

consists of i.i.d. weight vectors, the components of which are all bounded by a. Let us view $\mathbb{V}^<$ as the subtree of $\mathbb{V}$ with the same root (for $\mathbb{S}_0^< = \{\emptyset\}$) obtained by discarding all nodes in $\mathbb{V}\backslash\mathbb{V}^<$ and drawing edges between any v and vw with $v \in \mathbb{S}_n^<$ and $vw \in \mathbb{S}_{n+1}^<$ for some $n \geq 0$. In natural compliance with the original weighted branching model we interpret any component $L_v(w)$ from $T^<(v)$ as the weight attached to the edge from v to vw. This provides us with a new weighted branching model imbedded in the original one and derived from the ladder epochs $\sigma_n^<$, as announced in the section title. The WBP associated with this model is given by

$$Z_n^< \stackrel{\text{def}}{=} Z_{\mathbb{S}_n^<} = \sum_{v \in \mathbb{S}_n^<} L(v), \quad n \geq 0. \tag{5.1}$$

Put $\mathcal{F}_n^< \stackrel{\text{def}}{=} \mathcal{F}_{\mathbb{S}_n^<}$ for $n \geq 0$. The following result ensures that our basic assumptions (C1-4) carry over to the imbedded model including the important fact that the a.s. limit of $Z_n^<$ is still W.

Proposition 5.1. *In the previous notation, the imbedded weighted branching model based on $(T^<(v))_{v\in\mathbb{V}^<}$ also satisfies (C1-4). Moreover, $(Z_n^<, \mathcal{F}_n^<)_{n\geq 0}$ is a uniformly integrable martingale with a.s. limit W, thus $Z_n^< = \mathbb{E}(W|\mathcal{F}_n^<)$ a.s. for all $n \geq 0$, and*

$$W = \sum_{v \in \mathbb{S}_n^<} L(v) W^{(v)} \tag{5.2}$$

Proof. Let $T^< = (T_i^<)_{i\geq 1}$ denote a generic copy of $T^<$. Since each $\sigma_n^<$ is a.s. finite, we have $\mathbb{E}Z_n^< = \mathbb{P}(\sigma_n^< < \infty) = 1$ for all $n \geq 0$ by (4.6). This shows (C1) for the imbedded model. As to (C3), note that, since all $T_i^< < a \leq 1$ and $\sum_{i\geq 1} \mathbb{E}T_i^< = 1$, we infer

$$\mathbb{P}(T^< \in \{0,1\}^{\mathbb{N}}) = \mathbb{P}(T_i^< = 0 \text{ for all } i \geq 1) < 1.$$

The same facts further imply for $N^< \stackrel{\text{def}}{=} \sum_{i\geq 1} \mathbf{1}_{\{T_i^< > 0\}}$

$$a\,\mathbb{E}N^< > \mathbb{E}\left(\sum_{i=1}^{N^<} T_i^<\right) = \sum_{i\geq 1} \mathbb{E}T_i^< = 1$$

and so $\mathbb{E}N^< > 1/a > 1$ which proves (C4). Next use $\mathbb{E}\sigma_1^< < \infty$, $\mathbb{E}\log M_1 \in (-\infty, 0)$, (4.7) and Wald's identity to obtain

$$\sum_{i\geq 1} \mathbb{E}T_i^< \log T_i^< = \sum_{v \in \mathbb{S}_1^<} \mathbb{E}L(v) \log L(v) = \mathbb{E}\log M_{\sigma_1^<} = \mathbb{E}\sigma_1^<\, \mathbb{E}\log M_1 \in (-\infty, 0)$$

and thus the first half of (C2) for the imbedded model. Left with the verification of the second half, that is

$$\mathbb{E}Z_1^< \log^+ Z_1^< \ < \ \infty,$$

it suffices to invoke Theorem 1.1(a) if we still prove that W is also the a.s. limit of $(Z_n^<)_{n\geq 0}$. But since all $\sigma_n^<$ are a.s. finite, we infer from Lemma 4.2(c) that $Z_n^< = \mathbb{E}(W|\mathcal{F}_n^<)$ a.s. for all $n \geq 0$ as well as (5.2). In particular, $(Z_n^<, \mathcal{F}_n^<)_{n\geq 0}$ constitutes a uniformly integrable martingale with a.s. limit W. This completes the proof of the proposition. ♢

For the next lemma we first have to recall some facts from renewal theory as applied to $S_n = \log M_n - n\log a$, $n \geq 0$. Put $S^* \stackrel{\text{def}}{=} \sup_{n\geq 0} S_n$ and $M^* \stackrel{\text{def}}{=} e^{S^*} = \sup_{n\geq 0} a^{-n}M_n$. By Lemma 2 in [35],

$$\mathbb{E}f(S^*) \ = \ \frac{1}{\mathbb{E}\sigma_1^<}\mathbb{E}\left(\sum_{n=0}^{\sigma_1^<-1} f(S_n)\right) \tag{5.3}$$

for any nonnegative measurable function f as $\sigma_1^<$ is the first descending ladder epoch of $(S_n)_{n\geq 0}$. Moreover, S^* possesses a useful distributional representation in terms of the first strictly ascending ladder height distribution, known as the Pollaczek-Khintchine formula, see e.g. [6, Theorem IX.2.3]. To state it, let $\sigma^> \stackrel{\text{def}}{=} \inf\{n : S_n > 0\}$ be the defective first strictly ascending ladder epoch of $(S_n)_{n\geq 0}$, thus $\nu \stackrel{\text{def}}{=} \mathbb{P}(\sigma^> < \infty) \in (0,1)$, and put $Q^> \stackrel{\text{def}}{=} \mathbb{P}(S_{\sigma^>} \in \cdot|\sigma^> < \infty)$. Let further $(\widehat{S}_n)_{n\geq 0}$ be a zero-delayed renewal process with increment distribution $Q^>$ and ζ an independent geometric random variable with parameter ν, i.e. $\mathbb{P}(\zeta = n) = (1-\nu)\nu^n$ for $n \in \mathbb{N}_0$. Then S^* satisfies the distributional relation

$$S^* \ \stackrel{d}{=} \ \widehat{S}_\zeta \tag{5.4}$$

or, equivalently,

$$\mathbb{P}(S^* \in \cdot) \ = \ (1-\nu)\sum_{n\geq 0} \nu^n Q^{>*n} \tag{5.5}$$

where $Q^{>*n}$ denotes the n-fold convolution of $Q^>$ and $Q^{>*0} \stackrel{\text{def}}{=} \delta_0$. Putting $\widehat{M}_n \stackrel{\text{def}}{=} e^{\widehat{S}_n}$ for $n \geq 0$, (5.4) may of course be rewritten in multiplicative form as

$$M^* \ \stackrel{d}{=} \ \widehat{M}_\zeta. \tag{5.6}$$

By combining these facts with Lemma 4.2(b) specialized to $\mathcal{S} = \mathcal{S}_1^<$, the following result is immediate from (4.9).

LEMMA 5.2. *Let Π be a copy of the $\Pi(v)$, $v \in \mathbb{V}$, and independent of $(M_n)_{n\geq 0}$, $(\widehat{M}_n)_{n\geq 0}$ and ζ. Then*

$$\mathbb{E}\left(\sum_{v\prec\mathcal{S}_1^<} L(v)\Psi(L(v),\Pi(v))\right) \ = \ \mathbb{E}\sigma_1^<\,\mathbb{E}\Psi(M^*,\Pi) \ = \ \mathbb{E}\sigma_1^<\,\mathbb{E}\Psi(\widehat{M}_\zeta,\Pi). \tag{5.7}$$

The remainder of this section is devoted to a series of lemmata on the behavior of the moments of M_n and M^*.

LEMMA 5.3. *Given $\alpha \geq 0$, $\ell \in \mathfrak{R}_0^*$, and a nonnegative random variable X with $X \geq \eta$ a.s. for some $\eta > 1$ and $\mathbb{E}X^\alpha \ell(X) < \infty$, the following assertion holds true: For each $\mu > \mu_\alpha \stackrel{\text{def}}{=} \mathbb{E}X^\alpha$ there exists $b = b_\mu > 0$ such that*

$$\sup_{x \geq b} \mathbb{E}\left(X^\alpha \frac{\ell(xX)}{\ell(x)}\right) \; < \; \mu. \tag{5.8}$$

Moreover, if $(M_n)_{n\geq 0}$ denotes a multiplicative random walk with $M_0 = 1$ and $M_1 \stackrel{d}{=} X$, then

$$\mathbb{E}M_n^\alpha \ell(M_n) \; \leq \; C\mu^n \tag{5.9}$$

for all $n \geq 0$, $\mu > \mu_\alpha$ and some $C = C_\mu \in (0, \infty)$.

PROOF. The function ℓ being submultiplicative, we see that $\frac{\ell(xX)}{\ell(x)} \leq \ell(X)$ a.s. for all $x \geq 1$ whence the family $\{X^\alpha \frac{\ell(xX)}{\ell(x)}, x \geq 1\}$ is dominated by $X^\alpha \ell(X)$ and thus uniformly integrable. Since, furthermore, $\lim_{x\to\infty} X^\alpha \frac{\ell(xX)}{\ell(x)} = X^\alpha$ a.s., we infer (5.8) with the help of the dominated convergence theorem.

