Model reduction for multiscale problems

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In this contribution we present efficient numerical multiscale methods for flow in heterogeneous porous media, in particular also in situations where the resulting equations are to be solved repeatedly for varying parameters, as e.g. in the context of uncertainty quantification, time dependent scenarios or optimal control problems. We discuss a posteriori based discretization methods and suggest a suitable conceptual approach for an efficient numerical treatment of parameterized variational multiscale problems where the parameters are either chosen from a low dimensional parameter space or consists of parameter functions from some compact low dimensional manifold that is embedded in some high dimensional or even infinite dimensional function space. Our general approach [14] covers a large class of numerical multi-scale schemes based on an additive splitting of function spaces into macroscopic and fine scale contributions combined with a tensor decomposition of function spaces in the context of multi query applications.

In detail, let U, V denote suitable function spaces over a domain $\Omega \subset \mathbb{R}^d$ and let us look at solutions $u_\mu^\epsilon \in U$ of parameterized variational problems of the form

$$R_\mu^\epsilon[u_\mu^\epsilon](v) = 0 \qquad \forall v \in V.$$

with an ϵ and μ-dependent mapping $R_\mu^\epsilon : U \to V'$ where ϵ denotes a parameter that indicates the multiscale character of the problem, and $\mu : \Omega \to \mathbb{R}^p, p \in \mathbb{N}$ denotes a vector of parameter functions that do not depend on ϵ.

Numerical multiscale methods make use of a possible separation of scales in the underlying problem. The macroscopic scale is defined by a priori chosen macroscopic approximation spaces $U_H \subset U, V_H \subset V$, typically chosen as piecewise polynomial functions on a uniform coarse partition $\mathcal{T}_H$ of Ω. The fine scale in the multiscale problem is usually defined by a priori chosen microscopic approximation spaces $U_h \subset U, V_h \subset V$, also typically chosen as piecewise polynomial functions on a uniform fine partition $\mathcal{T}_h$ of Ω. For suitable choices of polynomial degrees and meshes the spaces should satisfy $U_H \subset U_h \subset U$, and $V_H \subset V_h \subset V$, respectively. In this setting, let us denote with $\pi_{U_H} : U \to U_H$, $\pi_{V_H} : V \to V_H$ projections into the coarse spaces. We then define fine parts of U_h, or V_h through

$$U_{f,h} := \{u_h \in U_h : \pi_{U_H}(u_h) = 0\}, V_{f,h} := \{v_h \in V_h : \pi_{V_H}(v_h) = 0\}.$$

The discrete solution $u_{\mu,h}^\epsilon \in U_h$ is then defined through its decomposition $u_{\mu,h}^\epsilon = u_H + u_{f,h} \in U_H \oplus U_{f,h}$, satisfying

$$R_\mu^\epsilon[u_H + u_{f,h}](v_H) = 0 \qquad \forall v_H \in V_H, \tag{1}$$

$$R_\mu^\epsilon[u_H + u_{f,h}](v_{f,h}) = 0 \qquad \forall v_{f,h} \in V_{f,h}. \tag{2}$$

In a further step, a localization of the fine scale correction $u_{f,h}$ is obtained. Thus, let a coarse partition $\mathcal{T}_H$ of Ω and macroscopic discrete function spaces $U_H(\mathcal{T}_H), V_H(\mathcal{T}_H)$ be given, e.g. by choosing globally continuous, piecewise polynomial finite element spaces on $\mathcal{T}_H$. Furthermore, we choose quadrature rules $(\omega_{T,q}, x_{T,q})_{q=1}^Q$ for $T \in \mathcal{T}_H$ and associate with each quadrature point $x_{T,q}$ a local

function space $U^{\delta}_{f,x_{T,q}}$ which might e.g. be given as

$$U^{\delta}_{f,x_{T,q}} := \{u_{f,x_{T,q}} = u_{f,h}|_{Y^{\delta}(x_{T,q})} : u_{f,h} \in U_{f,h})\}$$

where $Y^{\delta}(x_{T,q})$ is an appropriate discrete δ-environment of $x_{T,q}$ that can be decomposed with elements from the fine mesh $\mathcal{T}_h$. Local function spaces $V^{\delta}_{f,x_{T,q}}$ are defined analogously.