Turning to (5.9), fix any $\mu > \mu_\alpha$ and then b according with (5.8). Let $k \in \mathbb{N}$ be so large that $\eta^k \geq b$. Then for all $n > k$

$$\begin{aligned}
\mathbb{E}M_n^\alpha \ell(M_n) \; &= \; \int x^\alpha \ell(x)\, \mathbb{E}\bigg(M_1^\alpha \frac{\ell(xM_1)}{\ell(x)}\bigg)\; \mathbb{P}(M_{n-1} \in dx) \\
&\leq \; \int x^\alpha \ell(x) \sup_{x \geq b} \mathbb{E}\bigg(M_1^\alpha \frac{\ell(xM_1)}{\ell(x)}\bigg)\; \mathbb{P}(M_{n-1} \in dx) \\
&\leq \; \mu\, \mathbb{E}M_{n-1}^\alpha \ell(M_{n-1})
\end{aligned}$$

and therefore

$$\mathbb{E}M_n^\alpha \ell(M_n) \; \leq \; \mu^{n-k}\mathbb{E}M_k \ell(M_k)$$

for all $n > k$. Combining this with

$$\mathbb{E}M_k \ell(M_k) \; \leq \; \Big(\mathbb{E}X^\alpha \ell(X)\Big)^k \; < \; \infty,$$

where again the submultiplicativity of ℓ has been utilized, the assertion easily follows. $\diamondsuit$

LEMMA 5.4. *Let $\ell \in \mathfrak{R}_0^*$.*

(a) If $\alpha > 1$, then $\mathbb{E}\widehat{M}_1^{\alpha-1}\ell(\widehat{M}_1) < \infty$ iff $\mathbb{E}M_1^{\alpha-1}\ell(M_1) = \sum_{i\geq 1} \mathbb{E}T_i^\alpha \ell(T_1) < \infty$.

(b) $\mathbb{E}\ell(\widehat{M}_1) < \infty$ iff $\mathbb{E}\mathbb{U}\ell(M_1) < \infty$.

PROOF. We write $A \simeq B$ hereafter if $A < \infty$ holds iff $B < \infty$. Let us start by noting the following tail estimate from renewal theory for the (defective) ladder height $S_{\sigma^>}\mathbf{1}_{\{\sigma^> < \infty\}}$ (see

[2, eq. (2.15)]): As $t \to \infty$,

$$\mathbb{P}(S_{\sigma^>} > t, \sigma^> < \infty) \asymp \int_t^\infty \mathbb{P}(S_1 > s)\, ds.$$

As $\mathbb{P}(\widehat{M}_1 \in \cdot) = \mathbb{P}(e^{S_{\sigma^>}} \in \cdot | \sigma^> < \infty)$, this gives

$$\mathbb{P}(\widehat{M}_1 > t) = \nu^{-1}\mathbb{P}(S_{\sigma^>} > \log t, \sigma^> < \infty) \asymp \int_{\log t}^\infty \mathbb{P}(S_1 > s)\, ds$$

as $t \to \infty$. For $\alpha > 1$, we will further need that $x^{\alpha-1}\ell(x)$, ℓ in normalized form (2.9), has derivative $\alpha x^{\alpha-2}\ell(x) + \mathbf{1}_{(1,\infty)}(x) x^{\alpha-2}\varepsilon(x)\ell(x) \asymp x^{\alpha-2}\ell(x)$, as $x \to \infty$.

(a) With the help of these estimates, we infer

$$\begin{aligned}
\mathbb{E}\widehat{M}_1^{\alpha-1}\ell(\widehat{M}_1) &\simeq \int_0^\infty t^{\alpha-2}\ell(t)\,\mathbb{P}(\widehat{M}_1 > t)\, dt \\
&\simeq \int_1^\infty t^{\alpha-2}\ell(t) \int_{\log t}^\infty \mathbb{P}(S_1 > s)\, ds\, dt \\
&= \int_1^\infty t^{\alpha-2}\ell(t) \int_t^\infty x^{-1}\mathbb{P}(S_1 > \log x)\, dx\, dt \\
&= \int_1^\infty x^{-1}\mathbb{P}(M_1 > x) \int_1^x t^{\alpha-2}\ell(t)\, dt\, dx \\
&\simeq \int_1^\infty x^{\alpha-2}\ell(x)\,\mathbb{P}(M_1 > x)\, dx \\
&\simeq \mathbb{E}M_1^{\alpha-1}\ell(M_1).
\end{aligned}$$

(b) Using again the above tail estimate in combination with $(\mathbb{U}\ell)'(x) = x^{-1}\ell(x)$ for $x > 1$, we obtain by a similar estimation as before

$$\begin{aligned}
\mathbb{E}\ell(\widehat{M}_1) &\simeq \int_0^\infty \ell'(t)\,\mathbb{P}(\widehat{M}_1 > t)\, dt \\
&= \int_1^\infty \ell'(t) \int_t^\infty x^{-1}\mathbb{P}(S_1 > \log x)\, dx\, dt \\
&= \int_1^\infty x^{-1}\mathbb{P}(M_1 > x) \int_1^x \ell'(t)\, dt\, dx \\
&= \int_1^\infty x^{-1}\ell(x)\,\mathbb{P}(M_1 > x)\, dx \\
&\simeq \mathbb{E}\mathbb{U}\ell(M_1). \qquad \diamondsuit
\end{aligned}$$

With the help of the previous lemmata we are now able to derive a crucial moment result for $M^* = \sup_{n\geq 0} a^{-n}M_n$, in case $\alpha > 1$ for suitably chosen $a \in (0,1]$. Rewritten in terms of $S^* = e^{M^*}$, it may be viewed as an extension of Theorem 3 in [2] and Theorem 2 in [54].

LEMMA 5.5. *Let $\ell \in \mathfrak{R}_0^*$.*

(a) If $\alpha > 1$ and $a \in (0,1]$ is chosen such that $g(\alpha) = \mathbb{E}M_1^{\alpha-1} \leq \mathbb{E}(M_1/a)^{\alpha-1} < 1$, then $\mathbb{E}(M^*)^{\alpha-1}\ell(M^*) < \infty$ holds true iff $\mathbb{E}M_1^{\alpha-1}\ell(M_1) < \infty$.

(b) If ℓ is unbounded, then $\mathbb{E}\ell(M^*)<\infty$ holds true iff $\mathbb{E}\mathbb{U}\ell(M_1)<\infty$.

PROOF. (a) Given $\mathbb{E}M_1^{\alpha-1}\ell(M_1)<\infty$, Lemma 5.4 ensures $\mathbb{E}\widehat{M}_1^{\alpha-1}\ell(\widehat{M}_1)<\infty$. Since $\mathbb{E}(M_1/a)^{\alpha-1}<1$, $((M_n/a^n)^{\alpha-1})_{n\ge 0}$ forms a supermartingale converging a.s. to 0. Hence, by the optional sampling theorem in combination with Fatou's lemma,

$$\nu\,\mathbb{E}\widehat{M}_1^{\alpha-1} \;=\; \mathbb{E}\left(\frac{M_{\sigma^>}}{a^{\sigma^>}}\right)^{\alpha-1}\mathbf{1}_{\{\sigma^><\infty\}} \;\le\; \liminf_{n\to\infty}\mathbb{E}\left(\frac{M_{\sigma^>\wedge n}}{a^{\sigma^>\wedge n}}\right)^{\alpha-1} \;\le\; \mathbb{E}\left(\frac{M_1}{a}\right)^{\alpha-1} \;<\; 1.$$

Next, Lemma 5.3 allows us to pick any μ close enough to $\mathbb{E}\widehat{M}_1^{\alpha-1}$ such that $\nu\mu<1$ and

$$\mathbb{E}\widehat{M}_n^{\alpha-1}\ell(\widehat{M}_n) \;\le\; C\mu^n$$

for all $n\ge 0$ and some $C>0$. Now use (5.6) to conclude

$$\begin{aligned}\mathbb{E}(M^*)^{\alpha-1}\ell(M^*) \;&=\; \mathbb{E}\widehat{M}_\zeta^{\alpha-1}\ell(\widehat{M}_\zeta)\\ &=\; (1-\nu)\sum_{n\ge 0}\nu^n\,\mathbb{E}\widehat{M}_n^{\alpha-1}\ell(\widehat{M}_n)\\ &\le\; C(1-\nu)\sum_{n\ge 0}(\nu\mu)^n \;<\; \infty,\end{aligned}$$

as claimed. For the converse it suffices to note that

$$(1-\nu)\nu\,\mathbb{E}\widehat{M}_1^{\alpha-1}\ell(\widehat{M}_1)\le\; \mathbb{E}(M^*)^{\alpha-1}\ell(M^*) \;<\; \infty$$

implies $\mathbb{E}M_1^{\alpha-1}\ell(M_1)<\infty$ by another appeal to Lemma 5.4.

(b) Here, Lemma 5.3 allows us to pick a $1<\mu<\nu^{-1}$ such that $\mathbb{E}\ell(\widehat{M}_n)\le C\mu^n$ for all $n\ge 0$ and some $C>0$. The remaining arguments using (5.6) and Lemma 5.4 are very similar to those for part (a) and are therefore omitted. ◇

The final lemma of this section may be viewed as an extension of Lemma 5.3.

LEMMA 5.6. *Let $\ell\in\mathfrak{R}_0^*$ and $(M_n)_{n\ge 0}$ be a nonnegative multiplicative random walk with $M_0=1$. Put $\mu_\alpha\overset{\rm def}{=}\mathbb{E}M_1^\alpha$ and $\gamma_\alpha\overset{\rm def}{=}\mathbb{E}M_1^\alpha\ell(M_1)$ for $\alpha\ge 0$.*

(a) If $\alpha>0$, $\mu_\alpha<1$ and $\gamma_\alpha<\infty$, then there exists $\mu<1$ such that $\mathbb{E}M_n^\alpha\ell(M_n)\le C\mu^n$ for all $n\ge 0$ and some $C>0$.

(b) If $\gamma_0<\infty$, then $\mathbb{E}\ell(M_n)=o(\mu^n)$ as $n\to\infty$ for any $\mu>1$.

PROOF. (a) By a similar argument as in the proof of Lemma 5.3, we can pick $b\ge 1$ and $\mu\in(\mu_\alpha,1)$ such that

$$\sup_{x\ge b}\mathbb{E}\left(M_1^\alpha\frac{\ell(xM_1)}{\ell(x)}\right) \;\le\; \mu.$$

Consequently, by splitting up the range of integration into $\{M_{n-1} \le b\}$ and its complement,

$$\mu^{-n}\mathbb{E}M_n^\alpha \ell(M_n) \;\le\; \ell(b)\,\gamma_\alpha\left(\frac{\mu_\alpha}{\mu}\right)^{n-1} \;+\; \mu^{-n+1}\mathbb{E}M_{n-1}^\alpha \ell(M_{n-1})$$

follows for all $n \ge 1$, and this easily yields the assertion.

(b) Fix any $\mu > 1$ and $\varepsilon > 0$ and choose b so large that $\mathbb{E}\ell(M_1)\mathbf{1}_{\{M_1>b\}} < \varepsilon$ and

$$\sup_{x\ge b}\mathbb{E}\left(\frac{\ell(xM_1)}{\ell(x)}\right) \;\le\; \mu.$$

Let $X_1 = M_1, X_2, \ldots$ be the i.i.d. factors of the random walk $(M_n)_{n\ge0}$. By integrating separately over $\{X_n > b\}$, $\{M_{n-1} > b, X_n \le b\}$ and $\{M_{n-1} \le b, X_n \le b\}$, it is easily verified that

$$\mathbb{E}\ell(M_n) \;\le\; (\mu+\varepsilon)\,\mathbb{E}\ell(M_{n-1}) \;+\; \ell(b)^2$$

for all $n \ge 1$, and this in turn implies $\mathbb{E}\ell(M_n) = O((\mu+\varepsilon)^n)$, as $n \to \infty$. ◊

6. A pathwise renewal theorem

As a final prerequisite for the proof of our main results, this section will provide a pathwise renewal theorem which is also of interest in its own right. Under the stated conditions it is tailored to our needs but it actually belongs to a larger class of related results derived in [3] under more general assumptions. Earlier results of such type are due to Nerman [48] and Gatzouras [30].

Suppose that the multiplicative random walk $(M_n)_{n\ge0}$ as in the previous sections satisfies $M_n < 1$ a.s., thus $\sup_{i\ge1} T_i < 1$ a.s. As renewal theory is usually cast in the framework of additive random walks with positive drift, we put $S_n \stackrel{\text{def}}{=} -\log M_n$ for $n \ge 0$ (which up to a sign change equals the definition of Section 5 with $a = 1$) and consider the a.s. finite first passage time

$$\tau(b) \;\stackrel{\text{def}}{=}\; \inf\{n \ge 0 : S_n > b\} \;=\; \inf\{n \ge 0 : M_n < e^{-b}\}$$

and its overshoot $R_b \stackrel{\text{def}}{=} S_{\tau(b)} - b$ for $b > 0$. Let $\mathcal{T}_b$ denote the associated HSL and $S(v) \stackrel{\text{def}}{=} -\log L(v)$ for $v \in \mathbb{V}$. By (4.7) of Lemma 4.2,

$$\mathbb{P}(R_b \le t) \;=\; \mathbb{E}\left(\sum_{v\in\mathcal{T}_b} L(v)\delta_{S(v)-b}([0,t])\right), \quad t > 0, \tag{6.1}$$

and renewal theory asserts that, if S_1 is nonarithmetic, then (recall $\mathbb{E}S_1 = -\mathbb{E}\log M_1 = |\gamma|$)

$$\lim_{b\to\infty}\mathbb{P}(R_b \le t) \;=\; \zeta(t) \;\stackrel{\text{def}}{=}\; \frac{1}{|\gamma|}\int_{[0,t]}\mathbb{P}(S_1 > x)\;\lambda\!\!\!\lambda(dx) \tag{6.2}$$

for all $t > 0$. A corresponding result holds true in the d-arithmetic case ($d > 0$) if $b \to \infty$ only through the minimal lattice $d\mathbb{Z}$ on which S_1 is concentrated and $\lambda\!\!\!\lambda$ is replaced with d

times counting measure on that lattice in the definition of ζ. Our purpose is to derive a similar pathwise result for the empirical measure

$$\mathcal{R}_b \stackrel{\text{def}}{=} \sum_{v\in\mathcal{T}_b} L(v)\delta_{S(v)-b}$$

showing up under the expectation sign in (6.1).

PROPOSITION 6.1. *Suppose (C.1-4) and* $\sup_{i\ge 1} T_i < 1$ *a.s. Then*

$$\mathcal{R}_b([0,t]) - W\zeta(t) \ \stackrel{\mathbb{P}}{\to}\ 0 \tag{6.3}$$

for all $t>0$, where $b\to\infty$ only through $d\mathbb{Z}$ if S_1 is d-arithmetic.

PROOF. We confine ourselves to the case that S_1 has a nonarithmetic distribution and start by introducing some necessary notation. If $\boldsymbol{L} \stackrel{\text{def}}{=} (L(w))_{w\in\mathbb{V}}$ denotes the random vector of branch weights of our given weighted branching model, let $[\boldsymbol{L}]_v \stackrel{\text{def}}{=} (L_v(w))_{w\in\mathbb{V}}$ denote the corresponding vector for the subtree emanating from v for each $v\in\mathbb{V}$. So the bracket operator $[\cdot]_v$ acts as a shift and will be used for functionals $U=\Phi(\boldsymbol{L})$ as well by setting $[U]_v \stackrel{\text{def}}{=} \Phi([\boldsymbol{L}]_v)$. We write $\mathcal{R}_b(t)$ as shorthand for $\mathcal{R}_b([0,t])$ and put $\zeta_b(t) \stackrel{\text{def}}{=} \mathbb{E}\mathcal{R}_b(t)$ which, by (6.1) and (6.2), equals $\mathbb{P}(R_b\le t)$ and converges to $\zeta(t)$, as $b\to\infty$. Notice that all $[\mathcal{R}_b]_v$, $v\in\mathbb{V}$, are identically distributed with $\mathbb{E}[\mathcal{R}_b]_v(t)=\zeta_b(t)$ for all $t>0$.

Using $W=\sum_{v\in\mathcal{T}_b} L(v)W^{(v)}$ from Lemma 4.2(c), it is now readily seen that, for all $b,t>0$,

$$\begin{aligned}
\mathcal{R}_{2b}(t) - W\zeta(t) \ =\ & \sum_{v\in\mathcal{T}_b} L(v)\mathbf{1}_{\{2b<S(v)\le 2b+t\}} \ -\ \zeta(t)\sum_{v\in\mathcal{T}_b} L(v)W^{(v)}\mathbf{1}_{\{S(v)>2b\}}\\
&+\ \sum_{v\in\mathcal{T}_b} L(v)W^{(v)}(\zeta_{2b-S(v)}(t)-\zeta(t))\mathbf{1}_{\{S(v)\le 2b\}}\\
&+\ \sum_{v\in\mathcal{T}_b} L(v)\mathbf{1}_{\{S(v)\le 2b\}}\Big([\mathcal{R}_{2b-S(v)}]_v(t) - W^{(v)}\zeta_{2b-S(v)}(t)\Big).
\end{aligned}$$

By separately estimating the four terms $I_1(b,t),...,I_4(b,t)$, say, on the right-hand side, we will now verify that

$$\lim_{b\to\infty} \mathbb{E}|\mathcal{R}_{2b}(t)-W\zeta(t)| \ =\ 0 \tag{6.4}$$

which particularly implies (6.3). By utilizing (4.8) of Lemma 4.2 in all three assertions below, we easily see that

$$\begin{aligned}
\mathbb{E}I_1(b,t) \ &=\ \mathbb{P}(b<R_b\le b+t)) \ \to\ 0,\\
\mathbb{E}I_2(b,t) \ &=\ \zeta(t)\,\mathbb{P}(R_b>b)\,\mathbb{E}W \ \to\ 0,
\end{aligned}$$

and

$$\mathbb{E}|I_3(b,t)| \ \le\ \mathbb{E}\left(\sum_{v\in\mathcal{T}_b} L(v)W^{(v)}|\zeta_{2b-S(v)}(t)-\zeta(t)|\mathbf{1}_{\{S(v)\le 2b\}}\right)$$

$$= \mathbb{E}|\zeta_{b-R_b}(t) - \zeta(t)|\mathbf{1}_{\{R_b \le b\}}\, \mathbb{E}W \ \to\ 0.$$