Next, we define local corrector operators $\mathcal{Q}_{x_{T,q}} : U_H \to U^{\delta}_{f,x_{T,q}}$ through an appropriate localization of 2, e.g.

$$R^{\epsilon}_{\mu}[u_H + \mathcal{Q}_{x_{T,q}}(u_H)](v_{f,x_{T,q}}) = 0 \qquad \forall v_{f,x_{T,q}} \in V^{\delta}_{f,x_{T,q}}. \tag{3}$$

A corresponding local reconstruction operator $\mathcal{R}_{x_{T,q}}$ is then given as

$$\mathcal{R}_{x_{T,q}}(u_H) = u_H + \mathcal{Q}_{x_{T,q}}(u_H) \tag{4}$$

and we obtain the overall method using numerical quadrature in the coarse scale equation (1) and by replacing $u_H + u_{f,h}$ in (1) by the localized reconstruction $\mathcal{R}_{x_{T,q}}(u_H)$. Depending on the choice of trail and test functions, and on the choice of specific localizations of the function space for the fine scale correctors and by choosing corresponding localized corrector operators a variety of numerical multiscale methods can be recovered. For a detailed derivation of the multiscale finite element method (MsFEM), the variational multiscale method, and the heterogeneous multiscale method (HMM) in such a framework we refer to the expositions in [6] and [12]. We in particular focus on a posteriori error estimation and adaptivity for HMM approximations of elliptic problems [13, 7] and for approximation of immiscible two phase flow in porous media [9, 10]. We also refer to [9] for homogenization of degenerate two phase flow in porous media in a more complex situation where also jumps in the capillary pressure and relative permeability curves on the fine scale are allowed. Finally, we present an a posteriori error estimate for MsFEM that in particular is able to measure the error due to oversampling in heterogeneous scenarios [8].

To efficiently cope with two phase flow in porous media in multi-query scenarios, we introduce the reduced basis approach [5] with extensions for non-linear PDEs, based on the concept of empirical operator interpolation [3]. Numerical experiments are given for two phase flow in porous media [4]. Finally, we present a generalization of the classical projection based reduced basis approach to efficiently cope with multiscale problems in multi-query scenarios. Let thus suppose that in a first step we have computed snapshots, i.e. solutions $u^{\epsilon}_{\mu,h}$ with our favorite numerical multiscale method for suitable chosen parameters $\mu_i, i = 1, \ldots, N$. The choice of suitable parameters may for example be done by a Greedy algorithm based on efficient a posteriori error estimates. Let us denote Φ_N a orthogonalized Basis of $V_N := \text{span}(\mu_i, i = 1, \ldots, N)$. The classical reduced basis approach is based on approximating solutions by linear expansions of the form $u^{\epsilon}_{\mu,N}(x) = \sum_{i=1}^{N} a_i \phi_i(x), x \in \Omega$. Hence, the spatial variation of the solution is represented by the globally defined basis functions only. Here, we apply a generalization of this approach (see [14]) by replacing the linear combination of reduced basis functions by the nonlinear combination $u^{\epsilon}_{\mu,N} = \sum_{i=1}^{N} a_i(x)\phi_i(x)$, with the hope to significantly reduce the number N of reduced basis functions needed to represent the solution manifold of the underlying parameterized problem. Here the coefficients a_i are now supposed to be macroscopic functions that are able to take care of the macroscopic spatial variation of the solution manifold. Let us for

instance assume $a_i \in U_H$, while $\phi_i \in V_N \subset U_h$. The reduced multiscale solution space is then given as $U_{H,N} := \{u_{H,N}(x) = \sum_{i=1}^{N} a_i(x)\phi_i(x) | a_i \in U_H, \phi_i \in \Phi_N\}$ and a corresponding reduced scheme is obtained by suitable projection of the original problem onto such function spaces.

Particular realizations of this approach are the local reduced basis discontinuous Galerkin method [11] and the localized reduced basis multiscale method [2], but also other approaches such as the mixed multiscale finite element method using limited global information [1], the generalized finite element method or partition of unity methods fit into this framework.

Numerical experiments are given to demonstrate the efficiency of the new approach.

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