This leaves us with the estimation of the crucial term $I_4(b,t)$ which requires once again a martingale argument. Observe that, conditioned upon $\mathcal{F}_{\mathcal{T}_b}$, $I_4(b,t) = \sum_{v\in\mathcal{T}_b} L(v)Y(v)$ may be viewed as a martingale limit, because the

$$Y(v) \ \stackrel{\text{def}}{=}\ \mathbf{1}_{\{S(v)\le 2b\}}\Big([\mathcal{R}_{2b-S(v)}]_v(t) - W^{(v)}\zeta_{2b-S(v)}(t)\Big), \quad v \in \mathcal{T}_b, \tag{6.5}$$

are conditionally independent with mean 0. Moreover,

$$Y(v) \ = \ \Psi\Big(S(v) - b, W^{(v)}, ([\mathcal{R}_x]_v(t))_{0\le x\le 2b}\Big)$$

for all $v \in \mathbb{V}$ and a suitable function Ψ. Note that $\Pi(v) \stackrel{\text{def}}{=} (W^{(v)}, ([\mathcal{R}_x]_v(t))_{0\le x\le 2b})$ forms an independent copy of $(W, (\mathcal{R}_x(t))_{0\le x\le 2b})$ which is also independent of $\mathcal{F}_{\mathcal{T}_b}$ and thus $S(v) - b$ for each $v \in \mathcal{T}_b$. Let Π denote a generic copy of $\Pi(v)$ independent of all other occurring random variables. By another appeal to Lemma 4.2, the last observations allow us to infer that

$$\begin{aligned}
\mathbb{E}\left(\sum_{v\in\mathcal{T}_b} \phi(L(v)Y(v))\right) \ &= \ \mathbb{E}\left(\sum_{v\in\mathcal{T}_b} \phi\Big(L(v)\Psi\big(S(v) - b, \Pi(v)\big)\Big)\mathbf{1}_{\{S(v)\le 2b\}}\right) \\
&= \ \mathbb{E}\left(\Psi\big(R_b, \Pi\big)\overline{\phi}\Big(e^{-b-R_b}\Psi\big(R_b, \Pi\big)\Big)\mathbf{1}_{\{R_b\le b\}}\right)
\end{aligned}$$

for any even $\phi : \mathbb{R} \to [0,\infty)$ with $\overline{\phi}(x) \stackrel{\text{def}}{=} \frac{\phi(x)}{|x|}$. Choose $\phi(x) = x^2\mathbf{1}_{[0,1]}(x) + (2x-1)\mathbf{1}_{(1,\infty)}(x)$ for $x \ge 0$ and note that $\phi \in \mathfrak{C}_0^*$ and $\overline{\phi}(x) = x\mathbf{1}_{[0,1]}(x) + (2 - \frac{1}{x})\mathbf{1}_{(1,\infty)}(x) \sim x$, as $x \to 0$. By an appeal to the Topchii-Vatutin inequality [55], [4] (abbreviated as TV-inequality hereafter and a generalization of the von Bahr-Esseen inequality to convex functions with conacve derivatives), we infer

$$\begin{aligned}
\mathbb{E}\phi(I_4(b,t)) \ &\le\ 2\,\mathbb{E}\left(\sum_{v\in\mathcal{T}_b} \phi\Big(L(v)Y(v)\Big)\right) \\
&=\ 2\,\mathbb{E}\left(\big|\Psi\big(R_b, \Pi\big)\big|\overline{\phi}\Big(e^{-b-R_b}\Psi\big(R_b, \Pi\big)\Big)\mathbf{1}_{\{R_b\le b\}}\right) \\
&\le\ 2\Big(\overline{\phi}\big(ce^{-b}\big) + 2\,\mathbb{E}\big|\Psi\big(R_b, \Pi\big)\big|\mathbf{1}_{\{|\Psi(R_b,\Pi)|>c\}}\Big)
\end{aligned}$$

for each $c > 0$. Clearly, $\overline{\phi}(ce^{-b}) \sim ce^{-b} \to 0$, as $b \to \infty$, for any $c > 0$. Finally,

$$\mathbb{E}\Psi\big(R_b, \Pi\big)\mathbf{1}_{\{\Psi(R_b,\Pi)>c\}} \ = \ \int_{(0,b]} \mathbb{E}\big|\mathcal{R}_{b-x}(t) - W\zeta_{b-x}(t)\big|\mathbf{1}_{\{|\mathcal{R}_{b-x}(t)-W\zeta_{b-x}(t)|>c\}}\ \mathbb{P}(R_b \in dx)$$

can be made arbitrarily small for c sufficiently large because of

$$\big|\mathcal{R}_{b-x}(t) - W\zeta_{b-x}(t)\big| \ \le\ \mathcal{R}_{b-x}((0,\infty)) + W \ = \ Z_{\mathcal{T}_{b-x}} + W \ = \ \mathbb{E}(W|\mathcal{F}_{\mathcal{T}_{b-x}}) + W$$

and the uniform integrability of $(\mathbb{E}(W|\mathcal{F}_{\mathcal{T}_a}))_{a\ge 0}$. ◇

As to our purposes, the important consequence of the previous proposition is stated in the next corollary and should be viewed as a tightness result for the number of weights $L(v), v \in \mathcal{T}_b$, that are 'close' to the stopping boundary e^{-b}.

6.2. Corollary. *Given the situation of Proposition 6.1, fix any $\varepsilon \in (0,1)$ and $c > 0$ with $\zeta(c) > 1-\varepsilon$. Define*

$$N_b(c) \stackrel{\text{def}}{=} \sum_{v\in\mathcal{T}_b} \mathbf{1}_{[e^{-b-c},e^{-b}]}(L(v)) \tag{6.6}$$

for $b \geq 0$. Then

$$\lim_{b\to\infty} \mathbb{P}(e^{-b}N_b(c) - (1-\varepsilon)W \geq -\varepsilon) \;=\; 1. \tag{6.7}$$

Proof. We infer for each $b > 0$

$$\mathcal{R}_b(c) \;=\; \sum_{v\in\mathcal{T}_b} L(v)\mathbf{1}_{[e^{-b-c},e^{-b}]}(L(v)) \;\leq\; e^{-b}N_b(c)$$

and then upon using Proposition 6.1

$$\begin{aligned}\mathbb{P}(e^{-b}N_b(c) - (1-\varepsilon)W \geq -\varepsilon) \;&\geq\; \mathbb{P}(\mathcal{R}_b(c) - (1-\varepsilon)W \geq -\varepsilon)\\ &\geq\; \mathbb{P}(\mathcal{R}_b(c) - W\zeta(c) \geq -\varepsilon) \;\to\; 1,\end{aligned}$$

as $b \to \infty$, and this is easily seen to imply (6.7). ♢

7. Proofs of the main results

Besides our standing assumptions (C1-4), the notation introduced in Sections 4 and 5 will be in force throughout. In particular, $(M_n)_{n\geq 0}$ denotes the multiplicative random walk introduced at the beginning of Section 4. We point out that, by Lemma 4.2(c),

$$\mathcal{W}_n \stackrel{\text{def}}{=} Z_{\mathcal{S}_1^<\wedge n} \;=\; \mathbb{E}(W|\mathcal{F}_{\mathcal{S}_1^<\wedge n}), \quad n \geq 0 \tag{7.1}$$

forms a uniformly integrable martingale with limit $Z_1^< \stackrel{\text{def}}{=} Z_{\mathcal{S}_1^<}$ and increments

$$\mathcal{D}_n \stackrel{\text{def}}{=} Z_{\mathcal{S}_1^<\wedge n} - Z_{\mathcal{S}_1^<\wedge (n-1)} \;=\; \sum_{v\prec\mathcal{S}_1^<, |v|=n-1} L(v)(Z_1(v)-1), \quad n \geq 1. \tag{7.2}$$

Before proceeding with the proof of Theorem 1.2, some further notation and an extension of Lemma 3.4 must be given. Recalling the definitions of $Z_n^{(\alpha)}, W_n^{(\alpha)}, D_n^{(\alpha)}, \overline{D}_n^{(\alpha)}$ from Section 3, we further put

$$\mathcal{W}_n^{(\alpha)} \stackrel{\text{def}}{=} \sum_{v\in\mathcal{S}_1^<\wedge n} g(\alpha)^{-|v|}L(v)^\alpha,$$

$$\mathcal{D}_n^{(\alpha)} \stackrel{\text{def}}{=} \mathcal{W}_n^{(\alpha)} - \mathcal{W}_{n-1}^{(\alpha)} \;=\; g(\alpha)^{-n} \sum_{v\prec\mathcal{S}_1^<, |v|=n-1} L(v)^\alpha\big(Z_1^{(\alpha)}(v) - g(\alpha)\big),$$

and

$$\overline{\mathcal{D}}_n^{(\alpha)} \overset{\text{def}}{=} g(\alpha)^n \mathcal{D}_n^{(\alpha)}$$

for $n \ge 1$ and $\alpha > 0$ with $g(\alpha) < \infty$. For $n = 0$ these variables are defined as 1. One can then readily check that $(\mathcal{W}_n^{(\alpha)})_{n\ge 0}$, like $(W_n^{(\alpha)})_{n\ge 0}$, constitutes a mean one martingale and that the following counterpart of Lemma 3.4 holds true without further ado and is therefore stated without proof. Recall that $\mathfrak{Z}$ denotes the class of all even nonnegative functions ψ which are continuous, nondecreasing on $[0,\infty)$ and satisfy the growth condition (2.3). Recall further that we write $A \ll B$ if $B < \infty$ implies $A < \infty$.

LEMMA 7.1. *Let $m \in \mathbb{N}$ and $\psi \in \mathfrak{Z}$. Suppose that $\mathbb{E}\psi(Z_1) < \infty$, $\mu(2^m) < \infty$ and $g(2^m) < 1$. Then*

$$\sup_{n\ge 0} \mathbb{E}\psi(\mathcal{W}_n) \ \ll\ \mathcal{Q}(m,\psi) \overset{\text{def}}{=} \mathcal{Q}_1(m,\psi) + \mathcal{Q}_2(m,\psi),$$

where

$$\mathcal{Q}_1(m,\psi) \overset{\text{def}}{=} \mathbb{E}\mathbb{S}^{-m}\psi\left(\sum_{n\ge 0} \overline{\mathcal{D}}_n^{(2^m)}\right) \quad \text{and} \quad \mathcal{Q}_2(m,\psi) \overset{\text{def}}{=} \sum_{l=0}^{m-1}\sum_{n\ge 0} \mathbb{E}\mathbb{S}^{-l}\psi\left(\overline{\mathcal{D}}_n^{(2^l)}\right).$$

Furthermore,

$$0 \le \sum_{k\ge 0} \overline{\mathcal{D}}_k^{(2^m)} < \infty \quad \mathbb{P}\text{-a.s.}$$

PROOF OF THEOREM 1.2. "(a)⇒(b)" By Lemma 2.4(d), the assumption $\varepsilon \in \mathfrak{R}_0$ in case $\alpha \in \{2^m : m \in \mathbb{N}_0\}$ implies $\ell = \mathbb{U}\ell_0$ for some $\ell_0 \in \mathfrak{R}_0$. Hence, for any $\alpha \ge 1$ and $m \in \mathbb{N}_0$ determined by $2^m \le \alpha < 2^{m+1}$, Lemma 2.1 ensures the existence of a function $\hat{\phi}(x) = x^\alpha \hat{\ell}(x) \in \mathfrak{C}_m^* \cap \mathfrak{R}_\alpha$ such that $\phi \asymp \hat{\phi}$ or, equivalently, $\ell \asymp \hat{\ell}$. The latter implies $\sup_{x\ge x_0} \frac{\ell(x)}{\hat{\ell}(x)} < \infty$ for some $x_0 > 0$. Since $\ell \ge 1$ by (1.12), we also have $\sup_{x\ge 0} \frac{\hat{\ell}(x)}{\ell(x)} < \infty$. Consequently,

$$C_1\hat{\phi}(x) \le \phi(x) \le C_2\Big(x^\alpha \vee \hat{\phi}(x)\Big) \asymp \hat{\phi}(x) \tag{7.3}$$

for all $x \ge 0$ and suitable $C_1, C_2 > 0$. As before, $C \in (0,\infty)$ will denote a generic constant which may differ from line to line.

Instead of (b) we will in fact prove (and need) the extended assertion

$$\mathbb{E}\phi(Z_1^<) \le \sup_{n\ge 0} \mathbb{E}\phi(\mathcal{W}_n) < \infty \quad \text{and} \quad \mathbb{E}\phi(W) \le \sup_{n\ge 0} \mathbb{E}\phi(W_n) < \infty, \tag{7.4}$$

provided that in case $\alpha > 1$ the parameter $a \in (0,1]$ in the definition of $\mathcal{S}_1^<$ and thus of $Z_1^<$ is chosen such that $\mathbb{E}M_1^{\alpha-1} = g(\alpha) < a^{\alpha-1}$. Note that $\mathbb{E}\phi(W) > 0$ is guaranteed by Lemma 3.3. We will distinguish the cases $\alpha \in [2^m, 2^{m+1})$ and use an induction over m.

STEP 1. $\alpha \in [1,2)$. Once again the double martingale structure of $(Z_n)_{n\ge 0}$ will be utilized, more precisely that of $(\mathcal{W}_n)_{n\ge 0}$ as exhibited by (7.1) and (7.2). Since $\hat{\phi} \in \mathfrak{C}_0^*$, a double use of the TV-inequality in combination with (7.3) and the submultiplicativity of ϕ leads to

$$\begin{aligned}
\mathbb{E}\hat{\phi}(\mathcal{W}_n) \ &\le\ \sum_{k=0}^{n}\mathbb{E}\hat{\phi}(\mathcal{D}_k)\ \le\ \hat{\phi}(1)\ +\ 2\sum_{k=1}^{n}\mathbb{E}\left(\sum_{v\prec\mathcal{S}_1^<,|v|=k-1}\hat{\phi}(L(v)(Z_1(v)-1))\right)\\
&\le\ \hat{\phi}(1)\ +\ 2\,\mathbb{E}\left(\sum_{v\prec\mathcal{S}_1^<,|v|<n}\hat{\phi}(L(v)Z_1(v))\right)\\
&\le\ \hat{\phi}(1)\ +\ C\,\mathbb{E}\left(\sum_{v\prec\mathcal{S}_1^<,|v|<n}\phi(L(v)Z_1(v))\right)\\
&\le\ \hat{\phi}(1)\ +\ C\,\mathbb{E}\phi(Z_1)\,\mathbb{E}\left(\sum_{v\prec\mathcal{S}_1^<,|v|<n}\phi(L(v))\right)
\end{aligned} \tag{7.5}$$

for all $n \ge 0$ (and a constant C not depending on n) and then, upon taking the supremum over n and using (5.7),

$$\mathbb{E}\hat{\phi}(Z_1^<)\ \le\ \sup_{n\ge 0}\mathbb{E}\hat{\phi}(\mathcal{W}_n)\ \ll\ \mathbb{E}\phi(Z_1)\,\mathbb{E}\left(\sum_{v\prec\mathcal{S}_1^<}\phi(L(v))\right)\ =\ \mathbb{E}\sigma_1^<\,\mathbb{E}\phi(Z_1)\,\mathbb{E}\overline{\phi}(M^*),$$

where $\overline{\phi}(x) \stackrel{\text{def}}{=} x^{-1}\phi(x) = x^{\alpha-1}\ell(x)$. Since ℓ is supposed to be unbounded if $\alpha = 1$, we have $\mathbb{E}\overline{\phi}(M^*) = \mathbb{E}(M^*)^{\alpha-1}\ell(M^*) < \infty$ by Lemma 5.5. Moreover, $\mathbb{E}\phi(Z_1^<) \le C\,\mathbb{E}[(Z_1^<)^\alpha \vee \hat{\phi}(Z_1^<)] \ll \mathbb{E}\hat{\phi}(Z_1^<)$ by (7.3) and $\mathbb{E}\phi(Z_1) < \infty$, whence we arrive at the conclusion

$$\mathbb{E}\phi(Z_1^<)\ \ll\ \mathbb{E}\phi(Z_1)\,\mathbb{E}\overline{\phi}(M^*)\ <\ \infty,$$

that is the first half of (7.4). But a similar estimation as in (7.5) shows

$$\mathbb{E}\hat{\phi}(W_n)\ \ll\ \sum_{|v|=n-1}\mathbb{E}\phi(L(v)Z_1(v))\ \le\ \mathbb{E}\phi(Z_1)\,\mathbb{E}\overline{\phi}(M_{n-1})\ \le\ \mathbb{E}\phi(Z_1)\,\mathbb{E}\overline{\phi}(M^*)\ <\ \infty$$

for all $n \ge 1$ and therefore the second half of (7.4).

Step 2. Now assume claim (b) be true whenever $\alpha < 2^{m+1}$ for some $m \ge 0$, $\ell \in \mathfrak{R}_0^*$ and $(Z_n)_{n\ge 0}$ is any WBP satisfying the conditions of (a) for such α, ℓ (inductive hypothesis). Pick $\alpha \in [2^{m+1}, 2^{m+2})$.

Step 2a. Proof of first half of (7.4). Again, we begin with the proof of the first half of (7.4), that is of $\sup_{n\ge 0}\mathbb{E}\hat{\phi}(\mathcal{W}_n) < \infty$ which, by Lemma 7.1, reduces to the proof of $\mathcal{Q}_1(m+1,\hat{\phi}) < \infty$ and $\mathcal{Q}_2(m+1,\hat{\phi}) < \infty$. Put $s \stackrel{\text{def}}{=} 2^{m+1}$. Via a similar estimation as in the proof of Theorem 3.1 (there for $Q_1(m+1,\phi_\alpha)$) in combination with (7.3), $g(\alpha) < 1$ and

$\mathbb{S}^{-m-1}\hat\phi\in\mathfrak{C}_0^*$, $\mathbb{S}^{-m-1}\hat\phi(x^s)=\hat\phi(x)$

$$\begin{aligned}
\mathcal{Q}_1(m+1,\hat\phi) &= \mathbb{E}\mathbb{S}^{-m-1}\hat\phi\left(\sum_{k\ge 0}\overline{\mathcal{D}}_k^{(s)}\right)\\
&\le \hat\phi(1) + 4\sum_{k\ge 1}\mathbb{E}\left(\sum_{v\prec\mathcal{S}_1^<,|v|=k-1}\mathbb{S}^{-m-1}\hat\phi\Big(L(v)^s\,(1\vee Z_1(v))^s\Big)\right)\\
&= \hat\phi(1) + 4\,\mathbb{E}\left(\sum_{v\prec\mathcal{S}_1^<}\hat\phi\Big(L(v)\,(1\vee Z_1(v))\Big)\right)\\
&\le \hat\phi(1) + C\,\mathbb{E}\left(\sum_{v\prec\mathcal{S}_1^<}\phi\Big(L(v)\,(1\vee Z_1(v))\Big)\right)\\
&\le \hat\phi(1) + C\,\mathbb{E}\phi(1\vee Z_1)\,\mathbb{E}\left(\sum_{v\prec\mathcal{S}_1^<}\phi(L(v))\right)\\
&= \hat\phi(1) + C\,\mathbb{E}\phi(1\vee Z_1)\,\mathbb{E}\overline{\phi}(M^*) \;<\; \infty,
\end{aligned}\tag{7.6}$$

the finiteness being true by the same argument as at the end of Step 1 (invoking again Lemma 5.5(a)).

To show $\mathcal{Q}_2(m+1,\hat\phi)<\infty$ or, equivalently,

$$\mathcal{U}(l,\hat\phi)\;\stackrel{\text{def}}{=}\;\sum_{n\ge 1}\mathbb{E}\mathbb{S}^{-l}\hat\phi\Big(\overline{\mathcal{D}}_n^{(2^l)}\Big)\;<\;\infty\quad\text{for } l\in\{0,...,m\},$$

we start by pointing out that $\mathbb{E}\hat\phi(Z_1)<\infty$ gives

$$\mathbb{E}\mathbb{S}^{-l}\hat\phi\Big(\overline{\mathcal{D}}_1^{(2^l)}\Big)\;\le\;\mathbb{E}\mathbb{S}^{-l}\hat\phi\left(1\vee\sum_{i\ge 1}T_i^{2^l}\right)\;\le\;\mathbb{E}\mathbb{S}^{-l}\hat\phi(1\vee Z_1)\;=\;\mathbb{E}\hat\phi(1\vee Z_1)\;<\;\infty.$$

As for $\sum_{n\ge 2}\mathbb{E}\mathbb{S}^{-l}\hat\phi\Big(\overline{\mathcal{D}}_n^{(2^l)}\Big)$, an appeal to the BDG-inequality shows that it suffices to verify

$$\sum_{n\ge 2}\mathcal{J}_1(n,l,\hat\phi)\;+\;\sum_{n\ge 2}\mathcal{J}_2(n,l,\hat\phi)\;<\;\infty,\tag{7.7}$$

where

$$\mathcal{J}_1(n,l,\hat\phi)\;\stackrel{\text{def}}{=}\;\mathbb{E}\mathbb{S}^{-l-1}\hat\phi\left(\mu(2^{l+1})\sum_{v\prec\mathcal{S}_1^<,|v|=n-1}L(v)^{2^{l+1}}\right)$$

and

$$\mathcal{J}_2(n,l,\hat\phi)\;\stackrel{\text{def}}{=}\;\mathbb{E}\left(\sum_{v\prec\mathcal{S}_1^<,|v|=n-1}\mathbb{S}^{-l}\hat\phi\left(L(v)^{2^l}\left|\sum_{i\ge 1}T_i(v)^{2^l}-g(2^l)\right|\right)\right)$$

(cf. (3.3-5) in the proof of Theorem 3.1 for a similar estimation).

As for the simpler assertion $\sum_{n\geq 2}\mathcal{J}_2(n,l,\hat\phi)<\infty$, we obtain

$$\begin{aligned}
\sum_{n\geq 2}\mathcal{J}_2(n,l,\hat\phi) &\leq \sum_{n\geq 2}\mathbb{E}\left(\sum_{v\prec\mathcal{S}_1^<,|v|=n-1}\mathbb{S}^{-l}\hat\phi\left(L(v)^{2^l}\left(1\vee\sum_{i\geq 1}T_i(v)^{2^l}\right)\right)\right)\\
&\leq \sum_{n\geq 2}\mathbb{E}\left(\sum_{|v|=n-1}\mathbb{S}^{-l}\hat\phi\left(L(v)^{2^l}\left(1\vee\sum_{i\geq 1}T_i(v)\right)^{2^l}\right)\right)\\
&= \mathbb{E}\left(\sum_{n\geq 2}\sum_{v\prec\mathcal{S}_1^<,|v|=n-1}\hat\phi(L(v)(1\vee Z_1(v))\right)\\
&\leq C\,\mathbb{E}\phi(1\vee Z_1)\,\mathbb{E}\overline{\phi}(M^*)\ <\ \infty,
\end{aligned}\tag{7.8}$$

where finiteness of the last line has already been found in (7.6).

Turning to $\sum_{n\geq 2}\mathcal{J}_1(n,l,\hat\phi)<\infty$, use (2.3), (7.3) and the submultiplicativity of $\mathbb{S}^{-l-1}\phi\in$ to infer

$$\begin{aligned}
\sum_{n\geq 2}\mathcal{J}_1(n,l,\hat\phi) &\leq C\sum_{n\geq 2}\mathbb{E}\mathbb{S}^{-l-1}\hat\phi\left(\sum_{v\prec\mathcal{S}_1^<,|v|=n-1}L(v)^{2^{l+1}}\right)\\
&\leq C\sum_{n\geq 2}\mathbb{E}\mathbb{S}^{-l-1}\phi\left(Z_{n-1}^{(2^{l+1})}\right)\\
&= C\sum_{n\geq 2}\mathbb{E}\mathbb{S}^{-l-1}\phi\left(g(2^{l+1})^{n-1}W_{n-1}^{(2^{l+1})}\right)\\
&\leq C\sum_{n\geq 2}\phi(g(2^{l+1})^{n/2^{l+1}})\,\mathbb{E}\mathbb{S}^{-l-1}\phi(W_n^{(2^{l+1})})\\
&\leq C\sup_{n\geq 0}\mathbb{E}\mathbb{S}^{-l-1}\phi(W_n^{(2^{l+1})})\sum_{k\geq 1}\phi(g(2^{l+1})^{k/2^{l+1}}).
\end{aligned}\tag{7.9}$$

If $\alpha>2^{l+1}$, we will show the last line be finite, whereas in the case $\alpha=2^{l+1}$ we will do so for the penultimate line.

Suppose first $\alpha>2^{l+1}$, so that we must verify

(i) $\sup_{n\geq 0}\mathbb{E}\mathbb{S}^{-l-1}\phi(W_n^{(2^{l+1})})<\infty$,

(ii) $\sum_{k\geq 1}\phi(g(2^{l+1})^{k/2^{l+1}})<\infty$.

As for (ii), it is enough to notice that $g(2^{l+1})<1$ by Lemma 3.2(b) from which it is not difficult to infer

$$\phi(g(2^{l+1})^{k/2^{l+1}})\ =\ o(c^n),\quad n\to\infty,\tag{7.10}$$

for any $c>g(2^{l+1})^{k/2^{l+1}}$.

Turning to (i), we want to apply the inductive hypothesis to the normalized WBP $(W_n^{(2^{l+1})})_{n\geq 0}$ with generic weight vector $(T_i^{2^{l+1}}/g(2^{l+1}))_{i\geq 1}$ and the pair $(\alpha/2^{l+1},\mathbb{S}^{-l-1}\ell)$ in place of (α,ℓ), as $\psi_{l+1}(x)\overset{\text{def}}{=}\mathbb{S}^{-l-1}\phi(x)=x^{\alpha/2^{l+1}}\ell(x^{1/2^{l+1}})$. So we must verify the pertinent hypothesis (a). Observe that $\mathbb{S}^{-l-1}\ell \in \mathfrak{R}_0^*$ and

$$\mathbb{S}^{-l-1}\ell(x^{2^{l+1}}/g(2^{l+1})) \;\le\; C\ell(x)$$

for all $x \ge 0$ and some $C > 0$ (in fact $C = \ell(1/g(2^{l+1}))$ will do). Furthermore, by Lemma 3.5 applied to the WBP $(Z_n^{(2^{l+1})})_{n\ge 0}$ with generic weight sequence $(T_i^{2^{l+1}})_{i\ge 1}$ we may assume w.l.o.g. that

$$g(\alpha) \;=\; \sum_{i\ge 1}\mathbb{E}T_i^{\alpha} \;<\; \left(\sum_{i\ge 1}\mathbb{E}T_i^{2^{l+1}}\right)^{\alpha/2^{l+1}} \;=\; g(2^{l+1})^{\alpha/2^{l+1}}. \tag{7.11}$$

With this we get

$$\mathbb{E}\psi_{l+1}\left(W_1^{(2^{l+1})}\right) \;=\; \mathbb{E}\phi\left(\left(\sum_{i\ge 1}\frac{T_i^{2^{l+1}}}{g(2^{l+1})}\right)^{1/2^{l+1}}\right) \;\le\; C\,\mathbb{E}\phi(Z_1) \;<\; \infty,$$

$$\sum_{i\ge 1}\mathbb{E}\left(\frac{T_i^{2^{l+1}}}{g(2^{l+1})}\right)^{\alpha/2^{l+1}} \;=\; \frac{g(\alpha)}{g(2^{l+1})^{\alpha/2^{l+1}}} \;<\; 1,$$

thus confirming validity of (a) for the WBP $(W_n^{(2^{l+1})})_{n\ge 0}$ and $(\alpha/2^{l+1}, \mathbb{S}^{-l-1}\ell)$ in place of (α,ℓ).

Now suppose $\alpha = 2^{l+1}$ and note that (7.10) remains valid. In order to verify that the penultimate line of (7.9) is finite it therefore suffices to prove that

$$\mathbb{E}\psi_{l+1}\left(W_n^{(2^{l+1})}\right) \;=\; \mathbb{E}W_n^{(2^{l+1})}\overline{\psi}(W_n^{(2^{l+1})}) \;=\; o(c^{-n}), \quad n\to\infty,$$

for any $c \in (0,1)$, where $\overline{\psi}_{l+1}(x) = \ell(x^{1/2^{l+1}}) \in \mathfrak{R}_0^*$. But $\psi_{l+1} \asymp \hat{\psi}$ for some $\hat{\psi} \in \mathfrak{C}_0^*$ and the martingale property of $(W_n^{(2^{l+1})})_{n\ge 0}$ yields as in Step 1

$$\mathbb{E}\psi_{l+1}\left(W_n^{(2^{l+1})}\right) \;\le\; C\sum_{|v|=n-1}\mathbb{E}\psi_{l+1}(L(v)Z_1(v)) \;\le\; C\,\mathbb{E}\psi_{l+1}(Z_1)\,\mathbb{E}\overline{\psi}_{l+1}(M_{n-1}),$$

and since $\mathbb{E}\overline{\psi}_{l+1}(M_1) \ll \mathbb{E}\psi_{l+1}(Z_1) < \infty$, Lemma 5.6(b) provides us with the desired conclusion $\mathbb{E}\overline{\psi}_{l+1}(M_n) = o(c^{-n})$ for any $c \in (0,1)$. We have thus completed the proof of the first half of (7.4).

Step 2b. Proof of second half of (7.4). Suppose a appearing in the definition of $Z_1^<$ be fixed strictly less than 1 and satisfying $\mathbb{E}M_1^{\alpha-1} < a^{\alpha-1}$. We will take advantage of what has been proved so far, namely that $\mathbb{E}\phi(Z_1) < \infty$ and $g(\alpha) < 1$ implies $\mathbb{E}\phi(Z_1^<) < \infty$. Moreover, by Lemma 4.2,

$$g^<(\alpha) \;\stackrel{\text{def}}{=}\; \sum_{i\ge 1}\mathbb{E}(T_i^<)^{\alpha} \;=\; \sum_{v\in\mathcal{S}_1^<}\mathbb{E}L(v)^{\alpha} \;=\; \mathbb{E}M_{\sigma_1^<}^{\alpha-1} \;<\; a \;\le\; 1,$$

so that, if $(Z_n)_{n\ge 0}$ satisfies hypothesis (a), then so does the WBP $(Z_n^<)_{n\ge 0}$. By Proposition 5.1, the latter also satisfies the standing assumptions (C1-4), and W is also its a.s. limit. However, $(Z_n^<)_{n\ge 0}$ possesses the additonal property that its generic weights $T_i^<$, $i\ge 1$, are all bounded by a. Based on these remarks, it suffices to prove the second half of (7.4) when substituting W_n with $W_n^< \stackrel{\text{def}}{=} Z_n^<$. Namely, having done so, we infer $0<\mathbb{E}\phi(W)<\infty$ and from this $\sup_{n\ge 0}\mathbb{E}\phi(W_n)<\infty$ by an appeal to the tail inequality (1.18) which in fact even yields $\mathbb{E}\phi(W^*)<\infty$, where $W^*=\sup_{n\ge 0}W_n$ should be recalled. In order to not overburden the necessary notation, we assume w.l.o.g. that $(Z_n)_{n\ge 0}$ itself already has generic weights strictly bounded by some $a<1$, thus giving $Z_n=Z_n^<$ and $M_n\le a^n$ for all $n\ge 0$.

We will again distinguish the cases $\alpha\in[2^m,2^{m+1})$, $m\ge 0$, and use an induction over m. The case $\alpha\in[1,2)$ has already been proved in Step 1. So let us make the inductive hypothesis that the assertion holds true whenever $\alpha\le 2^{m+1}$ for some $m\ge 0$, $\ell\in\mathfrak{R}_0^*$ and $(Z_n)_{n\ge 0}$ is any WBP satisfying assertion (a) for such α,ℓ and having generic weights bounded by some constant strictly less than 1. By Lemma 3.4, we must show $Q_1(m+1,\hat\phi)<\infty$ and $Q_2(m+1,\hat\phi)<\infty$. By a similar estimation as in (7.6), we obtain (with $s=2^{m+1}$)

$$\begin{aligned} Q_1(m+1,\hat\phi) &= \mathbb{E}\mathbb{S}^{-m-1}\hat\phi\left(\sum_{k\ge 0}\overline{D}_k^{(s)}\right)\\ &\le \hat\phi(1) + C\,\mathbb{E}\phi(1\vee Z_1)\sum_{k\ge 0}\sum_{|v|=k}\mathbb{E}\phi(L(v))\\ &= \hat\phi(1) + C\,\mathbb{E}\phi(1\vee Z_1)\sum_{k\ge 0}\mathbb{E}\overline{\phi}(M_k)\\ &\le \hat\phi(1) + C\,\mathbb{E}\phi(1\vee Z_1)\sum_{k\ge 0}\overline{\phi}(a^k) < \infty. \end{aligned} \tag{7.12}$$

As for $Q_2(m+1,\hat\phi)<\infty$ or, equivalently,

$$U(l,\hat\phi) \stackrel{\text{def}}{=} \sum_{n\ge 1}\mathbb{E}\mathbb{S}^{-l}\hat\phi\left(\overline{D}_n^{(2^l)}\right) < \infty \quad \text{for } l\in\{0,...,m\},$$

the procedure is similar to that in Step 2a for $\mathcal{Q}_2(m+1,\hat\phi)$. We have

$$\mathbb{E}\mathbb{S}^{-l}\hat\phi(\overline{D}_1^{(2^l)}) \le \mathbb{E}\hat\phi(1\vee Z_1) < \infty$$

and, by an appeal to the BDG-inequality,

$$\sum_{n\ge 2}\mathbb{E}\mathbb{S}^{-l}\hat\phi\left(\overline{D}_n^{(2^l)}\right) \ll \sum_{n\ge 2}J_1(n,l,\hat\phi) + \sum_{n\ge 2}J_2(n,l,\hat\phi),$$

where

$$J_1(n,l,\hat\phi) \stackrel{\text{def}}{=} \mathbb{E}\mathbb{S}^{-l-1}\hat\phi\left(\mu(2^{l+1})\sum_{|v|=n-1}L(v)^{2^{l+1}}\right)$$

and

$$J_2(n,l,\hat{\phi}) \stackrel{\text{def}}{=} \sum_{|v|=n-1} \mathbb{E}\mathbb{S}^{-l}\hat{\phi}\left(L(v)^{2^l}\left|\sum_{i\ge 1} T_i(v)^{2^l} - g(2^l)\right|\right).$$

But

$$\sum_{n\ge 2} J_1(n,l,\hat{\phi}) \le C\sum_{n\ge 2} \mathbb{E}\mathbb{S}^{-l-1}\hat{\phi}\left(Z_{n-1}^{(2^{l+1})}\right)$$

and the latter sum has already been shown to be finite in the estimation of $\sum_{n\ge 2}\mathcal{J}_2(n,l,\hat{\phi})$ in Step 2A (see (7.9)). Finally,

$$\begin{aligned}
\sum_{n\ge 2} J_2(n,l,\hat{\phi}) &\le C\sum_{n\ge 2}\sum_{|v|=n-1} \mathbb{E}\mathbb{S}^{-l}\phi\left(L(v)^{2^l}\left(1\vee\sum_{i\ge 1} T_i(v)\right)^{2^l}\right)\\
&\le C\,\mathbb{E}\phi(1\vee Z_1)\sum_{n\ge 2}\sum_{|v|=n-1}\mathbb{E}\phi(L(v))\\
&= C\,\mathbb{E}\phi(1\vee Z_1)\sum_{n\ge 1}\mathbb{E}\overline{\phi}(M_n)\\
&\le C\,\mathbb{E}\phi(1\vee Z_1)\sum_{n\ge 2}\overline{\phi}(a^n) < \infty,
\end{aligned} \tag{7.13}$$

and this finally completes our proof of "(a)⇒(b)".

"(b)⇒(a)" Let us first consider the case $\alpha > 1$ which is very simple. As before, write $\phi(x) = x^\alpha \ell(x)$. Since $0 < \mathbb{E}\phi(W) < \infty$ implies $\sup_{n\ge 0}\mathbb{E}\phi(W_n) \le \mathbb{E}\phi(W^*) < \infty$ (Lemma A.1), we particularly infer $\mathbb{E}\phi(Z_1) < \infty$. But $\mathbb{E}W^\alpha \ll \mathbb{E}\phi(W)$ further implies $g(\alpha) < 1$ by an appeal to Theorem 3.1.

The case $\alpha = 1$, for which $\mathbb{E}Z_1\mathbb{U}\ell(Z_1) < \infty$ must be proved, is more difficult and requires a combination of Corollary 6.2 with an argument appearing in a similar form in [5] for the Galton-Watson process. By assumption on ℓ, there exists $\phi \in \mathfrak{C}_0^*$ with $x\ell(x) \asymp \phi(x)$. Following Section 6, let $\mathcal{T}_b$ denote the HSL associated with $\tau(b) = \inf\{n \ge 0 : M_n < e^{-b}\}$ for $b > 0$. Fix any $b > 0$ and let $\mathcal{S}_n$ be the HSL associated with $\tau(bn)+1$. Define $\mathcal{W}_n \stackrel{\text{def}}{=} Z_{\mathcal{S}_n}$, $\mathcal{W}_n^* \stackrel{\text{def}}{=} \max_{0\le k\le n}\mathcal{W}_n$ for $n \ge 0$ and $\mathcal{W}^* \stackrel{\text{def}}{=} \sup_{n\ge 0}\mathcal{W}_n$. Note that

$$\mathcal{W}_n = \sum_{v\in\mathcal{T}_{bn}} L(v)Z_1(v).$$

Clearly, $\mathcal{S}_n \uparrow \infty$ as $n\to\infty$, whence (1.18) ensures $\mathbb{P}(\mathcal{W}^* > t) \le C\,\mathbb{P}(W > at)$ for all $t > 1$, $a \in (0,1)$ and a suitable constant $C = C_a > 0$. Since $\mathbb{E}W = 1$ and $\mathcal{W}_n = \mathbb{E}(W|\mathcal{F}_{\mathcal{S}_n}) \to W$ a.s., there exists $0 < \rho < 1$ such that $\inf_{n\ge 0}\mathbb{P}(\rho \le \mathcal{W}_n^* \le \rho^{-1}, W \ge \rho) > 0$. By Corollary 6.2, we can further choose $c > 0$ such that

$$N_{bn}(c) = \sum_{v\in\mathcal{T}_{bn}} \mathbf{1}_{[e^{-bn-c},e^{-bn}]}(L(v))$$

satisfies $\mathbb{P}(e^{-bn}N_{bn}(c) \geq (1-\rho/2)W - \rho/2) \to 1$, as $n \to \infty$. Hence, by considering the event $\{\rho \leq \mathcal{W}_n^* \leq \rho^{-1}, W \geq \rho, e^{-bn}N_{bn}(c) \geq (1-\rho/2)W - \rho/2\}$ and setting $\kappa \stackrel{\text{def}}{=} (1-\rho/2)\rho - \rho/2 > 0$, we see that

$$\eta \ \stackrel{\text{def}}{=} \ \inf_{n \geq m} \mathbb{P}(\rho \leq \mathcal{W}_{n-1}^* \leq \rho^{-1}, e^{-bn}N_{bn}(c) \geq \kappa) \ > \ 0$$

for $m \geq 1$ sufficiently large. With these observations we infer

$$\begin{aligned}
\mathbb{P}(\mathcal{W}^* > t) \ &= \ \mathbb{P}(\mathcal{W}_0 > t) \ + \ \sum_{n \geq 1} \mathbb{P}(\mathcal{W}_{n-1}^* \leq t, \mathcal{W}_n > t) \\
&\geq \ \sum_{n > m} \mathbb{P}\left(\rho \leq \mathcal{W}_{n-1}^* \leq t, N_{bn}(c) \geq \kappa e^{bn}, \sum_{v \in \mathcal{T}_{bn}} \mathbf{1}_{[e^{-bn-c}, e^{-bn}]}(L(v))L(v)Z_1(v) > t\right) \\
&\geq \ \sum_{n > m} \mathbb{P}(\rho \leq \mathcal{W}_{n-1}^* \leq t, N_{bn}(c) \geq \kappa e^{bn}) \, \mathbb{P}(e^{-bn-c}U_{\kappa e^{bn}} > t) \\
&\geq \ \eta \sum_{n > m} \mathbb{P}(\overline{U}_{\kappa e^{bn}} > te^c/\kappa)
\end{aligned}$$

for all $t \geq \rho^{-1}$, where $U_s \stackrel{\text{def}}{=} X_1 + ... + X_{\lceil s \rceil}$ and $\overline{U}_s \stackrel{\text{def}}{=} s^{-1}U_s$ for $s > 0$ with $X_1, X_2, ...$ being i.i.d. copies of Z_1. The remaining argument can be copied from [5, p. 927] and leads to the inequality

$$\mathbb{P}(\mathcal{W}^* > t) \ \geq \ \eta \, \mathbb{P}(\overline{U}^* > at), \quad \overline{U}^* \ \stackrel{\text{def}}{=} \ \sup_{k \geq 1} \overline{U}_k,$$

for all $t \geq \rho^{-1}$ and some $a > 0$. Consequently, $\mathbb{E}\mathcal{W}^*\ell(\mathcal{W}^*) \ll \mathbb{E}\phi(\mathcal{W}^*) < \infty$ implies $\mathbb{E}\phi(\overline{U}^*) < \infty$ which in turn holds iff $\mathbb{E}\mathbb{L}\phi(Z_1) < \infty$ by Lemma 4.4 in [5]. But $\mathbb{L}\phi$, defined as in (2.6), satisfies $\mathbb{L}\phi(x) \sim x\mathbb{U}\ell(x)$, as $x \to \infty$, by (2.7) of Lemma 2.3. Thus we finally conclude $\mathbb{E}Z_1\mathbb{U}\ell(Z_1) < \infty$ which completes the proof of Theorem 1.2. $\diamondsuit$

PROOF OF THEOREM 1.3. The proof of this result, as compared to that of Theorem 1.2, differs only in those places where, given $\phi(x) = x^\alpha\ell(x)$ with $\ell \in \mathfrak{R}_0$, the submultiplicativity of ℓ has been utilized before and must now be replaced with a use of a submultiplicative cap $\ell^* \in \mathfrak{R}_0^*[\ell]$ satisfying $\sum_{i \geq 1} \mathbb{E}T_i^\alpha \ell^*(T_i) < \infty$ $(\alpha > 1)$, resp. $\sum_{i \geq 1} \mathbb{E}T_i^\alpha \mathbb{U}\ell^*(T_i) < \infty$ $(\alpha = 1)$. These places are (7.5), (7.6), (7.8), (7.9), (7.12) and (7.13), and the necessary modification of the argument is always of the same form. We therefore restrict ourselves to a demonstration of this modification in (7.5).

With ℓ^* as stated, put $\phi^*(x) \stackrel{\text{def}}{=} x^\alpha\ell^*(x)$ and notice that, in extension of (7.3) and naturally keeping the notation from there, we have

$$C_1\hat{\phi}(x) \ \leq \ \phi(x) \ \leq \ C_2\Big(x^\alpha \vee \hat{\phi}(x)\Big) \ \asymp \ \hat{\phi}(x) \ \leq \ C_3\phi^*(x) \tag{7.14}$$

for all $x \geq 0$ and suitable $C_1, C_2, C_3 > 0$. W.l.o.g. suppose ℓ be normalized. By Lemma 2.5, we infer $\ell(xy) \leq C\ell(x)\ell^*(y)$ and thus $\phi(xy) \leq C\phi(x)\phi^*(y)$ for all $x, y \geq 0$ and some $C > 0$.

Turning to (7.5), hence assuming $\alpha \in [1,2]$ and $\hat{\phi} \in \mathfrak{C}_0^*$, we then obtain

$$\begin{aligned} \mathbb{E}\hat{\phi}(\mathcal{W}_n) &\le \hat{\phi}(1) + C\,\mathbb{E}\left(\sum_{v \prec \mathcal{S}_1^<, |v|<n} \phi(L(v)Z_1(v))\right) \\ &\le \hat{\phi}(1) + C\,\mathbb{E}\phi(Z_1)\,\mathbb{E}\left(\sum_{v \prec \mathcal{S}_1^<, |v|<n} \phi^*(L(v))\right) \end{aligned}$$

for all $n \ge 0$ (and a constant C not depending on n) and therefrom (as $\mathbb{E}\phi(Z_1) < \infty$)

$$\mathbb{E}\hat{\phi}(Z_1^<) \le \sup_{n\ge 0} \mathbb{E}\hat{\phi}(\mathcal{W}_n) \ll \mathbb{E}\left(\sum_{v \prec \mathcal{S}_1^<} \phi(L(v))\right) = \mathbb{E}\sigma_1^< \, \mathbb{E}\overline{\phi^*}(M^*).$$

But Lemma 5.5 ensures $\mathbb{E}\overline{\phi^*}(M^*) < \infty$ iff $\sum_{i\ge 1} \mathbb{E}T_i^\alpha \ell^*(T_i) < \infty$ in case $\alpha > 1$, respectively $\sum_{i\ge 1} \mathbb{E}T_i \mathbb{U}\ell^*(T_i) < \infty$ in case $\alpha = 1$. ♢

Proof of Theorem 1.4. "(a)⇒(b)" Here it suffices to note that, if $\lim_{x\to\infty} \ell(x) = 0$, then $\ell^* \equiv 1 \in \mathfrak{R}_0^*[\ell]$ satisfies the extra condition of Theorem 1.3 as reducing to $g(\alpha) < \infty$.

"(b)⇒(a)" If $\mathbb{E}W^\alpha \ell(W) < \infty$, then $\mathbb{E}W^\beta < \infty$ for all $\beta < \infty$ whence, by Theorem 3.1, $g(\beta) < 1$ for all such β. But then, by Fatou's lemma,

$$\sum_{i=1}^n \mathbb{E}T_i^\alpha \le \liminf_{\beta\uparrow\alpha} \sum_{i=1}^n \mathbb{E}T_i^\beta \le \liminf_{\beta\uparrow\alpha} g(\beta) \le 1$$

for all $n \ge 1$ and thus

$$g(\alpha) = \lim_{n\to\infty} \sum_{i=1}^n \mathbb{E}T_i^\alpha \le 1.$$

So once again we infer the extra condition of Theorem 1.3 with $\ell^* \equiv 1$ and thus validity of (a) by that theorem. ♢

Appendix. A tail inequality

Given a weighted branching model satisfying (C1-4), let $(\mathcal{S}_n)_{n\ge 0}$ be an increasing sequence of a.s. finite HSL with $\mathcal{S}_n \uparrow \infty$. This means that $\mathcal{S}_0 \preceq \mathcal{S}_1 \preceq ...$ and

$$\{v \in \mathbb{V} : |v| \le k, \mathcal{S}_n \preceq v\} \downarrow \emptyset, \quad n \to \infty$$

for all $k \ge 0$. Note that these conditions include the cases when $\mathcal{S}_n = \{v : |v| = n\}$ or $\mathcal{S}_n = \mathcal{S}_1^< \wedge n$ (see (7.1)) for $n \ge 0$. Put $\mathcal{W}_n \stackrel{\text{def}}{=} Z_{\mathcal{S}_n}$ for $n \ge 0$ and $\mathcal{W}^* \stackrel{\text{def}}{=} \sup_{n\ge 0} \mathcal{W}_n$. By Lemma 4.2(c), $\mathcal{W}_n = \mathbb{E}(W|\mathcal{F}_{\mathcal{S}_n})$ a.s. and thus forms a martingale with a.s. limit W as $\mathcal{S}_n \uparrow \infty$. The proof of the following tail inequality may be found in [38, Theorem 1.1.4] and is based on an adaptation of an argument given by Biggins in [16].

LEMMA A.1. *For any $0 < a < 1$, there exists a constant $C = C(a) \in (0, \infty)$ such that*

$$\mathbb{P}(W > at) \;\geq\; C\,\mathbb{P}(\mathcal{W}^* > t) \tag{A.1}$$

for all $t > 1$. In particular,

$$\mathbb{E}\phi(W) < \infty \quad \Leftrightarrow \quad \mathbb{E}\phi(\mathcal{W}^*) < \infty \tag{A.2}$$

for any nondecreasing convex $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ and $\phi(2x) \leq c\phi(x)$ for some $c > 0$ and all $x \geq 0$.

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